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A FREE PRODUCT OF FINITELY GENERATED NILPOTENT GROUPS AMALGAMATING A CYCLE THAT IS NOT SUBGROUP SEPARABLE

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ABSTRACT. We exhibit a counterexample to a recent assertion concerning the subgroup separability of groups in the title. The example also serves as a simplification of work of Gitik and Rips.

Denote the generalized free product of groups A and B amalgamating the common subgroup H by $A *_H B$. Let $\mathbf{Q}$ be any class of groups, for example the finite groups, the finite p-groups or the nilpotent groups. A group G is said to be *residually* $\mathbf{Q}$ iff it has the following property where $g \in G : g = 1$ *if and only if* $\overline{g} = 1$ *in all homomorphic images* $\overline{G}$ *of* G *in* $\mathbf{Q}$.

Likewise G is called *subgroup separable* (also called LERF) iff for all $g \in G$ and all *finitely generated* subgroups H of $G : g \in H$ *if and only if* $\overline{g} \in \overline{H}$ *in all finite images* $\overline{G}$ *of* G.

Separation properties such as these are often shared by nilpotent groups and free groups, in part because free groups are residually finitely generated torsion free nilpotent [9]. Furthermore, the proof that each free product F of free groups with cyclic amalgamation is residually finite [2] and potent [1] uses the passage from F down to generalized free products of nilpotent groups. Now in [3] it was shown that each generalized free product F as above is subgroup separable. Consequently one might well expect a similar result for generalized free products of nilpotent groups. The paper [13] attempts to prove this result in the same fashion as [3] by establishing two propositions and then appealing to Lemma 1 of [3]. Unfortunately Proposition 3.5 of [13], whilst true, seems to be insufficient.

Counterexamples of the type claimed can readily be obtained by selecting suitable subgroups of the generalized free products constructed in [5]. Consider, for instance, $G = A *_{\langle a \rangle} B$ where $B = \langle a, c : [a, c] = 1\rangle$ and

$$A = \langle a, d_1, \ldots, d_8 : [d_i, d_j] = 1, d_1^a = d_1 d_2, d_2^a = d_2, d_3^a = d_4,$$
$$d_4^a = d_3, d_5^a = d_6, d_6^a = d_5, d_7^a = d_7, d_8^a = d_8\rangle.$$

Let $G^\dagger = A^\dagger *_{\langle a^2\rangle} B^\dagger \leq G$ where $A^\dagger = \langle a^2, d_1, \ldots, d_8\rangle \leq A$ and $B^\dagger = \langle a^2, c\rangle \leq B$. Thus $G^\dagger$ is the free product of a finitely generated torsion free nilpotent group of class 2 with a free abelian group amalgamating $\langle a^2\rangle$, a maximal cyclic subgroup

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(and retract) of $A^\dagger$. $G^\dagger$ is easily checked to be normal of index 2 in G. Now every finite extension of a subgroup separable group is again subgroup separable [11], [12], and so, if $G^\dagger$ were subgroup separable then G would be too. Hence $G^\dagger$ cannot be subgroup separable.

However, it is possible to modify such examples to product much simpler counterexamples with fewer generators, which, in turn, allow us to check our counterexamples' properties by what is a direct proof.

We prove:

Theorem. *Let $G = A *_{\langle a\rangle} B$ where $A = \langle a, d_1, d_2, d_3 : [d_i, d_j] = 1, d_1^a = d_1 d_2^2, d_2^a = d_2, d_3^a = d_3\rangle$ and $B = \langle a, c : [a, c] = 1\rangle$. Then A and B are nilpotent, but G is not subgroup separable.*

Proof. Clearly A is nilpotent of class 2. Let C be the subgroup

$$C = \langle ad_2^2, cd_1, d_2cd_1d_2^{-2}, d_3cd_3, d_2^3d_3cd_3d_2^{-1}\rangle.$$

Call the final four generators of C "c-generators". We claim that $a \notin C$ but that $a \in H$ for every subgroup H of finite index in G which contains C.

Perhaps the easiest way to see that $a \notin C$ is to write each element γ of C in the form

$$X_0(ad_2^2)^{\alpha_1}X_1(ad_2^2)^{\alpha_2}\cdots X_{k-1}(ad_2^2)^{\alpha_k}X_k$$

where α_i is possibly 0 and each X_i is a c-generator or its inverse. Of course X_0 or X_k may be the identity element, and $X_{i-1} \neq X_i^{-1}$ if $\alpha_i = 0$. In looking at a segment $X_{i-1}(ad_2^2)^{\alpha_i}X_i$ of γ we find (whether $\alpha_i = 0$ or not) a middle section of the form $c^{\pm 1}u(ad_2^2)^{\alpha_i}vc^{\pm 1}$ where u and v^{-1} are members of

$$\{1, d_1, d_2^{-1}, d_1d_2^{-2}, d_3^{-1}, d_3, d_3^{-1}d_2^{-3}, d_3d_2^{-1}\}.$$

Let $M = u(ad_2^2)^{\alpha_i}v$. Then $M = ua^{\alpha_i}d_2^{2\alpha_i}v = a^{\alpha_i}u^*d_2^{2\alpha_i}v = a^{\alpha_i}(d_2^{2\alpha_i}u^*v)$ where $u^* = a^{-\alpha_i}ua^{\alpha_i}$. It is easy to see that $M \in A \setminus \langle a\rangle$ except perhaps if $u = d_1$ or $d_1d_2^{-2}$. In these cases, we obtain $M = a^{\alpha_i}(d_2^{4\alpha_i}d_1v)$ or $a^{\alpha_i}(d_2^{4\alpha_i-2}d_1v)$ respectively. It follows immediately from the normal form theorem for generalized free products that $\gamma \neq a$.

To see that $a \in H$ for any finite index subgroup H of G containing C, note that $a^m \in H$ for some positive integer m and apply (a) and (b) repeatedly:

(a) $a^{2n} \in H \Rightarrow a^n \in H$.

This follows immediately because $(cd_1)(ad_2^2)^{-n}(a^{2n})(cd_1)^{-1} = a^n$.

(b) $a^{2n+1} \in H \Rightarrow a \in H$.

This follows from the implications:

$$\begin{aligned}
&a^{2n+1} \in H\\
&\qquad \Rightarrow d_2^{2(2n+1)} \in H\\
&\qquad \Rightarrow d_2^{2n+1} \in H \text{ since } (ad_2^2)^n(d_2cd_1d_2^{-2})(ad_2^2)^{-n} = (d_2^{2n+1})(cd_1)(d_2^{-4n-2})\\
&\qquad \Rightarrow d_2^2 \in H \text{ since } (ad_2^2)^n(d_2^3d_3cd_3d_2^{-1})(ad_2^2)^{-n} = (d_2^2)(d_2^{2n+1})(d_3cd_3)(d_2^{-2n-1})\\
&\qquad \Rightarrow a \in H.
\end{aligned}$$

This completes the proof. □

Note that G happens to be *cyclic* subgroup separable (also called π_c) [4], [8], residually p-finite for all primes p [7] and potent [1]. Generalized free products where the factors are isomorphic are often rather tractable. Yet here we could have chosen B to be an isomorphic copy of A and the same proof serves to show the resulting generalized free product is not subgroup separable. Indeed, the only property of B we used is that there exists $c \in B \setminus \langle a \rangle$ such that $[a, c] = 1$. Thus the main theorem of [5] is proved with a notably simpler choice of factor A and subgroup C.

Finally, making a trifling variation of the main proof in [10] (namely change all occurrences of LERF to π_{c}), gives an example of a free product of π_{c} groups amalgamating a cycle that is not π_{c} thus answering negatively a question posed in [4], [6]. The π_{c} factors here are countable though not finitely generated. Can a counterexample be found where these π_{c} factors are finitely generated?

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