Characterizations of simple linear groups in the class of periodic groups

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Intro

Theorem (Jordan, 1878)

There exists an integral function $J(n)$, such that, for every field k of characteristic 0 and every finite simple group $G \leq GL(n, k)$

$$|G| \leqslant J(n).$$

Intro

Theorem (Schur, 1911)

A periodic linear group is locally finite.

Intro

Theorem (Kegel, 1967)

A countable locally finite simple linear group is the union of an ascending sequence of finite simple subgroups.

Theorem (Winter, 1968)

A periodic linear group has a unipotent normal subgroup of countable index.
In particular, every simple periodic linear group is countable.

Corollary

A periodic simple linear group is the union of an ascending sequence of finite simple subgroups.

Intro

Corollary (mod CFSG)

An infinite periodic simple linear group is the union of an ascending sequence of finite simple groups of Lie type whose ranks are bounded.

Intro

The condition mod CFSG is redundant:

Theorem (Larsen, Pink, 1998 – preprint, 2011 – journal publication)

There exists an integral function $J_1(n)$ such that, for any finite simple group G posessing a faithful linear representation of dimension at most n over a field k we have either

(a) $|G| \leqslant J_1(n)$, or

(b) $p = \mathrm{char}(k)$ is positive and G is a group of Lie type in characteristic p.

Intro

Theorem (Belyaev; Borovik; Hartley and Shute; Thomas, 1983–84)

The union of an ascending sequence of finite simple groups of Lie type of bounded Lie ranks is isomorphic to a simple group of Lie type over a locally finite field.

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Corollary

An infinite simple periodic linear group is isomorphic to a group of Lie type over a locally finite field.

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Intro

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&\{\text{Infinite simple periodic linear groups}\} = \\
&= \{\text{Simple groups of Lie type over infinite locally finite fields}\} = \\
&= \{{}^{*}X_n(Q) \mid X \in \{A, B, C, D, E, F, G\}; n \in \mathbb{N}; \\
&\quad Q \text{ is a locally finite field}; \\
&\quad * \text{ is an empty sign, or one of the numbers } 2, 3\}.
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- In particular, any infinite simple periodic linear group posesses a local system consisting of finite simple groups of Lie type whose ranks are bounded. It is locally finite and countable.

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- Remind that a family $\mathfrak{L}$ of subgroups of a group G is said to be a local system of G, if for any element of G there is a member of $\mathfrak{L}$ containing it, and every two members of $\mathfrak{L}$ are contained in a member of $\mathfrak{L}$.

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- Every finite subgroup of a group ${}^*X_n(Q)$ lies in a subgroup isomorphic to ${}^*X_n(F)$, where F is a finite subfield of Q. In other words, every group ${}^*X_n(Q)$ is saturated with groups from the set

  $$\mathfrak{M} = \{{}^*X_n(F) \mid F \text{ is a finite subfield of the field } Q\},$$

  in the sense of the following definition.

Groups saturated with given groups

Definition

Let $\mathfrak{M}$ be a set of groups. We say that a group G is saturated with groups from $\mathfrak{M}$, if any finite subgroup of G is contained in a subgroup isomorphic to some member of $\mathfrak{M}$.

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    - For example, any free periodic group $B(2, n)$ of prime period $n \geqslant 665$ is saturated with groups from the set $\mathfrak{M}$, which consists of a single cyclic group of order n, but $B(2, n)$ is not locally finite due to Novikov and Adian.

Groups saturated with given groups

Conjecture (Shlöpkin)

A periodic group saturated with finite simple groups of Lie type of bounded Lie ranks is isomorphic to a group of Lie type over a locally finite field.

Groups saturated with given groups

Question

Let Q be a locally finite field, and ${}^*X_n(Q)$ be a group of Lie type ${}^*X$ of rank n over Q. Suppose that

$$\mathfrak{M} = \{{}^*X_n(Q_\alpha) \mid Q_\alpha \text{ is a finite subfield of } Q\}.$$

Is it true that any periodic group G saturated with groups from $\mathfrak{M}$ is isomorphic to ${}^*X_n(Q_0)$, where Q_0 is a subfield of Q?

Groups of Lie type

Finite simple groups of Lie rank 1

$A_1 = L_2$	Rubashkin, Filippov; 2005
${}^2A_2 = U_3$	Lytkina, Al. Shlöpkin; 2016
${}^2B_2 = Sz$	Filippov; 2005–2012
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Groups of Lie rank 2

$A_2 = L_3$	Lytkina, Al. Shlöpkin; 2016
$B_2 = S_4 = O_5$	Lytkina, Mazurov; 2017–2018

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- Groups ${}^2A_3 = U_4$; ${}^2A_4 = U_5$; G_2, ${}^2F_4$; ${}^3D_4$ and also all groups of Lie rank at least 3 are queueing for being investigated.

Groups of Lie type

Theorem (Lytkina, Mazurov; to appear)

Suppose that any finite subgroup of a periodic group G lies in a subgroup of G isomorphic to $G_2(q)$ for some odd number q. Then $G \simeq G_2(Q)$ for some locally finite field Q.

In particular, G is locally finite.

Theorem (Lytkina, Mazurov; to appear)

Suppose that any finite subgroup of a periodic group G lies in a subgroup of G isomorphic to ${}^3D_4(q)$ for some odd number q. Then $G \simeq {}^3D_4(Q)$ for some locally finite field Q.

In particular, G is locally finite.

Groups of Lie type

Theorem (Li, Lytkina; 2017)

Let G be a periodic group saturated with finite simple groups of Lie type of odd characteristics whose ranks are bounded.

Then any 2-subgroup of G is locally finite.