Groebner-Shirshov bases for linear (Ω-)algebras

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I. History

A.I.Shirshov (1962)

H.Hironaka (1964)

B.Buchberger (1965, 1970)

Associative algebra

II. Composition-Diamond lemma for associative algebras (A.I.Shirshov (implicitly, 1962), L.A.B. (1976), G.Bergman (1978))

$S \subset k\langle X\rangle$, k - field, $(X^*, \leq)$ - monomial well ordering
$f = \bar{f} + r_f$, $r_f < \bar{f}$, $f \in S$—monic polynomial

Compositions

$$(f, g)_w = fa - bg \quad (w = \bar{f}a = b\bar{g}, \bar{f} \cap \bar{g} \neq 1)$$

or

$$f - agb \quad (w = \bar{f} = a\bar{g}b)$$

S - GSB:

$$(f, g)_w = \sum \alpha_i a_i s_i b_i, \text{ with } a_i \bar{s}_i b_i < w, \; f, g, s_i \in S.$$

CD Lemma for associative algebras. $S \subset k\langle X \rangle$. TFAE

(i) S is a GSB,

(ii) $f \in \text{Ideal}(S) \Longrightarrow \bar{f} = a\bar{s}b, s \in S$,

(iii) $\text{Irr}(S) = \{u \mid u \neq a\bar{s}b, s \in S\}$ is a linear basis of $k\langle X|S\rangle$.

CD-lemma for Lie algebras (A.I.Shirshov, 1962)

$S \subset \mathrm{Lie}(X) \subset k\langle X\rangle$
Lie Compositions—special Shirshov bracketing of associative compositions of Lie polynomials,

$$(f,g)_w = [fa] - [bg], \quad f,g \in S \quad (w = \bar{f}a = b\bar{g}, a, b \in X^*)$$

$$\text{or} \qquad f - [agb] \quad (w = \bar{f} = a\bar{g}b)$$

S— Lie GSB:

$$(f,g)_w = \sum \alpha_i [a_i s_i b_i],$$

where $a_i\bar{s}_i b_i < w, f, g, s_i \in S$.

CD-lemma for Lie algebras (A.I.Shirshov, 1962)

$S \subset Lie(X) \subset k\langle X\rangle$. TFAE

(i) S is a Lie GSB,

(ii) $h \in \mathrm{LieIdeal}(S) \Longrightarrow \bar{h} = a\bar{s}b, s \in S$,

(iii) $\mathrm{Irr}(S) = \{[u] \mid u \neq a\bar{s}b, u \in \mathrm{ALSW}(X), s \in S\}$ is a linear basis of $Lie(X|S)$.

Associative Ω-algebra

CD Lemma for associative Ω-algebras (L.A.Bokut, Y.Q..Chen, J.J.Qiu,2010)

$S \subset k\langle X, - \cdot -, \Omega\rangle$ free associative Ω-algebra,
$(X, - \cdot -, \Omega)^*$ - associative Ω-words:

$u = \omega^{(3)}(x_2x_1x_1, x_1, \omega^{(1)}(x_2x_2x_1))x_2x_1$ word of the degree $deg(u) = 11$, the breadth ("shirina") $bre(u) = 3$, and the depth ("glubina") $dep(u) = 2$,

$wt(u) = (deg(u), bre(u), dep(u), u_1, ..., u_n)$, where $u = u_1...u_n$, u_i - simple words,

$$u < v \iff wt(u) < wt(v) \text{ lexicographically.}$$

Compositions of intersection and including.

Theorem (Composition-Diamond lemma for Ω-algebras).
$S \subset k\langle X, - \cdot -, \Omega\rangle$. TFAE.

(i) S is a Gröbner-Shirshov basis.

(ii) $f \in \mathrm{Id}(S) \Longrightarrow \bar{f} = u|_{\bar{s}}, s \in S$.

(iii) $\mathrm{Irr}(S) = \{w \mid w \neq u|_{\bar{s}} \text{ for any } s \in S\}$ is a linear basis of $k\langle X, - \cdot -, \Omega | S\rangle$.

Free Lie Ω-algebra

$\mathrm{Lie}(X, [-,-], \Omega)$ is a set of Lie polynomials in the free associative Ω-algebra $k\langle X, - \cdot -, \Omega\rangle$.

$u = \omega^{(3)}(x_2x_1x_1, x_1, \omega^{(1)}(x_2x_2x_1))x_2x_1$ - associative LS Ω-word: all max associative subwords are LS in the simple words alphabet —$x_2x_1x_1, x_1$, $x_2x_2x_1$, $\omega^{(1)}(x_2x_2x_1)$, u
(it is greater then any its cyclic permutation).

$[u] = [[\omega^{(3)}([[x_2x_1]x_1], x_1, \omega^{(1)}([x_2[x_2x_1]]))[x_2x_1]]$ —Lie LS Ω-word (a result of unique bracketing of u).

Property 1.The set of LS Lie Ω-words is a set of linear generators of a free Lie Ω-algebra.

Property 2. The set of LS Lie Ω-words is linear independent in $k\langle X, -\cdot-, \Omega\rangle$.

CD Lemma for Lie Ω-algebras (Y.Q.Chen, J.J.Qiu, JAA, 2017).
$S \subset \mathrm{Lie}(X, [-,-], \Omega) \subset k\langle X, - \cdot -, \Omega\rangle$

Lie compositions of Lie Ω-polynomials are the result of special Shirshov bracketing of associative compositions of that polynomials.

Theorem. TFAE.

(i) S is a Gröbner-Shirshov basis.

(ii) $f \in$Lie-Id$(S) \Longrightarrow \bar{f} = u|_{\bar{s}}$ for some $s \in S$.

(iii) The set $\mathrm{Irr}(S) = \{[w] \mid w \in \mathrm{ALSW}(X, \Omega), w \neq u|_{\bar{s}}, s \in S\}$ is a linear basis of the Lie Ω-algebra $\mathrm{Lie}(X, [-,-], \Omega|S)$.

Applications

Free λ-Rota-Baxter Lie algebras,
$[P(x)P(y)] = P([P(x)y]) + P([xP(y)]) + \lambda P([xy]), x, y \in L, \lambda \in k.$

Free modified λ-Rota-Baxter Lie algebras,
$[P(x)P(y)] = P([P(x)y]) + P([xP(y)]) + \lambda[xy], x, y \in L, \lambda \in k.$
M. Semonov-Tian-Shansky, What is a classical R-matrix? Funct. Anal. Appl. (1983) 259-272

Free Nijenhuis Lie algebras,
$[P(x)P(y)] = P([P(x)y]) + P([xP(y)]) - P^2([xy]), x, y \in L.$
I. Dorfman, Dirac structures and integrability of nonlinear evolution equations, Wiley, Chichester, 1993.

In all cases the defining relations are Lie Ω GS bases.

Jianjun Qiu, Yuqun Chen, Gröbner-Shirshov bases for Lie Ω-algebras and free Rota-Baxter algebras, J. Algebra Appl., 16 (10) (2017).

GS bases for Gelfand-Dorfman-Novikov (GDN) algebras (L.A.B., Y.Q.Chen, Z.R.Zhang, JAA, 2017)

(right) GDN algebra $(A, \circ)$:

$$(x, y, z) = (x, z, y), \quad x \circ (y \circ z) = y \circ (x \circ z).$$

where (x, y, z) is the associator.

Example (S.I.Gelfand) – every commutative differential algebra under new multiplication $x \circ y := D(x)y$.

Gelfand-Dorfman-Novikov algebras

$k[X;D]$ — free differential polynomial algebra.

Free GDN algebra (Dzhumadildaev, Löfwall, 2002).

$$\mathrm{GDN}(X) = \mathrm{span}\{u \mid \mathrm{wt}(u) = 0\} \subset k[X, D],$$

where for $u = D^{r_1}(a_1)D^{r_2}(a_2)...D^{r_n}(a_n)$, $a_i \in X$, the weight of u is defined to be

$$wt(u) = (\sum r_i) - n + 1.$$

Theorem (Composition-Diamond lemma for GDN algebras) Let $S \subset \mathrm{GDN}(X) \subset k[X; D]$, $\leq$—deg-lex. TFAE

(i) S is a Gröbner-Shirshov basis.

(ii) $f \in \text{GDN-Id}(S) \Longrightarrow \bar{f} = u\overline{D^i(s)}$ for some $s \in S$, $u \in [X; D]$.

(iii) $\mathrm{Irr}(S) = \{w \in [X; D] \mid \mathrm{wt}(w) = 0, w \neq u\overline{D^i(s)}, u \in [X; D], s \in S\}$ is a linear basis of $GDN(X|S)$.

Theorem (PBW theorem for GDN-algebras).
Every GDN algebra is a GDN-subalgebra of its universal enveloping commutative differential algebra, i.e. $\mathrm{GDN}(X|S) \subseteq k[X; D|S]$.

L.A. Bokut, Yuqun Chen and Zerui Zhang, Groebner-Shirshov bases method for Gelfand-Dorfman-Novikov algebras, Journal of Algebra and Its Applications, 16(1) (2017)(22 pages).

Gelfand-Dorfman-Novikov Poisson algebras

GS bases for GDN Poisson algebras (L.A.B., Y.Q.Chen, Z.R.Zhang, JA, 2017).

$(A, \circ, \cdot, e)$—GDN Poisson algebra if:

(i) $(A, \cdot, e)$ – commutative algebra with the unit e,

(ii) $(A, \circ)$ GDN algebras,

(iii) $x \circ (y \cdot z) = (x \circ y) \cdot z$, $x \cdot (y \circ z) - (x \cdot y) \circ z = x \cdot (z \circ y) - (x \cdot z) \circ y$, $x, y, z \in A$.

Xiaoping Xu, On simple Novikov algebras and their irreducible modules, J. Algebra 185(3) (1996) 905-934 . Xiaoping Xu, Novikov-Poisson algebras, J. Algebra 190(2) (1997) 253-279.

Definition (B.-C.-Z.). Special GDN-Poisson admissible algebra $(A, \cdot, *, D)$:

(i) $(A, \cdot)$ – commutative associative algebra with unit e,

(ii)$(A, *)$ – commutative associative algebra,

(iii) $(x \cdot y) * z = x \cdot (y * z)$,

$D(x * y) = (Dx) * y + x * (Dy)$,

$D(x \cdot y) = (Dx) \cdot y + x \cdot (Dy) - x \cdot y \cdot (De)$.

Example. $(A, \cdot, *, D, e)$ - a special GDN-Poisson admissible algebra, then $(A, \cdot, \circ), x \circ y := Dx * y, x, y \in A$,- a GDN-Poisson algebra.

Gelfand-Dorfman-Novikov Poisson algebras

$k[X;\cdot,*,D]$— a free special GDN Poisson admissible algebra (linear span of monomilals:

$[X;\cdot,*,D]=\{w=D^{i_1}a_1*\cdots*D^{i_j}a_j\cdot D^{i_{j+1}}a_{j+1}\cdot D^{i_{j+2}}a_{j+2}\cdots D^{i_n}a_n \mid n\geq 1,$ $1\leq j\leq n,(i_1,a_1)\geq\cdots\geq(i_n,a_n)$ lexicographically, and if $j<n$, then $(i_n,a_n)\neq(0,e)\}$,

$\mathrm{wt}(w)=(\sum_{1\leq t\leq j} i_t)-(j-1)$

Theorem. $\mathrm{GDN}-\mathrm{P}(X) = \mathrm{span}\{w \in [X;\cdot,*,D,e] \mid \mathrm{wt}(w) = 0\}$ is a free GDN- Poisson subalgebra of $k[X;\cdot,*,D,e]$.

Theorem (PBW theorem for GDN-Poisson algebras)
Every GDN-Poisson algebra $\mathrm{GDN}-\mathrm{P}(X|S)$ is a GDN-Poisson subalgebra of its universal enveloping special GDN-Poisson admissible algebra $k[X;\cdot,*,D,e|S]$.

L.A. Bokut, Yuqun Chen and Zerui Zhang, On free Gelfand-Dorfman-Novikov-Poisson algebras and a PBW theorem, J. Algebra, 500 (2018), 153-170, the special issue dedicated to 60-th birthday of E.I.Zelmanov.