Group of virtual braids and homotopy groups of 2-sphere

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The i-th homotopy group $\pi_i(S^n)$ summarizes the different ways in which the i-dimensional sphere S^i can be mapped continuously into the n-dimensional sphere S^n. It is known that if $i < n$, then $\pi_i(S^n) = 0$.

Hurewicz theorem says that for a simply-connected space X, the first nonzero homotopy group $\pi_k(X)$, with $k > 0$, is isomorphic to the first nonzero homology group $H_k(X)$. For the n-sphere, this immediately implies that for $n \geq 2$, $\pi_n(S^n) = H_n(S^n) = \mathbb{Z}$. Note that the homology groups $H_i(S^n)$, with $i > n$, are all trivial.

The question of computing the homotopy group $\pi_{n+k}(S^n)$ for positive k turned out to be a central question in algebraic topology.

$$
\begin{array}{lll}
\pi_2(S^2) = \pi_3(S^2) = \mathbb{Z}, & \pi_4(S^2) = \pi_5(S^2) = \mathbb{Z}_2, & \pi_6(S^2) = \mathbb{Z}_{12}, \\
\pi_7(S^2) = \pi_8(S^2) = \mathbb{Z}_2, & \pi_9(S^2) = \mathbb{Z}_3, & \pi_{10}(S^2) = \mathbb{Z}_{15}, \\
\pi_{11}(S^2) = \mathbb{Z}_2, & \pi_{12}(S^2) = \mathbb{Z}_2^2, & \pi_{13}(S^2) = \mathbb{Z}_{12} \oplus \mathbb{Z}_2, \\
\pi_{14}(S^2) = \mathbb{Z}_{84} \oplus \mathbb{Z}_2, & \pi_{15}(S^2) = \mathbb{Z}_2^2, & \dots
\end{array}
$$

A sequence of sets $\mathcal{X} = \{X_n\}_{n \geq 0}$ is called a simplicial set if there are face maps:

$$d_i : X_n \longrightarrow X_{n-1} \text{ for } 0 \leq i \leq n$$

and degeneration maps

$$s_i : X_n \longrightarrow X_{n+1} \text{ for } 0 \leq i \leq n.$$

This maps satisfy the following simplicial identities:

$$\begin{array}{lll} d_i d_j = d_{j-1} d_i & \text{if} & i < j, \\ s_i s_j = s_{j+1} s_i & \text{if} & i \leq j, \\ d_i s_j = s_{j-1} d_i & \text{if} & i < j, \\ d_j s_j = id = d_{j+1} s_j, & & \\ d_i s_j = s_j d_{i-1} & \text{if} & i > j+1. \end{array}$$

A simplicial group $\mathcal{G} = \{G_n\}_{n \geq 0}$ consists of a simplicial set $\mathcal{G}$ for which each G_n is a group and each d_i and s_i is a group homomorphism. The Moore complex $N\mathcal{G} = \{N_n\mathcal{G}\}_{n \geq 0}$ of a simplicial group $\mathcal{G}$ is defined by

$$N_n\mathcal{G} = \bigcap_{i=1}^{n} \mathrm{Ker}(d_i : G_n \longrightarrow G_{n-1}).$$

Then $d_0(N_n\mathcal{G}) \subseteq N_{n-1}\mathcal{G}$ and $N\mathcal{G}$ with d_0 is a chain complex of groups. An element in

$$\mathrm{B}_n\mathcal{G} = d_0(N_{n+1}\mathcal{G})$$

is called a Moore boundary and an element in

$$\mathrm{Z}_n\mathcal{G} = \mathrm{Ker}(d_0 : N_n\mathcal{G} \longrightarrow N_{n-1}\mathcal{G})$$

is called a Moore cycle. The nth homotopy group $\pi_n(\mathcal{G})$ is defined to be the group

$$\pi_n(\mathcal{G}) = H_n(N\mathcal{G}) = \mathrm{Z}_n\mathcal{G}/\mathrm{B}_n\mathcal{G}.$$

Milnor's $F[S^1]$-construction gives a possibility to define the homotopy groups $\pi_n(S^2)$ combinatorially, in terms of free groups. More accurately, consider the simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$:

$$S^1_0 = \{*\},\ \ S^1_1 = \{*, \sigma\},\ \ S^1_2 = \{*, s_0\sigma, s_1\sigma\}, \ldots, S^1_n = \{*, x_0, \ldots, x_n\}, \ldots,$$

where $x_i = s_n \ldots \widehat{s}_i \ldots s_0\sigma$. The $F[S^1]$-construction has the following terms

$$\begin{aligned} &F[S^1]_0 = 0,\\ &F[S^1]_1 = F(\sigma),\\ &F[S^1]_2 = F(s_0\sigma, s_1\sigma),\\ &F[S^1]_3 = F(s_i s_j \sigma \mid 0 \leq j \leq i \leq 2),\\ &\ldots \end{aligned}$$

The face and degeneracy maps are determined with respect to the standard simplicial identities for these simplicial groups.

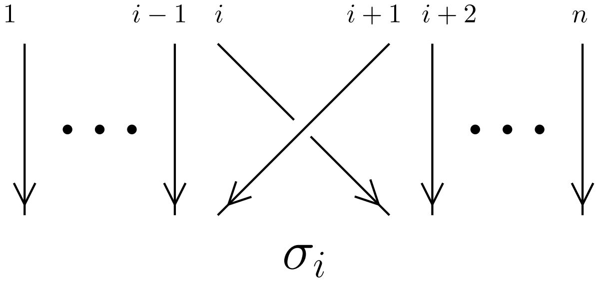
Milnor proved that the geometric realization of $F[S^1]$ is weakly homotopically equivalent to the loop space $\Omega S^2 = \Omega \Sigma S^1$. Hence, the homotopy groups of the Moore complex of $F[S^1]$ are naturally isomorphic to the homotopy groups $\pi_n(S^2)$:

$$\pi_n(F[S^1]) = Z_n(F[S^1])/B_n(F[S^1]) \simeq \pi_{n+1}(S^2).$$

Braid group B_n on $n \geq 2$ strands is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and is defined by relations

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } i = 1, 2, \ldots, n-2, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{for } |i-j| \geq 2. \end{aligned}$$

The generators σ_i have the following geometric interpretation:



There is a homomorphism $\varphi : B_n \longrightarrow S_n$, $\varphi(\sigma_i) = (i, i+1)$, $i = 1, 2, \dots, n-1$. Its kernel $\mathrm{Ker}(\varphi)$ is called the pure braid group and is denoted by P_n. Note that P_2 is infinite cyclic group.

Markov proved that P_n is a semi-direct product of free groups:

$$P_n = U_n \leftthreetimes U_{n-1} \leftthreetimes \dots \leftthreetimes U_2,$$

where $U_k \simeq F_{k-1}$, $k = 2, 3, \dots, n$, is a free group of rank k.

F. Cohen and J. Wu (2011) constructed cabling for classical braids, that is the simplitial group $P_* = \{P_n\}_{n \geq 1}$ with face and degeneracy maps corresponding to deleting and doubling of strands, respectively. They proved that P_* is contractible (hence $\pi_n(P_*)$ is trivial group for all n). On the other side, F. Cohen and J. Wu constructed an injective canonical map of simplicial groups

$$\Theta : F[S^1] \longrightarrow P_*,$$

This leads to the conclusion that the cokernel of Θ is homotopy equivalent to S^2. Hence, it is possible to present generators of $\pi_n(S^2)$ by braids.

The virtual braid group VB_n was introduced by L. Kauffman (1996).

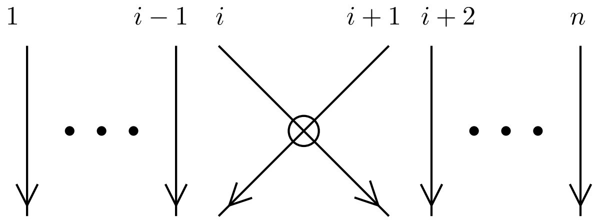
VB_n is generated by the classical braid group $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and the permutation group $S_n = \langle \rho_1, \ldots, \rho_{n-1} \rangle$. Generators $\rho_i, i = 1, \ldots, n-1$, satisfy the following relations:

$$\rho_i^2 = 1 \qquad \text{for} \quad i = 1, 2, \ldots, n-1,$$
$$\rho_i \rho_j = \rho_j \rho_i \qquad \text{for} \quad |i-j| \geq 2,$$
$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} \qquad \text{for} \quad i = 1, 2 \ldots, n-2.$$

Other defining relations of the group VB_n are mixed and they are as follows

$$\sigma_i \rho_j = \rho_j \sigma_i \qquad \text{for} \quad |i-j| \geq 2,$$
$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1} \qquad \text{for} \quad i = 1, 2, \ldots, n-2.$$

The generators ρ_i have the following diagram



ρ_i

As in classical case there is a homomorphism

$$\varphi : VB_n \longrightarrow S_n, \quad \varphi(\sigma_i) = \varphi(\rho_i) = \rho_i, \quad i = 1, 2, \ldots, n-1.$$

Its kernel $\mathrm{Ker}(\varphi)$ is called the virtual pure braid group and is denoted by VP_n.

Define the following elements in VB_n:

$$\lambda_{i,i+1} = \rho_i\, \sigma_i^{-1}, \quad \lambda_{i+1,i} = \rho_i\, \lambda_{i,i+1}\, \rho_i = \sigma_i^{-1}\, \rho_i, \quad i = 1, 2, \ldots, n-1,$$

$$\lambda_{ij} = \rho_{j-1}\, \rho_{j-2} \ldots \rho_{i+1}\, \lambda_{i,i+1}\, \rho_{i+1} \ldots \rho_{j-2}\, \rho_{j-1},$$

$$\lambda_{ji} = \rho_{j-1}\, \rho_{j-2} \ldots \rho_{i+1}\, \lambda_{i+1,i}\, \rho_{i+1} \ldots \rho_{j-2}\, \rho_{j-1}, \quad 1 \leq i < j-1 \leq n-1.$$

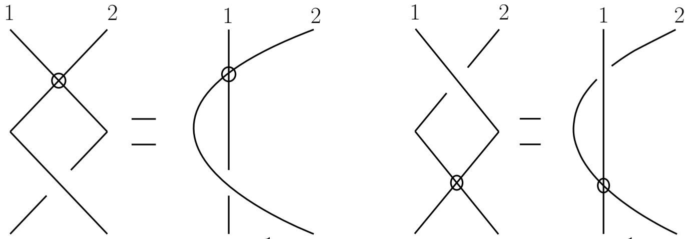
Theorem [V. B, 2004]

The group VP_n $(n \geq 2)$ admits a presentation with the generators $\lambda_{ij},\ 1 \leq i \neq j \leq n$, and the following relations:

$$\lambda_{ij}\lambda_{kl} = \lambda_{kl}\lambda_{ij},$$
$$\lambda_{ki}\lambda_{kj}\lambda_{ij} = \lambda_{ij}\lambda_{kj}\lambda_{ki},$$

where distinct letters stand for distinct indices.

Note that $VP_2 = \langle \lambda_{12}, \lambda_{21} \rangle$ is 2-generated free group. The generators have geometric interpretation:



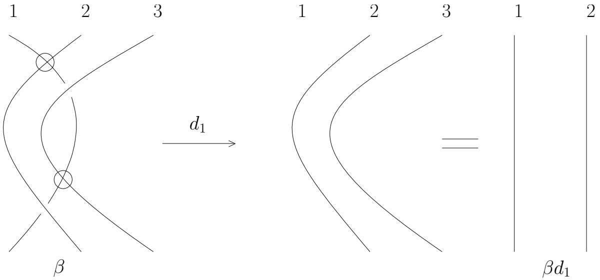
$\lambda_{12} = \rho_1 \sigma_1^{-1}$ $\lambda_{21} = \sigma_1^{-1} \rho_1$

Let $VP_* = \{VP_n\}_{n\geq 1}$ be the set of virtual pure braid groups.
Define the face map:

$$d_i : VP_n \longrightarrow VP_{n-1}, \quad i = 1, 2, \ldots, n,$$

what is the deleting of the ith strand.

Example:

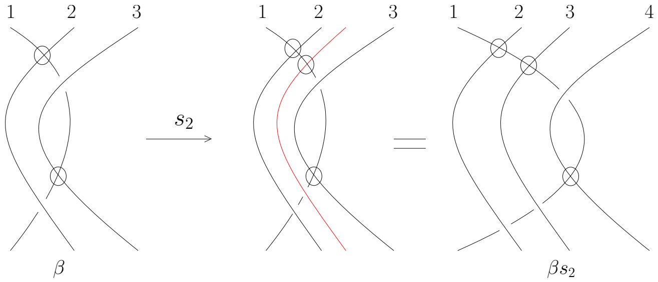


Define the degeneracy map:

$$s_i : VP_n \longrightarrow VP_{n+1}, \quad i = 1, 2, \ldots, n,$$

what is the doubling of the ith strand.

Example:



It is not difficult to see that we have the simplicial group

$$VP_* \quad : \quad \cdots \rightleftarrows VP_3 \rightleftarrows VP_2 \rightleftarrows VP_1.$$

Proposition

VP_* is contractible, i.e. $\pi_n(VP_*) = 0$ for all $n \geq 1$.

Define a simplicial group $T_* = \{T_n\}_{n \geq 1}$ that is a simplitial subgroup of VP_* and is generated by λ_{12} and λ_{21}:

$$T_* \quad : \quad \cdots \rightleftarrows T_3 \rightleftarrows T_2 \rightleftarrows T_1,$$

where T_n, $n = 1, 2, \ldots$, is defined by the following manner

$$T_1 = VP_2, \quad T_{n+1} = \langle s_1(T_n), s_2(T_n), \ldots, s_{n+1}(T_n) \rangle.$$

If we let $a_{11} = \lambda_{12}$, $b_{11} = \lambda_{21}$, and

$$a_{ij} = s_n \ldots \widehat{s}_i \ldots s_1 a_{11}, \quad b_{ij} = s_n \ldots \widehat{s}_i \ldots s_1 b_{11}, \quad i + j = n + 1.$$

Then

$$T_n = \langle a_{kl}, b_{kl} : k + l = n + 1 \rangle, \quad n = 1, 2, \ldots$$

It is easily to check that in T_n all elements a_{ij} (as all b_{ij}) pairwise commute. Hence one can formulates

Question

Is it true that $T_n = \mathbb{Z}^n * \mathbb{Z}^n$ for all $n \geq 1$?

If the answer is YES, then we have simplicial group

$$\cdots \rightleftarrows \mathbb{Z}^3 * \mathbb{Z}^3 \rightleftarrows \mathbb{Z}^2 * \mathbb{Z}^2 \rightleftarrows \mathbb{Z} * \mathbb{Z},$$

which has the same homotopy type as the space $\Omega(K(\mathbb{Z},2) \vee K(\mathbb{Z},2))$. It is known that

$$\pi_i(\Omega(K(\mathbb{Z},2) \vee K(\mathbb{Z},2))) = \begin{cases} \pi_i(S^2) & \text{for } i > 1, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } i = 1. \end{cases}$$

Hence $\pi_i(T_*) = \pi_i(S^2)$ for all $i > 1$.

Unfortunately the answer on this question is NO.
Put $c_{ij} = b_{ij} a_{ij}$.

Theorem [V. B., R. Mikhailov, V. V. Vershinin and J. Wu, 2016]

The group VP_3 is generated by elements

$$a_{11}, \quad c_{11}, \quad a_{21}, \quad a_{12}, \quad c_{21}, \quad c_{12}$$

and is defined by relations

$$[a_{21}, a_{12}] = [c_{21} a_{21}^{-1}, c_{12} a_{12}^{-1}] = 1,$$

$$a_{21}^{c_{11}} = a_{21}, \quad c_{21}^{c_{11}} = c_{21}, \quad a_{12}^{c_{11}} = a_{12}^{c_{12} c_{21}^{-1}}, \quad c_{12}^{c_{11}} = c_{12}^{c_{21}^{-1}},$$

i. e. $VP_3 = \langle T_2, c_{11} \rangle * \langle a_{11} \rangle$, $\langle T_2, c_{11} \rangle = T_2 \leftthreetimes \langle c_{11} \rangle$.

As a corollary of the previous theorem we have

Corollary

$T_2 = \langle a_{21}, a_{12}, b_{21}, b_{12} \rangle$ is defined by infinite set of relations

$$[a_{21}, a_{12}]^{c_{12}^k} = [b_{21}, b_{12}]^{c_{12}^k} = 1, \quad k \in \mathbb{Z}.$$

We have a good decomposition of VP_3. Using the maps s_1, s_2, s_3 we can find some relations in VP_4. On the other side VP_4 contains commutativity relations but VP_3 does not.
Denote

$$A_a = \langle a_{21} a_{22}^{-1} a_{31}, \quad a_{12} a_{13}^{-1} a_{22} a_{21}^{-1}, \quad a_{13} a_{12}^{-1} \rangle.$$

$$A_b = \langle b_{31} b_{22}^{-1} b_{21}, \quad b_{21}^{-1} b_{22} b_{13}^{-1} b_{12}, \quad b_{12}^{-1} b_{13} \rangle.$$

Also denote

$$B_a = \langle a_{21} a_{22}^{-1} a_{31}, \quad \left(a_{12} a_{13}^{-1} a_{22} a_{21}^{-1}\right)^{a_{12}}, \quad \left(a_{13} a_{12}^{-1}\right)^{a_{21}^{-1} a_{12}} \rangle$$

and

$$B_b = \langle b_{31} b_{22}^{-1} b_{21}, \quad \left(b_{21}^{-1} b_{22} b_{13}^{-1} b_{12}\right)^{a_{12}}, \quad \left(b_{12}^{-1} b_{13}\right)^{a_{21}^{-1} a_{12}} \rangle.$$

We see that $B_a = A_a^{a_{11}}$, $B_b = A_b^{a_{11}}$. Put $A = \langle A_a, A_b \rangle$, $B = \langle B_a, B_b \rangle$. Since B is conjugate with A, then A is isomorphic to B and we get

Proposition

$VP_4 = \langle G, a_{11} \mid\mid a_{11}^{-1} A a_{11} = B \rangle$ is the HNN-extension with the base group

$$G = \langle a_{21}, a_{12}, b_{21}, b_{12}, a_{31}, a_{22}, a_{13}, b_{31}, b_{22}, b_{13} \rangle,$$

associated subgroups A and B and stable letter a_{11}.

Let

$$\widetilde{T}_i = \langle \alpha_{kl}, \beta_{kl}, \quad 1 \leq k, l \leq i, k + l = i + 1 \rangle \simeq \mathbb{Z}^i * \mathbb{Z}^i,$$

where

$$\langle \alpha_{i1}, \alpha_{i-1,2}, \ldots, \alpha_{1i} \rangle \simeq \langle \beta_{i1}, \beta_{i-1,2}, \ldots, \beta_{1i} \rangle \simeq \mathbb{Z}^i$$

are free abelian group of rank i. There is an epimorphism $\widetilde{T}_i \longrightarrow T_i$, $i = 1, 2, \ldots$, defined by the rule

$$\alpha_{kl} \mapsto a_{kl}, \quad \beta_{kl} \mapsto b_{kl}.$$

Denote by K_i the kernel of this homomorphism. Put $\widetilde{T}_* = \{\widetilde{T}_i\}_{i \geq 1}$ and defining the face and degeneration maps as on T_* we get a simplicial group.

We have the following homomorphisms of simplicial groups

$$\begin{array}{ccccccc}
\cdots & \rightleftarrows & T_3 & \rightleftarrows & T_2 & \rightleftarrows & T_1 \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \rightleftarrows & \widetilde{T}_3 & \rightleftarrows & \widetilde{T}_2 & \rightleftarrows & \widetilde{T}_1 \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \rightleftarrows & K_3 & \rightleftarrows & K_2 & \rightleftarrows & K_1,
\end{array}$$

i.e. $K_* \longrightarrow \widetilde{T}_* \longrightarrow T_*$, where the left homomorphism is a monomorphism and the right homomorphism is an epimorphism. Consider the following simplicial group

$$K_*^{ab} : \cdots \rightleftarrows K_3^{ab} \rightleftarrows K_2^{ab} \rightleftarrows K_1^{ab},$$

that is the abelianization of the simplicial group K_*.

We have to prove that $H_i(K_*) = 0$ for all $i \geq 1$. To do it take the chain complex

$$K_*^{ab} : \cdots \longrightarrow K_3^{ab} \longrightarrow K_2^{ab} \longrightarrow K_1^{ab},$$

where the differential $\delta_n : K_n^{ab} \longrightarrow K_{n-1}^{ab}$, $n = 2, 3, \ldots$, is defined by the rule

$$\delta_n = \sum_{i=1}^{n+1} (-1)^{i-1} d_i.$$

Proposition

The group K_3^{ab} is generated by elements

1) $[\alpha_{31}, \alpha_{22}]^{\gamma_{21}^l} = [\alpha_{31}, \alpha_{22}^{\gamma_{22}^l \gamma_{31}^{-l}}], \quad [\beta_{31}, \beta_{22}]^{\gamma_{21}^l} = [\beta_{31}, \beta_{22}^{\gamma_{22}^l \gamma_{31}^{-l}}], \quad l \neq 0,$

2) $[\alpha_{31}, \alpha_{13}]^{\gamma_{12}^k \gamma_{11}^m} = [\alpha_{31}, \alpha_{13}^{\gamma_{13}^{k+m} \gamma_{22}^{-m} \gamma_{31}^{-k}}],$

$[\beta_{31}, \beta_{13}]^{\gamma_{12}^k \gamma_{11}^m} = [\beta_{31}, \beta_{13}^{\gamma_{13}^{k+m} \gamma_{22}^{-m} \gamma_{31}^{-k}}]$, where $|k| + |m| \neq 0,$

3) $[\alpha_{22}, \alpha_{13}]^{\gamma_{11}^m} = [\alpha_{22}, \alpha_{13}^{\gamma_{13}^m \gamma_{22}^{-m}}], \quad [\beta_{22}, \beta_{13}]^{\gamma_{11}^m} = [\beta_{22}, \beta_{13}^{\gamma_{13}^m \gamma_{22}^{-m}}], \quad m \neq 0.$

Theorem

For $n > 4$ all relations of VP_n come from relations of VP_{n-1} under the maps s_i, $i = 1, 2, \ldots, n-1$.

Using this theorem we can prove

Proposition

K_n^{ab} is generated by elements

$$[a_{i,n+1-i}, a_{j,n+1-j}]^{c_{j,i-j}^{l_1} c_{j,i-j-1}^{l_2} \ldots c_{j,1}^{l_{i-j}}}, \quad 1 \leq j < i \leq n,$$

$$[b_{i,n+1-i}, b_{j,n+1-j}]^{c_{j,i-j}^{l_1} c_{j,i-j-1}^{l_2} \ldots c_{j,1}^{l_{i-j}}}, \quad 1 \leq j < i \leq n.$$

Theorem

$H_n(K_*^{ab}) = 0$ for all $n \geq 1$.

From this theorem follows the main result

Corollary

$$\pi_1(T_*) = \mathbb{Z} \oplus \mathbb{Z}$$

and for $n > 1$ the following equality holds

$$\pi_n(T_*) = \pi_n(S^2).$$

Thank you!