Symmetries of Runge-Kutta methods

Sergey Khashin
Department of Mathematics, Ivanovo State University, Ivanovo, Russia
e-mail: khash2@gmail.com

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Abstract

A new (abstract algebraic) approach to the solution of the order conditions for Runge-Kutta methods (RK) and to the corresponding simplifying assumptions was suggested in [9, 10]. The approach implied natural classification of the simplifying assumptions and allowed to find new RK methods of high orders.

Here we further this approach. The new approach is based on the upper and lower Butcher's algebras. Here we introduce axillary varieties $\mathcal{M}_D$ and prove that they are projective algebraic varieties (Theorem 3.2). In some cases they are completely described (Theorem 3.5).

On the set of the 2-standard matrices (Definition 4.4) (RK methods with the property $b_2 = 0$) the one-dimensional symmetries are introduced. These symmetries allow to reduce consideration of the RK methods to the methods with $c_2 = 2/3\, c_3$, that is c_2 can be removed from the list of unknowns.

We formulate a hypothesis on how this method can be generalized to the case $b_2 = b_3 = 0$ where two-dimensional symmetries appear.

1 Introduction

The problem of construction of Runge-Kutta (RK) methods can be reduced to the solution of the order conditions (Butcher equations) that are the systems of large number of polynomial equations in large number of variables [1, 3, 7]. Numerous particular RK methods were obtained [2, 7, 4, 5, 13, 15, 17], but the general problem of the construction and the description of the RK methods remain open [2, 8, 15, 16].

One of the most important tool for obtaining RK methods are simplifying assumptions [2, 7, 15, 16]. Two of them are particular popular ($C(2)$ and $D(1)$ in [2, 7, 14]).

In papers [9, 10] we started a new abstract algebraic approach to the solution of Butcher's equations and developed new type of simplifying assumptions (based on the restriction of dimensions of some subspaces M_k, M'_k). In the present paper, this approach is refined and formalized. For this purpose, the projective algebraic varieties $\mathcal{M}_D$ (Definition 3.1) are introduced. Their dimensions are calculated in some cases (Theorem 3.5).

Some ideas suggested here are mentioned in the profound paper [14], which is concerned with the general theory of RK methods.

The simplifying assumption $D(1)$ is defined by the following system of equalities:

$$\begin{array}{ll} b_2 & = 0\,, \\ b_3(a_{32}c_2 - c_2^2/2) & = 0\,, \\ b_4(a_{42}c_2 + a_{43}c_3 - c_3^2/2) & = 0\,, \\ \dots & \\ b_s(a_{s2}c_2 + \cdots + a_{s,s-1}c_{s-1} - c_s^2/2) & = 0\,, \end{array} \tag{1}$$

Hence $b_2 = 0$ and for each $k > 2$ either $b_k = 0$, or

$$\frac{c_k^2}{2} = a_{k2}c_2 + \dots a_{k,k-1}c_{k-1}\,. \tag{2}$$

In the present paper we consider the case where for all $k > 2$ equality (2) holds. The matrix, that is satisfied this condition will be called a 2-standard matrix (Definition 4.4). Our construction is valid for both Butcher's and arbitrary matrices. Most of the known RK methods of high order (see, for example, [1, 7, 17]) satisfy this condition.

We suggest to consider one-dimensional symmetry on the set of the 2-standard matrices (2-transformation, Definition 4.6). The symmetry is the action of the group of nonzero reals $\mathbb{R}^*$ that is preserves the spaces M_k and M_k. From the point of view of the RK methods, this means that for each method satisfying the second simplifying assumption one can get a family of new RK methods (Theorem 4.6) with c_2 as a non-zero parameter (the remaining c_i are unchanged). That is by a suitable 2-transformations one can transform any RK method to the one of the same order with $c_2 = 2/3\, c_3$. This means, that it is enough to consider RK methods with $c_2 = 2/3\, c_3$.

In conclusion, we formulate a Hypothesis 4.9, that the method can be generalized to the case of $b_2 = b_3 = 0$. More complicated two-dimensional symmetries will be acting on the corresponding set of matrices. It would be also interesting to find out the structures of the upper and lower Butcher's algebras in this case. We do not expect them to be the same as for the 2-standard matrices.

The paper is organized as follows. In Sec. 2 we fix some notations and highlight some needed results from the previous papers of the author. In Sec. 3 we introduce varieties $\mathcal{M}_D$ and prove their properties. In Sec. 4 we introduce 2-standard matrices and their symmetries. We study their properties and formulate hypothesis on its non-trivial generalization to the case $b_2 = b_3 = 0$.

2 Preliminaries

Let an s-stage RK methods be defined by its *Butcher tableau*

$$\begin{array}{c|ccccc} c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ & \dots & & & & \\ c_s & a_{s1} & a_{s2} & \dots & a_{s,s-1} & \\ \hline & b_1 & b_2 & \dots & b_{s-1} & b_s \end{array} \tag{3}$$

Then the corresponding order conditions (*Butcher equations*) for a RK method [1, 7] of order p are as follows.

$$\sum_{j=1}^{s} b_j \Phi_{tj}(A) = b^T \Phi_t(A) = (b, \Phi_t(A)) = 1/\gamma(t)\ , \tag{4}$$

where t is an arbitrary tree of order $\rho(t) \le p$, and $\Phi_t(A)$ is a homogeneous polynomial of degree $(w(t) - 1)$ of coefficients from the matrix A.

Let A and $b = (b_1, \dots b_s)$ be the matrix and the row-vector defining an s-stage RK method of order p,

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ a_{21} & 0 & \dots & 0 \\ & & \dots & \\ a_{s1} & a_{s2} & \dots & 0 \end{pmatrix}\ . \tag{5}$$

As in [9] we consider instead of the pair (A, b), the extended matrix, which is matrix A with an extra row b at the bottom, and an extra column of zeros on the right to make it square. We denote the extended matrix by A, which should not lead to a confusion, since we are not using the initial matrix (5).

Several algebraic constructions of the present paper are valid for arbitrary lower-triangular with a zero diagonal matrices:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 \\ & \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} & 0 \end{pmatrix}, \tag{6}$$

the sum of elements of a string we will denote as

$$c_k = a_{k1} + \cdots + a_{k,k-1}\,.$$

The dimension of the space of such matrix is $n(n-1)/2$. Therefore, for the sake of generality, we consider extended matrix A in the form (6). That is $n = s+1$ and $b_1 = a_{n1}, \dots, b_s = a_{n,n-1}$.

In ([10], Theorem 3) we found a convenient form for Butcher equations. Namely, an extended matrix A defines a RK method of order p if and only if

$$(d, \Phi_t(A)) = \frac{1}{\delta(t)} \tag{7}$$

holds for every tree t of the weight less or equal to p.

By e we denote column vector $e = (1, \dots, 1)^t$. Then

$$Ae = (0, c_2, \dots, c_n)^t\,.$$

By $*$ we denote the coordinate-wise multiplication in $\mathbb{R}^n$:

$$(x_1, \dots, x_n)^t * (y_1, \dots, y_n)^t = (x_1y_1, \dots, x_ny_n)^t\,.$$

Recall some algebraic constructions introduced in [9]. We considered subspaces $L_k = <\Phi_t(A)>$ of $\mathbb{R}^n$ where t is a tree of weight k, and the filtration [1] of the space $\mathbb{R}^n$ for every given matrix A: $M_k = L_0 + \cdots + L_k$. Then

$$M_0 \subset M_1 \subset \cdots \subset \mathbb{R}^n$$

In [10] we introduced vectors $\Phi'_t(A)$ as follows. For an arbitrary tree t

$$\Phi'_t(A) = \delta(t)\Phi_t(A) - \underbrace{Ae * \cdots * Ae}_{d}\,,$$

where $d = w(t)$ is the weight of the tree. For $t = t_0$, $d = 0$ and $\Phi'(t_0) = 0$. Then for a given matrix A we considered subspaces L'_k, $k = 0, 1, \dots$ generated by vectors $\Phi'_t(A)$ for all trees t of weight k, and the corresponding filtration of the space $\underline{\mathbb{R}}^n$ by subspaces $M'_k = L'_2 + \cdots + L'_k$.

3 Varieties $\mathcal{M}_D$

The essential result of [9, 10] for the theory of RK methods, is that the restrictions of the dimensions of the vector spaces M_k and M'_k give simplifying assumptions which hold for almost all known RK methods of high order. Using two of these new simplifying assumptions we obtained several new families of RK methods of high orders. We hypothesize that using new simplifying assumptions one can obtain the general solution of Butcher equations.

By definition, the space M_0 for arbitrary matrix A is generated by one vector e, so $\dim M_0 = 1$ always. The space M_1 is generated by two vectors e and Ae. So, $\dim M_1 = 2$ if at least one of the c_k are non-zero. Spaces M'_0 and M'_1 are zero by definition. When $k \geq 2$ the dimensions of spaces M_k M'_k depend on the matrix.

Definition 3.1. For a given set of integers $D = (d_1, \dots, d_k)$ denote by $\mathcal{M}_D(n)$ the set of $n\times n$-matrices, such that $\dim M_i \leq d_i$, and by $\mathcal{M}'_D(n)$ the set of matrices, such that $\dim M'_i \leq d_i$.

Theorem 3.2. *The sets $\mathcal{M}_D(n)$ and $\mathcal{M}'_D(n)$ are projective algebraic varieties* [2] *in the space of lower-triangular matrices with zero diagonal.*

Proof. The coefficients of vectors $\Phi_t(A)$ are polynomially expressed through the coefficients of the matrix A. Vectors $\Phi_t(A)$ are linearly dependent, if the corresponding determinants of their coefficients are zeros. Therefore, the sets $\mathcal{M}_D(n)$ and $\mathcal{M}'_D(n)$ are given by some systems of polynomial equations of the coefficients of the matrix A and, therefore, are affine algebraic varieties.

To prove that $\mathcal{M}_D(n)$ and $\mathcal{M}'_D(n)$ are projective, it is sufficient to prove, that if the matrix $A \in \mathcal{M}_D(n)$ (or $\mathcal{M}'_D(n)$), then $\lambda A \in \mathcal{M}_D(n)$ ($\mathcal{M}'_D(n)$ correspondingly) for all nonzero $\lambda \in \mathbb{R}$. When matrix A are multiplied by λ, the vector $\Phi_t(A)$ are multiplied by λ^d where $d = \rho(t)$ is the weight of the tree. Then $\Phi'_t(A)$ is also multiplied by λ^d. Therefore, the spaces generated by any set of vectors of the form $\Phi_t(A)$, $\Phi'_t(A)$ are not changed. □

[1] The definitions of filtration and of adjoint graded algebras are standard and can be found for example in [11], p.172, [12], p.37.

[2] The definitions of algebraic and projective algebraic vaireties can be found for example in [6]

Remark 3.3. If $D_1 \le D_2$ (termwise), then $\mathcal{M}_{D_1} \subseteq \mathcal{M}_{D_1}$ and $\mathcal{M}'_{D_1} \subseteq \mathcal{M}'_{D_1}$.

Theorem 3.4.

- *Variety* $\mathcal{M}_{(0)}(n)$ *is empty.*
- *Variety* $\mathcal{M}_{(1)}(n)$ *are given by equations* $c_2 = c_3 = \cdots = c_n = 0$ *and hence have codimension* $n-1$.
- *Variety* $\mathcal{M}_{(k)}(n)$ *coincides with the set of all matrices when* $k \ge 2$.
- $\mathrm{codim}\mathcal{M}_{(2,2)}(\mathrm{n}) = 2(\mathrm{n}-2)$.
- $\mathrm{codim}\mathcal{M}_{(2,3)}(\mathrm{n}) = \mathrm{n}-2$.
- $\mathrm{codim}\mathcal{M}_{(2,\mathrm{k})}(\mathrm{n}) = 0$ *for* $k \ge 4$.
- $\mathrm{codim}\mathcal{M}'_{(\mathrm{k})}(\mathrm{n}) = 0$ *for* $k \ge 0$.
- $\mathrm{codim}\mathcal{M}'_{(0,0)}(\mathrm{n}) = \mathrm{n}-1$.
- $\mathrm{codim}\mathcal{M}'_{(0,\mathrm{k})}(\mathrm{n}) = 0$ *for* $k \ge 1$.

Proof. All the statements can be verified by direct computations. □

The following theorem is one of the main results of the present paper.

Theorem 3.5. *For* $n \ge 4$ *we have*

$$a)\quad \mathrm{codim}\ \mathcal{M}'_{(0,1,2)}(\mathrm{n}) = \frac{\mathrm{n}(\mathrm{n}-1)}{2} - \dim \mathcal{M}'_{(0,1,2)}(\mathrm{n}) = 2(\mathrm{n}-3)\,,$$
$$b)\quad \mathrm{codim}\ \mathcal{M}'_{(0,1,3)}(\mathrm{n}) = \frac{\mathrm{n}(\mathrm{n}-1)}{2} - \dim \mathcal{M}'_{(0,1,3)}(\mathrm{n}) = \mathrm{n}-4\,.$$

Proof. The statement of the theorem mean that the dimensions of spaces M'_1, M'_2, M'_3 satisfy the following conditions.

- $\dim M'_1 \le 0$;
- $\dim M'_2 \le 1$;
- $\dim M'_3 \le 2$ in case (a) and ≤ 3 in case (b).

The space M'_1 is zero by definition, and, therefore, the first condition is true. The space M'_2 is generated by one vector $w_2 = 2A^2e - Ae{*}Ae$, and hence its dimension is less or equal 1. The space M'_3 is generated by 4 vectors

$$M'_3 =< 2A^2e - Ae * Ae,\ 3A(Ae{*}Ae) - Ae{*}Ae{*}Ae,\ 2A^2e{*}Ae - Ae{*}Ae{*}Ae,\ 6A^3e - Ae{*}Ae{*}Ae > .$$

which can be written in simple form

$$M'_3 =< w_2,\ w_3,\ w_2{*}Ae,\ 3Aw_2 + w_3 >,$$

where $w_3 = 3Ae{*}Ae{*}Ae - A^3e$. The same space M'_3 can be generated as

$$M'_3 =< w_2, w_3, w_2{*}Ae, Aw_2 >,$$

where

$$w_2 = \begin{pmatrix} 0 \\ -c_2^2 \\ 2a_{32}c_2 - c_3^2 \\ 2(a_{42}c_2 + a_{43}c_3) - c_4^2 \\ \cdots \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 \\ -c_2^3 \\ 3a_{32}c_2^2 - c_3^3 \\ 3(a_{42}c_2^2 + a_{43}c_3^2) - c_4^3 \\ \cdots \end{pmatrix},$$

$$w_2{*}Ae = \begin{pmatrix} 0 \\ -c_2^3 \\ (2a_{32}c_2 - c_3^2)c_3 \\ (2(a_{42}c_2 + a_{43}c_3) - c_4^2)c_4 \\ \cdots \end{pmatrix}, \quad Aw_2 = \begin{pmatrix} 0 \\ 0 \\ -a_{32}c_2^2 \\ -a_{42}c_2^2 + a_{43}(2a_{32}c_2 - c_3^2) \\ \cdots \end{pmatrix},$$

To prove statement (b) it is sufficient to show, that between the four vectors there is at least one linear relation:

$$x_1 w_2 + x_2 w_3 + x_3 w_2{*}Ae + x_4 A w_2 = 0 \tag{8}$$

for some real numbers x_1, x_2, x_3, x_4.

By construction, the first coordinates of vectors $w_2, w_3, w_2{*}Ae, Aw_2$ are zeros. Comparing the corresponding second, third and fourth components of the vector equality (8), coefficients x_i can be expressed in terms of the second, third and fourth components of vectors $w_2, w_3, w_2{*}Ae, Aw_2$, that is, in terms of the coefficients a_{2i}, a_{3i}, a_{4i} of the initial matrix A. These expressions are huge. However, the explicit expressions are not important. Comparing the rest of the corresponding components in vector equality (8), we obtain $n-4$ relations between the elements of the matrix A. Thus, the statement (b) is proved.

To prove statement (a), it is sufficient to find all matrices A, such that vectors $w_2, w_3, w_2{*}Ae, Aw_2$ generate a two-dimensional space. That is between these four vectors there are at least two linear relations, two vector equalities. Comparing the corresponding second and third components of these vector equalities, we obtain the following relations.

$$\begin{aligned} d \cdot Aw_2 &= a_{32}c_2^2(c_2 \cdot w_2 - w_3)\,, \\ d \cdot Ae * w_2 &= (3c_2 - 2c_3)c_2^2 a_{32} \cdot w_2 - (c_2 - c_3)(2a_{32}c_2 - c_3^2) \cdot w_3\,, \end{aligned} \tag{9}$$

where $d = a_{32}c_2^2 + c_3^2(c_2 - c_3)$.

Relations (9) are linear relations between the elements of the k-th row of the matrix A for all $k \geq 4$. The total number of relations equals to $2(n-3)$. This finishes the prove of statement (a). □

Remark 3.6. Vector equality (9) can be written directly in terms of vectors $\Phi_t(A)$:

$$\begin{aligned} (Aw_2):\quad & 2(3c_2 - 2c_3)A^3e &&= c_2 Ae{*}Ae{*}Ae - 2c_3 A(Ae{*}Ae) + c_2^2(2A^2e - Ae{*}Ae), \\ (Ae{*}w_2):\quad & 2Ae{*}A^2e &&= Ae{*}Ae{*}Ae + 2c_2A^2e - c_2Ae{*}Ae. \end{aligned}$$

It follows that the dimension of the space M_3, generated by 8 vectors

$$e,\ Ae,\ A^2e,\ Ae{*}Ae,\ A^3e,\ A(Ae{*}Ae),\ A^2e{*}Ae,\ Ae{*}Ae{*}Ae,$$

equals to 6.

4 2-transformations

In the following definition we construct a set of transitions (symmetries) of RK methods.

Definition 4.1. Let A be a lower triangular matrix with zero diagonal. The transition with parameter $\lambda \neq 0$ from a matrix $A = (a_{ij})$ to a matrix $A' = (a'_{ij})$ is called 2-transformation, if all of the following holds.

- All the columns of the matrix A' but the first two coincide with the columns of the matrix A.
- The second column of the matrix is divided by λ: $a'_{k2} = a_{k2}/\lambda$.
- $a'_{k1} = a_{k1} + (1 - 1/\lambda)a_{k2}$ for $k \geq 3$.
- $a'_{21} = \lambda a_{21}$.

Remark 4.2. Matrix A' from Definition 4.1 and vector $C' = A'e$ can be given explicitly as follows.

$$A' = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \lambda a_{21} & 0 & 0 & \dots \\ a_{31} + (1-1/\lambda)a_{32} & a_{32}/\lambda & 0 & \dots \\ a_{41} + (1-1/\lambda)a_{42} & a_{42}/\lambda & a_{43} & \dots \\ a_{51} + (1-1/\lambda)a_{52} & a_{52}/\lambda & a_{53} & \dots \\ & \dots & & \end{pmatrix}, \quad C' = \begin{pmatrix} 0 \\ \lambda c_2 \\ c_3 \\ c_4 \\ c_5 \\ \dots \end{pmatrix}$$

Theorem 4.3. *(Properties of 2-transformations)*

a) $c'_2 = \lambda c_2$, $c'_3 = c_3, \ldots, c'_n = c_n$. *In the vector form*

$$C' = A'e = C + c_2(\lambda - 1)e_2$$

where $e_2 = (0, 1, 0, \ldots, 0)^t$.

b) $A'^{\,2}e = A^2 e$.

c) for $k \geq 1$*:*

$$A'(C'^k) = A(C^k) + (\lambda^{k-1} - 1)c_2^k w,$$

where $w = (0, a_{32}, \ldots, a_{n2})^t$ *is a second column of matrix* A.

Proof. The statements (a) can be verified by the direct computation.

(b) We have

$$A'^2 e = A'C' = \begin{pmatrix} 0 \\ 0 \\ c_3 \\ c_4 \\ c_5 \\ \ldots \end{pmatrix} = A^2 e.$$

(c) The k-th ($k \geq 3$) coordinate of the vector $A'(C'^k)$ equals to

$$a_{k2} c_2^k \lambda^{k-1} + a_{k3} c_3^k + \ldots ,$$

that is it differs from the k-th component of the vector $A(C^k)$ by $a_{k2} c_2^k (\lambda^{k-1} - 1)$. □

In the following definition, we introduce 2-standard matrices, that are some of the matrices, satisfying second simplifying assumption $(D(1))$.

Definition 4.4. Let the lower triangular matrix A with zero diagonal be called 2-standard, if for all $k \geq 3$

$$c_k^2/2 = \sum_{i=2}^{k-1} a_{ki} c_i \,,$$

where c_k be a sum of the elements of k-th row of matrix A.

For 2-standard matrix we have

$$w_2 = 2A^2 e - Ae * Ae = \begin{pmatrix} 0 \\ -c_2^2 \\ 0 \\ 0 \\ \ldots \end{pmatrix},$$

and the elements of the second column of the matrix A can be expressed in terms of the rest of the elements:

$$\begin{aligned} &a_{32} = c_3^2/(2c_2), \\ &a_{42} = (c_4^2/2 - a_{43}c_3)/c_2, \\ &a_{52} = (c_5^2/2 - a_{53}c_3 - a_{54}c_4)/c_2, \\ &\ldots \end{aligned}$$

In addition, in this case

$$Aw_2 = -c_2^2 \begin{pmatrix} 0 \\ 0 \\ a_{32} \\ a_{42} \\ \ldots \end{pmatrix}.$$

In [10] Butcher equations are formulated in terms of subspaces $L_k(A)$ and $L'_k(A)$ (Theorem 5 and Theorem 6) in terms of subspaces $L_k(A)$ and $L'_k(A)$. For a more detailed study of the properties of the 2-transformations it is convenient to present this conditions in the following form.

Theorem 4.5. *(Order conditions in terms M'_k) The extended matrix A defines a RK method of order p if and only if*

$$(1) \quad (Ae, d) = 1,$$
$$(2) \quad M'_p \perp d\,,$$

where $d = (0, \ldots, 0, 1)^t$.

Proof. Let matrix A defines a RK method of order p. Then it satisfies the order conditions ([10], Theorem 3): for each tree t of weight $\le p$

$$(\Phi_t(A), d\,) = \frac{1}{\delta(t)}. \tag{10}$$

For the tree with one edge this means that $(Ae, d) = 1$. Hence, the last coordinate of the vector Ae is equaled to 1. This proves statement (1).

From $(Ae, d) = 1$ it follows, that $(Ae{*}\ldots{*}Ae, d) = 1$. To prove statement (2) it is sufficient to verify that each vector generating space M'_p is orthogonal to vector d. By definition, the space $M'_p = L'_2 + \cdots + L'_p$ is generated by the vectors

$$\Phi'_t(A) = \delta(t)\Phi_t(A) - \underbrace{Ae{*}\ldots{*}Ae}_{d},$$

for all trees t of weight $d \le p$. Then relation $(\Phi'_t(A), d) = 0$ immediately follows from (10) for tree t.

Now let us prove the statement of the theorem in the opposite direction. Let conditions (1) and (2) of the theorem hold simultaneously.

As $(Ae, d) = 1$, then $(Ae{*}\ldots{*}Ae, d) = 1$. Condition (2) means that for an arbitrary tree t of weight $d \le p$

$$(\Phi'_t(A), d) = (\delta(t)\Phi_t(A) - Ae{*}\ldots{*}Ae, d) = 0,$$

that is

$$(\delta(t)\Phi_t(A), d) = 1\,.$$

The latter is exactly the order condition (10). □

In the following theorem we prove that 2-transformation preserve Butcher equations.

Theorem 4.6. *Let A be a 2-standard matrix and A' be the result of its 2-transformation with parameter $\lambda \neq 0$. Then*

a) the spaces $M_k(A')$ coincide with the spaces $M_k(A)$ for $k \neq 1$,

b) the spaces $M'_k(A')$ coincide with the spaces $M'_k(A)$,

c) the matrix A' is an extended matrix of RK method of order of p if and only if the matrix A is an extended matrix of RK method.

Proof. a) For $k = 0$ we have $M_0(A') = M_0(A) = e$.

By definition, the space $L_2(A)$ is generated by vectors A^2e and $Ae{*}Ae$. As A is a 2-standard matrix, only the second component of the vector $w_2 = 2A^2e - Ae{*}Ae \in L_2$ is different from zero. According to the properties of the 2-transformations (Theorem 4.3), $A'^2e = A^2e$ and the vector $A'e{*}A'e$ differs from $Ae{*}Ae$ only in 2nd component. Hence, $L_2(A') = L_2(A)$. Since the vector $A'e$ differs from the vector Ae only in the second component, the spaces $M_2(A) = L_0(A) + L_1(A) + L_2(A)$ and $M_2(A') = L_0(A') + L_1(A') + L_2(A')$ coincide.

We conclude the prove of statement (a) by induction for k. In ([10]) we derived the following recurrent relation:

$$L_k = A(L_{k-1}) + Ae{*}L_{k-1} + \sum_{i+j=k} L_i{*}L_j\,.$$

The first components of all the vectors in L_k equal to zero, hence, $L_k(A')$ differs from $L_k(A)$ only in the second component. Taking into account that $(0, 1, 0, \ldots, 0)^t \in L_2(A) = L_2(A')$, we obtain

$$M_k(A) = L_0(A) + \cdots + L_k(A) = M_k(A') = L_0(A') + \cdots + L_k(A')$$

Statement (b) can be proved analogously.

c) Under the transition from matrix A to A', the space M'_p does not change, hence, the order conditions in the form given in Theorem 4.5, for both matrices are satisfied simultaneously. □

Relations (9) for the 2-standard matrix look significantly simpler than for a general matrix. Namely, the second relation becomes trivial,

$$Ae * w_2 = c_2 w_2 \,, \tag{11}$$

since the vector w_2 in this case has the only one nonzero component. The first relation can be written as

$$w_3 = c_2 w_2 - \frac{3c_2 - 2c_3}{c_2} A w_2 \,,$$

or, equivalently,

$$(3c_2 - 2c_3) A w_2 = c_2 (c_2 w_2 - w_3) \,. \tag{12}$$

Corollary 4.7. *For a 2-standard matrix A coefficients a_{k1}, a_{k2}, a_{k3} can be expressed in terms of the rest a_{ij} and c_i by the following formulas.*

$$a_{k1} = c_k - \sum_{i=2}^{k-1} a_{ki} \,,$$

$$a_{k2} = \left(c_k^2/2 - \sum_{i=3}^{k-1} a_{ki} c_i \right) / c_2 \,,$$

$$a_{k3} = \left(c_k^2 (c_k - c_3) - \sum_{i=4}^{k-1} a_{ki} c_i (3c_i - 2c_3) \right) / c_3^2 \,,$$

Corollary 4.8. *Let A be a 2-standard matrix.*

- *If $3c_2 \neq 2c_3$, then vectors w_2 and w_3 are linearly independent and generate the space M_3', and the vector Aw_2 is expressed in terms of them by the relation* (12)*.*
- *If $3c_2 = 2c_3$, then $w_3 = c_2 w_2$ and the two-dimensional space M_3' is generated by vectors $w_2 = -c_2^2 \cdot (0, 1, 0, \ldots, 0)^t$ and $Aw_2 = -c_2^2 \cdot (0, 0, a_{32}, \ldots, a_{n2})^t$.*

Let A be an extended matrix of RK methods, satisfying the simplifying assumption $D(1)$. By a suitable 2-transformations one can transform a RK method into a new one of the same order and with $c_2 = 2/3c_3$. For matrices of RK methods with $c_2 = 2/3c_3$ we have

$$\begin{array}{llll} w_2 & = 2A^2 e & -Ae{*}Ae & = (0, -c_2^2, 0, \ldots, 0)^t \,, \\ w_3 & = 3A(Ae{*}Ae) & -Ae{*}Ae{*}Ae & = (0, -c_2^3, 0, \ldots, 0)^t \,, \\ Aw_2 & = & & -c_2^2 (0, 0, a_{32}, \ldots, a_{92})^t \,. \end{array}$$

This means that the theory of RK methods can be reduced to the consideration of RK methods with $c_2 = 2/3c_3$.

We note, that all the above results are true under the simplifying assumption $D(1)$ (1) with $b_2 = 0$ and $b_3, b_4, \ldots, b_s$ are not necessarily zero. For many of the known RK methods of high order $b_2 = b_3 = 0$. For these methods, one can expect much stronger results.

Hypothesis 4.9. There is a two-dimensional symmetry group, which is operating on RK methods, satisfying the conditions $b_2 = b_3 = 0$. By means of this symmetry group one can make c_2 and c_3 equal to any values, while the remaining c_i are unchanged under these transformations.

5 Conclusions

The main results of the present paper are the Corollary 4.7 which allows to express the first three columns of the Butcher tableau in terms of the others; and the Theorem 4.6, especially statement (c), which allows to exclude variable (c_2) from the list of unknowns.

Hypothesis 4.9 states that substantial progress in the development of RK methods can be made by generalization of the symmetry methods to satisfying the conditions $b_2 = b_3 = 0$ and others (further generalizations are possible too).

Overall, the abstract algebraic approach to the RK methods started by the author in [9, 10] and the present paper seems to be mathematically "correct" way to study the RK methods.

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