A Symbolic-Numeric Approach to the Solution of the Butcher Equations

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Abstract

A new approach based on the introduction of new simplifying assumptions of a novel kind is introduced. The approach is based on the construction of a graduated finite-dimensional algebra for a given Butcher tableau. This approach allowed us to discover some new families of Runge-Kutta (RK) methods of orders less than or equal to 8. Most of the methods constructed have new features different from those of previously known methods. A new order 9 method has been found having only 13 stages. For all of these families we have found representatives numerically and introduced a method to find their local dimensions. Using numerical information we additionally derive analytical solutions in some cases.

1 Introduction

In the beginning of 1960s, Butcher [3, 5, 4] suggested a convenient form for the systems of equations (Butcher's systems) defining Runge-Kutta (RK) methods. A good exposition can be found in [1, 9, 11].

The problem received overwhelming attention and many significant contributions were made (e.g. [1, 7, 8, 13, 9, 12, 10, 6, 2] and many others). Most of the scientists approach the problem symbolically, and indeed in many cases analytic solution have been found.

On the other hand,the numerical approach has not previously received much attention because in the interesting cases the task is time computing and the result, that is finding of partial solutions of Butcher's systems is not useful enough. In the present paper we apply the direct numerical approach which finds partial solutions of Butcher's systems, and many of these new solutions do not belong to any previously known family. Having a new partial (numerical) solution, we derive its properties including the local dimension of the solution variety at this point. A special method is suggested for computation this local dimension using numerical data only. Using this additional information we impose some addition properties on the variables, which allow us to simplify the Butcher system in this case and solve it analytically.

Starting with the RK methods of order 8 and 9 the numerical approach fails, and we suggest a new kind of simplifying assumptions basing on the construction of a graduated finite-dimensional algebra for a Butcher tableau. These assumptions are reasonable because all known solutions obey these new assumptions, and are useful because new solutions can be found. In other words, relative to the older assumptions, the new assumptions lead to a smaller set of equations to be solved, in a new (smaller) set of variables, and yet all previously known solutions are reproduced, with new solutions now being accessible.

A particular benefit is the obtaining of a 13-staged RK method of order 9, while the previous best known method of order 9 has 15 steps [13].

The method for finding the local dimensions of the solution varieties is not restricted to the Butcher equations but can be used for any polynomial or analytical system of equations.

2 Preliminaries

The n-stage RK methods are defined by its *Butcher tableau*, which puts the coefficients of the method in a table as follows:

$$\begin{array}{c|ccccc} c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ & \dots & & & & \\ c_n & a_{n1} & a_{n2} & \dots & a_{n,n-1} & \\ \hline & b_1 & b_2 & \dots & b_{n-1} & b_n \end{array} \tag{1}$$

Butcher tableau determines a RK method [1, 9] of order p if and only if

$$\sum_{j=1}^{n} b_j \Phi_{tj}(A) = (b, \Phi_t(A)) = 1/\gamma(t) \tag{2}$$

holds for each tree t of order $\leq p$ (*Butcher equations* or *order conditions*).

The next table shows the number of order conditions. One can see that the construction of higher order RK methods is not an easy task.

order	1	2	3	4	5	6	7	8	9	10
number of equns	1	2	4	8	17	37	85	200	486	1205
minimum number of stages :				4	6	7	9	11	≤ 15	≤ 17

In the search for RK schemes of maximal order, Butcher derived relations between individual equations. These simplifying assumptions may be applied to reduce further the number of equations.

Usually one uses one or two simplifying assumptions: first one ($C(2)$, with $c_n = 1$) and second one ($D(1)$, with $b_2 = 0$). The table below shows the number of variables and equations for the following cases:

- (none): without any simplifying assumptions;
- (1): with simplifying assumption $C(2)$;
- (2): with simplifying assumptions $C(2)$ and $D(1)$.

order/stages	4/4	5/6	6/7	7/9	8/11	9/13
(none): eqs/vars	8/10	17/21	37/28	85/45	200/66	486/91
(1):eqs/vars	4/6	9/15	20/21	48/36	115/55	286/78
(2):eqs/vars		6/11	13/16	32/29	79/46	202/67

3 Our Approach

3.1 Numerical Solution

Given a Butcher system $F(X) = 0$ in vector notation with hundreds of equations and dozens of variables, we use Newton method for solution.

Let n-th step of Newton method have linear equations

$$A \cdot dX = b \ ,$$

where $A = D_F = (\partial f_i / \partial x_j)$ and $b = -F(X_n)$, where f_i is the i-th equation, and x_j is the j-th variable of the Butcher system, and X_n is the current approximation for solution $X = (x_1, \dots)$. Since the system is not square, we apply regularization method of Tikhonov [14],

$$(A^t A + \lambda E) * dX = A^t b$$

for some small λ, where E is the identical matrix, and A^t is the transpose of matrix A.

Experimental Observations. *Our computations show that*

1. *Solutions of Butcher systems are degenerate for order p larger than* 6*. Recall that a point of a variety is called degenerate if all partial derivatives of the equations of the system are zero at this point.*

2. *In the case of degenerate solution first-order approximation formulae for the numerical differentiation gives better convergence than those of second-order. Moreover, this happens in absolute majority of the cases. It is a challenge to prove this analytically.*

3. *Choosing numbers from* 0 *to* 1 *as initial guess for all the variables so that the the length of vector X sufficiently small improves convergency of the method.*

4. *Convergence improves if we choose the epsilon in the approximation formulae for the numerical differentiation from the interval from* 10^{-7} *to* 10^{-9}.

5. *For better convergency it is essential to change parameter λ of Tikhonov regularization method dynamically, that is to decrease its value from* 10^{-2} *to* 10^{-12} *as we approach to the solution.*

Owing to the degeneracy of the solutions of the Butcher systems of orders larger than 6, Newton method leads to no result. However, using experimental facts described above Newton iterations converge after $10000 - 20000$ iterations, which takes several hours on average computer.

Our implementation language is Pascal.

3.2 Local Dimensions

Let v be a solution of a Butcher system, and $v + v_i$ are $K \approx 200$ solutions in its small neighborhood. Then $v + v_i$ solutions belong approximately to the tangent subspace to the variety of solutions at the point v. In our case all the solutions are numerical, so $v+v_i$ solutions belong even more approximately to the tangent subspace.

Theorem below takes into account these two noise problems and computes the dimension of the tangent space.

Theorem 3.1. *Let v be the point on a smooth manifold $V \subset \mathbf{R}^n$ of dimension d, and $w_i = v+v_i$, $i = 1, \dots K$ be some points on V in ε-vicinity of v. Consider the non-negative square form*

$$q(X) = \frac{\sum_{i=1}^{K} (X, v_i)^2}{\varepsilon^2} \ .$$

in $\mathbf{R}^n$ and let $\mu_i, i = 1, \dots, K$ be its eigenvalues. Let $C(\delta, \varepsilon)$ be the number of the μ_i values which are greater than δ. Then there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$ and for any $\varepsilon < \varepsilon_0$, $C(\delta, \varepsilon) \leq d$.

Proof. Let L be the tangent vector subspace for manifold V at the point v and L' be its orthogonal complement, $L \oplus L' = \mathbf{R}^n$ and $v_i = u_i + w_i$, $u_i \in L$, $w_i \in L'$. Then there exist $\varepsilon_0 > 0$ and constant C_0, such that for any $\delta > 0$ and for any $\varepsilon < \varepsilon_0$,

$$|w_i| < C_0 \cdot \varepsilon \cdot |v_i| \ .$$

Therefore, we have

$$q(w) < C_0 \cdot K \cdot \varepsilon \cdot |w|, \quad \forall w \in L' \ .$$

Therefore, all eigenvalues of the restriction of the square form $q(X)$ on the subspace L' are less than $C_0 \cdot K \cdot \varepsilon$ and the number of eigenvalues of the initial square form $q(X)$ that are greater than $C_0 \cdot K \cdot \varepsilon$ is not greater than $d = \dim L$. □

Therefore, if we find the eigenvalues μ_i (using Jacobi rotations method) of the non-negative square form $q(X)$ and count the number of those of them that are greater than ε, then we get a lower bound for the local dimension of the solution family.

Remark 3.2. In fact, it is not difficult to choose v_i vectors in such a way that that lower bound would be equal to the local dimension: one should make sure that the projections of all v_i vectors together generate the tangent space L.

Example 3.3. If the eigenvalues values are

$$1, \quad 0.85, \quad 0.77, \quad 1.1 \times 10^{-9}, \dots$$

then one may assume that the local dimension of this variety at the point v equals 3. If

$$1, \quad 0.68, \quad 0.48, \quad 0.40, \quad 0.40, \quad 0.29, \quad 9.3 \cdot 10^{-10}, \quad \dots \ ,$$

then the local dimension is equal to 6.

3.3 Filtrated Simplifying Assumptions

We suggest to rewrite Butcher equations as follows.

First of all, we suggest to consider $(n+1)\times(n+1)$ matrix $\widetilde{A}$, which consists of Butcher tableau A with the extra row at the bottom, consisting of b_i (1), and with extra zero-column on the right to make the new matrix square. Correspondingly, we have new vectors $\Phi_t(\widetilde{A})$.

Let $L = \mathbf{R}^{n+1}$ with coordinate-wise multiplication ("$*$"), and $e = (1,\dots,1)$, $e_{n+1} = (\underbrace{0,\dots,0}_{n},1)$.

Consider subspaces $L_k =< \Phi_t(\widetilde{A})|\text{t is a tree of weight k} >\subset L$, that is

$$\begin{aligned}
L_0 &=< e > \ , \\
L_1 &=< \widetilde{A}e > \ , \\
L_2 &=< \widetilde{A}^2e,\ \widetilde{A}e * \widetilde{A}e > \ , \\
L_3 &=< \widetilde{A}^3e,\ \widetilde{A}^2(\widetilde{A}e*\widetilde{A}e),\ \widetilde{A}(\widetilde{A}e*\widetilde{A}e*\widetilde{A}e),\ \widetilde{A}(\widetilde{A}^2e*\widetilde{A}e) > \ , \\
&\dots
\end{aligned}$$

Then the Butcher equations can be written as

$$(\widetilde{A}v, e_{n+1}) = \frac{(v, e_{n+1})}{k+1}, \quad \forall v \in L_k \ .$$

Consider the following filtration of the space L for every given matrix $\widetilde{A}$:

$$\begin{aligned}
M_0 &= L_0 \ , \\
M_1 &= L_0 + L_1 \ , \\
M_2 &= L_0 + L_1 + L_2 \ , \\
M_3 &= L_0 + L_1 + L_2 + L_3 \ , \\
&\dots
\end{aligned}$$

It follows from the Butcher equations that, the filtration corresponds to the multiplication, that is

$$M_i * M_j \subset M_{i+j}, \quad \widetilde{A}(M_i) \subset M_{i+1} \ .$$

Definition 3.4. We say that the adjoint algebra with respect to this filtration,

$$B(A) = \bigoplus_{k=0}^{n} B_k(\widetilde{A}) = \bigoplus_{k=0}^{n} M_k/M_{k-1}$$

is an *upper Butcher algebra*.

Now, consider $L'_k = L_k \cap < e_{n+1} >^{\perp}$, where $L' =< e_{n+1} >^{\perp}$ is the subspace of vectors that have the last coordinate equaled to zero. Then

$$\begin{aligned}
L'_0 &= 0 \ , \\
L'_1 &= 0 \ , \\
L'_2 &=< \widetilde{A}^2e - \tfrac{1}{2}\widetilde{A}e*\widetilde{A}e > \ , \\
&\dots
\end{aligned}$$

Again we consider a filtration of the space L':

$$\begin{aligned} M'_0 &= L'_0 \ , \\ M'_1 &= L'_0 + L'_1 \ , \\ M'_2 &= L'_0 + L'_1 + L'_2 \ , \\ M'_3 &= L'_0 + L'_1 + L'_2 + L'_3 \ , \\ \dots \end{aligned}$$

Definition 3.5. We say that the adjoint algebra with respect to this filtration,

$$B'(\widetilde{A}) = \bigoplus_{k=0}^{n} B'_k(\widetilde{A}) = \bigoplus_{k=0}^{n} M'_k/M'_{k-1}$$

is a *lower Butcher algebra*.

By construction, both algebras are commutative, associative, graduated, and finite dimensional.
By construction, the 0-th and 1-st components of "lower" Butcher algebra have dimension zero, and the 2-nd components have dimension one.

Definition 3.6. We call the restrictions on the dimensions of the 3-rd and the 4-th components of the lower Butcher algebra *filtrated simplifying assumptions*.

Remark 3.7. The dimension of the subspace B'_3 equals to $(\dim M'_3 - \dim M'_2)$. Since the subspaces L'_2 and L'_3 are generated by one and four vectors, correspondingly, the restriction $\dim B'_3 = 1$ means that $\dim M'_3 = 2$, that is four vectors generates just two-dimensional subspace.

We use the filtrated simplifying assumptions together with the classical ones ($C(2)$ and $D(1)$). Importantly, the filtrated simplifying assumptions hold for all known RK methods, that is for all known solutions of Butcher systems, which means that using these new assumptions we do not lose at least known solutions. On the other hand, the new simplifying assumption significantly reduce the number of equations in Butcher systems of high orders.

Remark 3.8. $\dim B'_3 = 1$ holds for all known methods of orders from 1 to 8 excluding order 5. For known methods of order 5 we always have $\dim B'_3 = 2$. Also for all known method either $\dim B'_4 = 1$, or $\dim B'_4 = 2$ holds.

Example 3.9. Consider the following family of 7-stage method of order 6 found by Butcher (we mention this family as 1. in Subsec. 4.1). The dimensions of the subspaces mentioned above are as follows.

$i:$	0	1	2	3	4	5	6
$dim(M_i):$	1	2	4	6	8	8	8
$dim(B_i):$	1	1	2	2	2	0	0
$dim(M'_i):$	0	0	1	2	4	5	6
$dim(B'_i):$	0	0	1	1	2	1	1

Example 3.10. If we use only one assumption $\dim B_3' = 1$, then the number of equations decreases relatively to the order of a RK method as follows:

order	usual simpl.assump.	with $\dim B_3' = 1$
4	3	3
5	6	5
6	13	10
7	32	23
8	79	54
9	202	133

3.4 Useful Proposition

The following result turned out to be useful for computations above. Up to our knowledge it has not appeared before in the literature.

Proposition 3.11. . *Given a Butcher tableau by its values (a_{ij}, b_i, c_i), which defines a RK method of some order satisfying two (classical) simplifying assumptions $C(2)$ and $D(1)$. Consider a new Butcher tableau, where c_2 is arbitrary and nonzero,*

$$\begin{array}{ll} a_{i2} & = (c_i^2/2 - a_{i3}c_3 - \cdots - a_{i,n-2}c_{n-2})/c_2 \ , \\ a_{i1} & = c_i - a_{i2} - \cdots - a_{i,i-1} \ , \end{array} .$$

and the rest of a_{ij}, b_i, c_i values remain unchanged. Then this tableau defines a RK method of the same order.

In other words, given a RK method, one can obtain a 1-dimensional family of RK methods parameterized by $c_2 \neq 0$.

Proof. Using 1st and 2nd simplifying assumption, we can express coefficients a_{i1}, a_{i2} through c_i, b_i and rest a_{ij}. After that, all the remaining equations will not depend on c_2. □

Corollary. For the further investigations of such methods one can simply put $c_2 = 1$.

4 Results

4.1 Methods of order 6

Consider Butcher system of order 6. Butcher [1] has found a family of 7-stages solutions of order 6. He has also proved that solutions with less number of stages are not possible. The parameters of the Butcher system in the 7-stages case are as follows: 37 equations and 28 variables, when we do not use simplifying assumptions, 20 equations and 21 variables if we use one simplifying assumption $C(2)$, and 13 equations and 16 variables if we use two simplifying assumptions (both $C(2)$ and $D(1)$).

Using our approach we discovered the following 7-stages solutions families. Since the families are large, it is convenient to use the major property (we shall call it "key property") of the family to distinguish them from each other.

1. Family with the key property $b_2 = 0$, in fact this case is the case with two simplifying assumptions. This family is exactly the family found by Butcher (dim = 4, and c_2, c_3, c_5, c_6 are free variables).

2. New family with the key property $b_2 \neq 0$ (so, $D(1)$ not hold), $b_1 = b_7 = 1/12$, $a_{61} = 1/6$. Local dimension is 4. We can find this solution family also analytically.

Example 4.1. Below is one representative of the family with the key property $b_2 \neq 0, b_1 = b_7 = 1/12$, $a_{71} = 1/6$. Notations $a(i,j) = a_{ij}$, $b(i) = b_i$ are used.

$a(2,1) =$	$0.092546683747426861;$	$a(6,5) =$	$0.252734125810753910;$
$a(3,1) =$	$-2.691723487894211920;$	$a(7,1) =$	$0.166666666666661427;$
$a(3,2) =$	$3.498440208912323260;$	$a(7,2) =$	$-1.087161772135447480;$
$a(4,1) =$	$-0.253228673413545159;$	$a(7,3) =$	$-1.439658909532307060;$
$a(4,2) =$	$0.486946470506554487;$	$a(7,4) =$	$2.115682066785066460;$
$a(4,3) =$	$-0.032589538034097808;$	$a(7,5) =$	$-0.980534445595906839;$
$a(5,1) =$	$1.266099168769624020;$	$a(7,6) =$	$2.225006393811933490;$
$a(5,2) =$	$-1.813678883893583040;$	$b(1) =$	$0.083333333333333245;$
$a(5,3) =$	$0.100690134574407749;$	$b(2) =$	$-0.047019299666198284;$
$a(5,4) =$	$0.857476050925974915;$	$b(3) =$	$-0.488931671883268965;$
$a(6,1) =$	$-1.711411739321930800;$	$b(4) =$	$0.294862881206967411;$
$a(6,2) =$	$2.383574928383134820;$	$b(5) =$	$0.225413862094639929;$
$a(6,3) =$	$0.014583794759490532;$	$b(6) =$	$0.849007561581193328;$
$a(6,4) =$	$-0.157873981349818600;$	$b(7) =$	$0.083333333333333336;$

3. New family with the key property $a_{62} = 0, b_2 \neq 0$ (so, $D(1)$ not hold). The local dimension is 4. We assume that such can be found analytically also.

Example 4.2. Below is one representative of the family with the key property $a_{62} = 0, b_2 \neq 0$. Notations $a(i,j) = a_{ij}$, $b(i) = b_i$ are used.

$a(2,1) =$	$0.84232731875896394;$	$a(6,5) =$	$0.05403744178074496;$
$a(3,1) =$	$0.22176793098704824;$	$a(7,1) =$	$-1.37990563691314763;$
$a(3,2) =$	$0.04097942043558803;$	$a(7,2) =$	$-1.66316707279261268;$
$a(4,1) =$	$-1.17706918659920457;$	$a(7,3) =$	$2.47436677928874809;$
$a(4,2) =$	$-0.46785430863828377;$	$a(7,4) =$	$0.30847520796140411;$
$a(4,3) =$	$2.28880313771194906;$	$a(7,5) =$	$0.19463441420057583;$
$a(5,1) =$	$1.99620306265929484;$	$a(7,6) =$	$1.06559630825503227;$
$a(5,2) =$	$0.72108861202663995;$	$b(1) =$	$0.07973197393356595;$
$a(5,3) =$	$-2.48269997856082245;$	$b(2) =$	$-0.50263462295926127;$
$a(5,4) =$	$0.55969720790019506;$	$b(3) =$	$0.39217619545543645;$
$a(6,1) =$	$0.81679069139657866;$	$b(4) =$	$0.25131510370831655;$
$a(6,2) =$	$0.00000000000000000;$	$b(5) =$	$0.20240358195781663;$
$a(6,3) =$	$0.06446645111760932;$	$b(6) =$	$0.50263462295926126;$
$a(6,4) =$	$-0.09296726553596899;$	$b(7) =$	$0.07437314494486442;$

4. Some other families are possible as we found some individual solutions that do not belong to any of the families above. We have not investigate them enough to derive their key properties.

4.2 Methods of order 7, 9 stages

Consider Butcher system of order 7. Butcher [1] has found a family of 9-stages solutions of order 7. He has also proved that solutions with less number of stages are not possible. The parameters of the Butcher system in the 9-stages case are as follows: 85 equations and 45 variables, when we do not use simplifying assumptions, 48 equations and 36 variables if we use one simplifying assumption, and 32 equations and 29 variables if we use two simplifying assumptions (both $C(2)$ and $D(1)$).

1. Family with the key property $b_2 = 0$, in fact this case is the case with two simplifying assumptions. This family is exactly the family found by Butcher (dim = 4, and c_4, c_5, c_6, c_7 are free variables).

2. A new family with unknown key property. Below is a representative of such family:

$$
\begin{array}{rlrl}
a(2,1) = & 0.055709745557893838; & a(8,3) = & 1.488453166149889360; \\
a(3,1) = & 0.026263635087658255; & a(8,4) = & -1.124712130835941090; \\
a(3,2) = & 0.042761323341521044; & a(8,5) = & 0.232807907805008520; \\
a(4,1) = & 0.180713934065073522; & a(8,6) = & 0.589620688908153016; \\
a(4,2) = & -1.724630252644998760; & a(8,7) = & -0.065896980303874902; \\
a(4,3) = & 1.773392326210328340; & a(9,1) = & 0.576596684335809870; \\
a(5,1) = & 0.122184739594819391; & a(9,2) = & -0.038206758304919255; \\
a(5,2) = & -0.476495863566640026; & a(9,3) = & -0.543418436743487204; \\
a(5,3) = & 0.470646779056013970; & a(9,4) = & -0.650089629457115554; \\
a(5,4) = & 0.218727173747420283; & a(9,5) = & 1.515186292771789430; \\
a(6,1) = & -0.668326023395610138; & a(9,6) = & 0.357327203429111682; \\
a(6,2) = & 0.798848097403175316; & a(9,7) = & 1.848052699961051090; \\
a(6,3) = & 0.567530742143598450; & a(9,8) = & -2.065448055992240060; \\
a(6,4) = & -1.074052801061825440; & b(1) = & -0.021131703541562841; \\
a(6,5) = & 0.907660014000541363; & b(2) = & 0.000000000000000000; \\
a(7,1) = & -0.667425091647899129; & b(3) = & 0.218257837154959567; \\
a(7,2) = & 0.289845464918453625; & b(4) = & -0.011716324536766582; \\
a(7,3) = & 0.969502991873638334; & b(5) = & 0.246207518407523748; \\
a(7,4) = & -0.244508963214403731; & b(6) = & 0.319797128012205456; \\
a(7,5) = & -0.211390924656335116; & b(7) = & 0.601609315645574299; \\
a(7,6) = & 0.619876986535953368; & b(8) = & -0.417596709342784309; \\
a(8,1) = & -1.252873635170588490; & b(9) = & 0.064572938200850633; \\
a(8,2) = & 0.813220953162738650; & &
\end{array}
$$

The local dimension of solution family at this point is equal to 6.

3. A new family of dimension 8. Below is a representative of such family:

$a(2,1) =$	$0.034363715721150246;$	$a(8,3) =$	$0.19911974070542774 2;$
$a(3,1) =$	$0.011408906068230086;$	$a(8,4) =$	$0.421107413657664193;$
$a(3,2) =$	$0.042873648688569631;$	$a(8,5) =$	$-0.542703355965776147;$
$a(4,1) =$	$0.093957708054522383;$	$a(8,6) =$	$0.712341382801249113;$
$a(4,2) =$	$-0.287918426922017676;$	$a(8,7) =$	$0.149159777373627983;$
$a(4,3) =$	$0.313174533344502095;$	$a(9,1) =$	$-0.313143378191429612;$
$a(5,1) =$	$0.529064287945578277;$	$a(9,2) =$	$0.000705719707173451;$
$a(5,2) =$	$-0.314579863958881281;$	$a(9,3) =$	$0.993996434424775352;$
$a(5,3) =$	$-0.685910969958008961;$	$a(9,4) =$	$-0.579823368233022578;$
$a(5,4) =$	$0.765359510917423582;$	$a(9,5) =$	$-0.138878024906600228;$
$a(6,1) =$	$-0.276850654629649011;$	$a(9,6) =$	$0.860606821512522765;$
$a(6,2) =$	$0.054653953415982125;$	$a(9,7) =$	$-1.186131874156395390;$
$a(6,3) =$	$0.711337439351539684;$	$a(9,8) =$	$1.362667669842976240;$
$a(6,4) =$	$-0.381649949076486322;$	$b(1) =$	$-0.044160962796119653;$
$a(6,5) =$	$0.301303725620202506;$	$b(2) =$	$0.000000000000000000;$
$a(7,1) =$	$-2.116797922601723170;$	$b(3) =$	$0.208220627606838437;$
$a(7,2) =$	$0.373069905871642721;$	$b(4) =$	$0.038270672182914916;$
$a(7,3) =$	$3.102068941425127280;$	$b(5) =$	$0.034978734936192145;$
$a(7,4) =$	$-0.329903697109250135;$	$b(6) =$	$0.370718295634355222;$
$a(7,5) =$	$-2.119261163321491940;$	$b(7) =$	$-0.063318023219908099;$
$a(7,6) =$	$1.810866965331287640;$	$b(8) =$	$0.391156820405265850;$
$a(8,1) =$	$-0.204401722537450190;$	$b(9) =$	$0.064133835250461204;$
$a(8,2) =$	$0.041954598532237562;$		

4.3 Methods of Order 8, 11 stages

Consider Butcher system of order 8. Verner [13] has found a family of 11-stages solutions of order 8. Therefore, it is enough to have 11 stages in this case and the parameters of the Butcher system in this case are as follows: 200 equations and 66 variables, when we do not use simplifying assumptions, 115 equations and 55 variables if we use one simplifying assumption, and 79 equations and 46 variables if we use two simplifying assumptions. Using our approach we discovered four 11-stages solutions families named by their following key properties:

- case $b_2 = a_{11,2} = 0$, local dimension is 7 (dim = 7);
- case $b_2 = b_3 = 0$, dim = 7;
- case $b_2 = b_3 = a_{11,2} = a_{10,2} = a_{9,2} = a_{8,2} = a_{7,2} = a_{6,2} = a_{5,2} = a_{4,2} = 0$, dim = 9;
- case $b_2 = b_3 = b_4 = a_{11,2} = a_{10,2} = a_{9,2} = a_{8,2} = a_{7,2} = a_{6,2} = a_{5,2} = 0$, dim = 7;

4.4 Methods of Order 9, 13 stages

Using filtrated simplifying assumptions, 13-stages RK methods of order 9 has been obtained (at the moment numerically only). In comparison Verner [13] has described a 15-stage method of order 9.

$$
\begin{aligned}
b(1) &= 0.03864998346208471; \\
b(2) &= 0.00000000000000000; \\
b(3) &= 0.00006697474166654; \\
b(4) &= -0.00644462489124065; \\
b(5) &= -0.13834186387360676; \\
b(6) &= 0.45808280422265741; \\
b(7) &= 0.17220783318423803; \\
b(8) &= 0.30231913753516575; \\
b(9) &= 0.01123452128011847; \\
b(10) &= -0.32225389004297118; \\
b(11) &= 0.34725529781658727; \\
b(12) &= 0.10909525391742731; \\
b(13) &= 0.02812857264787309;
\end{aligned}
$$

$$
\begin{aligned}
c(2) &= 1.23456789012345679; \\
c(3) &= 0.83093125079368057; \\
c(4) &= 0.25307939689146609; \\
c(5) &= 0.73978526602389008; \\
c(6) &= 0.32121210081087191; \\
c(7) &= 0.89708311688417826; \\
c(8) &= 0.14772935943031283; \\
c(9) &= 0.24771823716447115; \\
c(10) &= 0.24771823716447103; \\
c(11) &= 0.70886696164310943; \\
c(12) &= 0.55354855354899890; \\
c(13) &= 1.00000000000000000;
\end{aligned}
$$

$$
\begin{array}{rrrr}
a(2,1) = & 1.23456789012345679; & a(10,4) = & -0.04610490029804602; \\
a(3,1) = & 0.55130031714105422; & a(10,5) = & 0.75253043498499616; \\
a(3,2) = & 0.27963093365262635; & a(10,6) = & -0.10441616111328677; \\
a(4,1) = & 0.24466518387153281; & a(10,7) = & 0.00147329075932362; \\
a(4,2) = & 0.06201854732004206; & a(10,8) = & -0.19778083926479465; \\
a(4,3) = & -0.05360433430010878; & a(10,9) = & 0.08923687549442470; \\
a(5,1) = & -0.00530803298883925; & a(11,1) = & -0.06306037528545690; \\
a(5,2) = & 0.01122706141068128; & a(11,2) = & 0.00538294453321659; \\
a(5,3) = & 0.12815417149486828; & a(11,3) = & 0.16035657574223180; \\
a(5,4) = & 0.60571206610717977; & a(11,4) = & 0.02340627199559249; \\
a(6,1) = & 0.20385530723008664; & a(11,5) = & -0.08789296531510297; \\
a(6,2) = & 0.00136346637020209; & a(11,6) = & 0.60403792745181988; \\
a(6,3) = & -0.33639699411752137; & a(11,7) = & 0.00076880185958083; \\
a(6,4) = & 0.01077366106275497; & a(11,8) = & 0.40581575264524803; \\
a(6,5) = & 0.44161666026534959; & a(11,9) = & 0.55229114187524298; \\
a(7,1) = & 0.20326619346351716; & a(11,10) = & -0.89223911385926331; \\
a(7,2) = & 0.00030757434940501; & a(12,1) = & 0.00916707519205579; \\
a(7,3) = & -0.22597306215359036; & a(12,2) = & -0.00533157131032308; \\
a(7,4) = & -0.32538442851325729; & a(12,3) = & 0.06085378313059202; \\
a(7,5) = & 0.65042205710933842; & a(12,4) = & -0.22364683008555630; \\
a(7,6) = & 0.59444478262876532; & a(12,5) = & -0.88422193839572302; \\
a(8,1) = & 0.13299848113276506; & a(12,6) = & 0.87059878101090858; \\
a(8,2) = & 0.00029546140345323; & a(12,7) = & 0.03359047564465317; \\
a(8,3) = & -0.16156305306080680; & a(12,8) = & 0.59052956197703567; \\
a(8,4) = & 0.01653216363090123; & a(12,9) = & -0.18969593856262461; \\
a(8,5) = & 0.21249028701914083; & a(12,10) = & -0.57064404757925808; \\
a(8,6) = & -0.05379813867195099; & a(12,11) = & 0.86234920252723876; \\
a(8,7) = & 0.00077415797681026; & a(13,1) = & -0.27709947154251589; \\
a(9,1) = & 0.39384668082978243; & a(13,2) = & 0.00000000000000000; \\
a(9,2) = & 0.04642114869757957; & a(13,3) = & 0.59063198402216033; \\
a(9,3) = & -0.43327524322219219; & a(13,4) = & 4.53614500824935334; \\
a(9,4) = & -0.09788311725633047; & a(13,5) = & -1.80880252398515069; \\
a(9,5) = & 0.51870168682340150; & a(13,6) = & -4.04580607620794161; \\
a(9,6) = & 0.02295547927552810; & a(13,7) = & 0.50042903704327684; \\
a(9,7) = & -0.00392203324977779; & a(13,8) = & -0.32658866525372163; \\
a(9,8) = & -0.19912636473352000; & a(13,9) = & -4.75966712951217952; \\
a(10,1) = & 0.32543103357747712; & a(13,10) = & 4.60967549313617009; \\
a(10,2) = & 0.00199180356077018; & a(13,11) = & 0.24954286915090418; \\
a(10,3) = & -0.57464330053639332; & a(13,12) = & 1.73153947489964456;
\end{array}
$$

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