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# Residual Finiteness of Outer Automorphism Groups of Certain Pinched 1-Relator Groups

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E. K. Grossman (1974, *J. London Math. Soc.* **2**, 160–164) showed that outer automorphism groups of fundamental groups of closed orientable surfaces are residually finite. Here we generalize her result by showing that outer automorphism groups of generalized free products of two free groups amalgamating a maximal cyclic subgroup are residually finite. From this it follows that mapping class groups of closed orientable and nonorientable surfaces are residually finite. The latter answers a question raised by A. M. Gaglione. © 2001 Elsevier Science

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## 1. INTRODUCTION

A well-known result of Baumslag [2] states that the automorphism group of a finitely generated residually finite group is residually finite (in brief, $\mathcal{RF}$). However, there is very little information about the residual finiteness of the outer automorphism group of a finitely generated $\mathcal{RF}$ group. Grossman [4] showed that outer automorphism groups of free groups and those of the fundamental groups of closed orientable surfaces of genus $k$ are $\mathcal{RF}$ from which it follows that mapping class groups of closed orientable surfaces are $\mathcal{RF}$. Her method applies only to surface groups with the presentation

$$\left\langle a_1, \ldots, a_k, b_1, \ldots, b_k;\ \prod_{i=1}^{k}[a_i, b_i]\right\rangle. \tag{1.1}$$

The proof cannot readily be extended to nonorientable surfaces with the presentation

$$\langle a_1, \ldots, a_k;\ a_1^2 \cdots a_k^2\rangle. \tag{1.2}$$

To prove the residual finiteness of outer automorphism groups of groups given by (1.1), Grossman showed the groups given by (1.1) have Property A (see below). In this paper we generalize this result by proving that generalized free products of two free groups amalgamating a maximal cyclic subgroup have Property A. From this it follows that outer automorphism groups of generalized free products of two free groups amalgamating a maximal cyclic subgroup are $\mathcal{RF}$. Thus, in particular, outer automorphism groups of groups given by (1.2) are $\mathcal{RF}$ for $k \neq 2, 3$. In the case when $k = 2$, residual finiteness follows from results in [1]. For $k = 3$, we need Theorem 3.4. It follows that mapping class groups of closed orientable and nonorientable surfaces are $\mathcal{RF}$.

## 2. PRELIMINARY RESULTS

Throughout this paper we use standard notation and terminology.

A group $G$ is *residually finite* ($\mathcal{RF}$) if, for each nontrivial element $x \in G$, there exists a finite homomorphic image $\overline{G}$ of $G$ such that the image of $x$ in $\overline{G}$ is not trivial.

A group $G$ is *conjugacy separable* if, for each pair of elements $x, y \in G$ such that $x$ and $y$ are not conjugate in $G$, there exists a finite homomorphic image $\overline{G}$ of $G$ such that the images of $x$ and $y$ in $\overline{G}$ are not conjugate in $\overline{G}$.

If $G = A *_H B$ and $g \in G$, we use $\|g\|$ to denote the generalized free product length of $g$ or free product length if $H$ is trivial.

We use $\operatorname{Inn} g$ to denote the inner automorphism of $G$ induced by $g \in G$.
$\operatorname{Out} G$ denotes the outer automorphism group, $\operatorname{Aut}(G)/\operatorname{Inn}(G)$, of $G$.
$\Gamma_i(G)$ denotes the $i$th term of the lower central series of $G$.
$C_G(g)$ denotes the centralizer of $g$ in $G$. $Z(G)$ is the center of $G$.
We also use $x \sim_G y$ if $x$ is conjugate to $y$ in $G$.

DEFINITION 2.1. By a *conjugating endomorphism/automorphism* of a group $G$ we mean an endomorphism/automorphism $\alpha$ which is such that, for each $g \in G$, there exists $k_g \in G$, depending on $g$, so that $\alpha(g) = k_g^{-1} g k_g$. Any such $k_g$ will be called a *conjugator* of $g$ for $\alpha$.

DEFINITION 2.2. (Grossman [4]). A group $G$ has *Property A* if for each conjugating automorphism $\alpha$ of $G$, there exists a single element $k \in G$ such that $\alpha(g) = k^{-1} g k$ for all $g \in G$; i.e., $\alpha = \operatorname{Inn} k$.

We will make use of the following results:

THEOREM 2.3 (Grossman [4]). *Let $B$ be a finitely generated, conjugacy separable group with Property A. Then* $\operatorname{Out} B$ *is* $\mathcal{RF}$.

THEOREM 2.4 [1]. *Let $G = A *_{\langle h \rangle} B$ and $h \in Z(B)$. Suppose $A$ is abelian and $A \neq \langle h \rangle$. Then $G$ has Property A.*

We will also need the following result on free groups.

THEOREM 2.5. *Let $F$ be a free group. Let $\alpha$ be a conjugating endomorphism of $F$. Then $\alpha$ is an inner automorphism of $F$.*

*Proof.* Let $F$ be a free group on $\{a, b, c, \ldots\}$. Consider $F = \langle a \rangle * \langle b \rangle * \langle c \rangle * \cdots$. Let $\alpha(a) = k_a^{-1} a k_a$ where $k_a \in F$. Define $\bar{\alpha} = \operatorname{Inn} k_a^{-1} \circ \alpha$. Then $\bar{\alpha}(a) = a$. Moreover, we can certainly take $\bar{\alpha}(b) = k_b^{-1} b k_b$. Thus

$$\bar{\alpha}(ab) = \bar{\alpha}(a)\bar{\alpha}(b) = a k_b^{-1} b k_b. \tag{2.1}$$

Without loss of generality $k_b$ can be taken to be a word on $\langle a \rangle * \langle b \rangle * \cdots$ which does not begin with $b^i$. We assume that either $k_b$ does not end in $\langle a \rangle$ or that $\|k_b\| > 1$ in order to reach a contradiction. Then in both cases we have $\|(\bar{\alpha}(a)\bar{\alpha}(b))^n\| \geq 4n$. Since $\bar{\alpha}((ab)^n) = k_{ab}^{-1}(ab)^n k_{ab}$, we must have

$$k_{ab}^{-1}(ab)^n k_{ab} = (a k_b^{-1} b k_b)^n. \tag{2.2}$$

This means the length of the L.H.S. of (2.2) $\leq 2\|k_{ab}\| + 2n$. On the other hand the length of the R.H.S. of (2.2) is greater or equal to $4n$. This is clearly impossible for sufficiently large $n$. Hence $k_b = a^r$. Similarly $k_c = a^s$. Now,

$$\bar{\alpha}(bc) = k_{bc}^{-1} bc k_{bc} = \bar{\alpha}(b)\bar{\alpha}(c) = a^{-r} b a^r a^{-s} c a^s.$$

As before if $r \neq s$ then $\|\bar{\alpha}((bc)^n)\| \neq \|(\bar{\alpha}(b)\bar{\alpha}(c))^n\|$ for sufficiently large $n$. This implies $r = s$. Hence $k_c = k_b$. Similarly, $k_d = k_b, \ldots,$ etc. Thus $\bar{\alpha}$ is an inner automorphism of $F$, whence $\alpha$ is an inner automorphism of $F$.

## 3. MAIN RESULTS

In this section we shall prove our main result that if $G = A *_H B$ where $A, B$ are free and $H$ is maximal cyclic in $A$ and $B$, then $\operatorname{Out} G$ is residually finite. From this we show that mapping class groups of closed surfaces are residually finite.

We first prove the following lemma:

LEMMA 3.1. *Let $G = A *_H B$, where $A, B$ are free groups and $H = \langle h \rangle$ is a maximal cyclic subgroup of $A$ and $B$. Let $\alpha$ be a conjugating endomorphism of $G$ such that $k_a = g \in G$ is fixed for all $a \in A$ and $k_b = 1$ for all $b \in B$. Then $\alpha$ is the identity automorphism on $G$.*

*Proof.* $H$ being maximal cyclic implies $C_G(h) = \langle h \rangle$. Since $h \in A \cap B$, $\alpha(h) = g^{-1}hg = h$. This means $g = h^s$. Let $a \in A \backslash H$ and $b \in B \backslash H$. Then $k_{ab}^{-1} ab k_{ab} = h^{-s} a h^s b$; i.e., $ab \sim_G h^{-s} a h^s b$. Since $(h^{-s} a h^s) b$ is cyclically reduced, by [6, p. 212], $ab \sim_H h^{-s} a h^s b$. This implies

$$a = h^{-i}(h^{-s} a h^s) h^j \tag{3.1}$$

and

$$b = h^{-j} b h^i \tag{3.2}$$

for some integers $i$ and $j$. Since $B$ is free and $b \notin H$ where $H = \langle h \rangle$ is a maximal cyclic subgroup of $B$, $\langle b, h \rangle$ is free on $\{b, h\}$. Thus (3.2) implies $i = j = 0$. Therefore (3.1) gives $a = h^{-s} a h^s$. Since $a \notin H$, we have $s = 0$. Thus $g = h^s = 1$. Hence $\alpha$ is the identity automorphism on $G$.

THEOREM 3.2. *Let $G = A *_H B$ where $A, B$ are free groups and $H$ is a maximal cyclic subgroup of $A$ and $B$. Then $G$ has Property $A$.*

*Proof.* Let $H = \langle h \rangle$. Since $H$ is maximal cyclic in both $A$ and $B$, we have $C_A(h) = C_B(h) = C_G(h) = H$. Since $A$ is residually torsion-free nilpotent, there exists $r$ such that $\Gamma_r(A) \cap H = 1$. Let $1 \neq a \in \Gamma_r(A)$. Let $\alpha$ be a conjugating automorphism of $G$ such that $\alpha(a) = k_a^{-1} a k_a$ where $k_a \in G$. Again replacing $\alpha$ by $\bar{\alpha} = \operatorname{Inn} k_a^{-1} \circ \alpha$, we may assume $\alpha$ to be a conjugating automorphism of $G$ such that $\alpha(a) = a$ for this particular $a \in A$ and $\alpha(g) = k_g^{-1} g k_g$ for $g \in G$ where $k_g \in G$ could vary with $g$. Let $b \in B$. We shall first show that $k_b$ must be in $BA$ and then that $k_b$ can be chosen in $A$.

(I) $k_b$ must be in $BA$.

Suppose there exists $b \in B$ such that $k_b = a_1 b_1$, where $a_1 \in A \backslash H$ and $b_1 \in B \backslash H$. Consider $\alpha((ab)^n) = (\alpha(a)\alpha(b))^n$. We have

$$k_{ab}^{-1}(ab)^n k_{ab} = (a \cdot b_1^{-1} a_1^{-1} b a_1 b_1)^n. \tag{3.3}$$

Thus the length of the L.H.S. of (3.3) is at most $2n + 2\|k_{ab}\|$. On the other hand the length of the R.H.S. of (3.3) is $6n$, if $b \in B\backslash H$, and it is $4n$ if $b \in H$ since $a_1^{-1}ba_1 \notin H$ because $a_1 \notin H$ and $C_A(h) = H$. This means for sufficiently large $n$ (3.3) cannot hold. Hence $k_b$ cannot be of the form $a_1b_1$ with $a_1 \in A\backslash H$ and $b_1 \in B\backslash H$. Similarly we can show $k_b$ cannot be of the form $a_1b_1a_2,\ b_1a_1b_2$, or any longer forms. Hence $k_b \in BA$.

(II) $k_b$ can be chosen in $A$.

By (I), we can assume $k_b = b_1a_1$ where $b_1 \in B$ and $a_1 \in A$. Since $\alpha(ab) = \alpha(a)\alpha(b)$, we have

$$k_{ab}^{-1}abk_{ab} = ak_b^{-1}bk_b = aa_1^{-1}b_1^{-1}bb_1a_1.$$

Thus

$$ab \sim_G a_1aa_1^{-1} \cdot b_1^{-1}bb_1. \tag{3.4}$$

Since $1 \neq a \in \Gamma_r(A)$ and $\Gamma_r(A) \cap H = 1$, we have $a_1aa_1^{-1} \notin H$.

(i) $b \in H$. If $b_1 \notin H$ then $b_1^{-1}bb_1 \notin H$. Thus $\|ab\| = 1$ and $\|a_1aa_1^{-1} \cdot b_1^{-1}bb_1\| = 2$. By [6, p. 212], (3.4) cannot hold. This implies $b_1 \in H$. Hence $k_b$ can be assumed to belong to $A$.

(ii) $b \in B\backslash H$. Since $\|ab\| = 2$, [6, p. 212] and (3.4) show that $b_1^{-1}bb_1 \notin H$. Again, by [6, p. 212], (3.4) implies $ab \sim_H a_1aa_1^{-1} \cdot b_1^{-1}bb_1$. It follows that $a = h^{-i}a_1aa_1^{-1}h^j$ and $b = h^{-j}b_1^{-1}bb_1h^i$ for some $i, j$. Since $a \in \Gamma_r(A)$ and $\Gamma_r(A) \cap H = 1$, we must have $\bar{h}^{j-i} = 1$ in $\overline{A} = A/\Gamma_r(A)$ where $\overline{A}$ is torsion-free with $\bar{h} \neq 1$. Thus $i = j$. This implies $h^ibh^{-i} = b_1^{-1}bb_1$. This means

$$\alpha(b) = k_b^{-1}bk_b = a_1^{-1}b_1^{-1}bb_1a_1 = a_1^{-1}h^ibh^{-i}a_1.$$

Therefore we can choose $k_b = h^{-i}a_1 \in A$.

Hence we have proved $k_b \in A$ for all $b \in B$. Now, let $c, d \in B\backslash H$. Consider

$$k_{cd}^{-1}(cd)^nk_{cd} = (\alpha(cd))^n = (k_c^{-1}ck_c \cdot k_d^{-1}dk_d)^n, \tag{3.5}$$

where we can choose $k_c, k_d \in A$. Since $k_c, k_d \in A$, by considering the lengths of each side of (3.5) we must have $k_ck_d^{-1} = h^r$, where $r$ depends on $c$. This implies $\alpha(c) = k_d^{-1}h^{-r}ch^rk_d$. For a fixed $d$, let $\bar{\alpha} = \operatorname{Inn} k_d^{-1} \circ \alpha$. Then $\bar{\alpha}(d) = d,\ \bar{\alpha}(c) = h^{-r}ch^r$ for all $c \in B\backslash H$, where $r$ may vary with $c$. Also $\bar{\alpha}(h) = k_dk_h^{-1}hk_hk_d^{-1}$. Then $x = k_hk_d^{-1} \in A$, since $k_h, k_d$ can be chosen in $A$. Next, we consider $\bar{\alpha}(hc) = x^{-1}hx \cdot h^{-r}ch^r$. Since $c \in B\backslash H,\ h^{-r}ch^r \in B\backslash H$. If $x^{-1}hx \in A\backslash H$, then $\|\bar{\alpha}(hc)^n\| = \|k_{hc}^{-1}(hc)^nk_{hc}\| \neq \|(x^{-1}hxh^{-r}ch^r)^n\|$ for sufficiently large $n$. Hence $x^{-1}hx \in H$. Therefore $x \in H$ and $\bar{\alpha}(h) = h$. We may then assume that, for all $b \in B$, $\bar{\alpha}(b) = k_b^{-1}bk_b$ where $k_b \in H$. Thus $\bar{\alpha}$ restricted to $B$ is a conjugating endomorphism of $B$.

Since $B$ is free by Theorem 2.5, $\bar{\alpha}$ is an inner automorphism of $B$. Hence there exists a fixed $k \in B$ such that $\bar{\alpha}(b) = k^{-1}bk$ for all $b \in B$. Let $\alpha' = \operatorname{Inn} k^{-1} \circ \bar{\alpha}$. Then $\alpha'(b) = b$ for all $b \in B$.

Let $\Gamma_n(B) \cap H = 1$ and let $1 \neq b \in \Gamma_n(B)$. Repeating the argument with $\alpha'(b) = b$ and $\alpha'(x) = k_x^{-1}xk_x$ for $x \in A$, as in (II) above, we can assume that $k_x \in B$ for all $x \in A$. Consider $k_{hx}^{-1}(hx)k_{hx} = \alpha'(hx) = \alpha'(h)\alpha'(x) = h \cdot k_x^{-1}xk_x$. Hence $hx \sim_G k_xhk_x^{-1} \cdot x$. Since $hx \in A$ and $k_x \in B$, $k_xhk_x^{-1} \in H$. Hence $k_x \in H$. Thus $\alpha'$ restricted to $A$ is a conjugating endomorphism of $A$. Since $A$ is free by Theorem 2.5, $\alpha'$ is an inner automorphism of $A$. Hence there exists a fixed $g \in A$ such that $\alpha'(x) = g^{-1}xg$ for all $x \in A$. Thus, by Lemma 3.1, $\alpha'$ is the identity automorphism on $G$. Since $\alpha'$ is obtained from the original $\alpha$ by a sequence of inner automorphisms of $G$, $\alpha$ is an inner automorphism of $G$.

This completes the proof.

Dyer [3] showed that generalized free products of two free groups amalgamating a cyclic subgroup are conjugacy separable. Thus applying Theorems 2.3 and 3.2, we immediately have:

THEOREM 3.3. *Let $G = A *_H B$ where $A, B$ are free and $H = \langle h \rangle$ is a maximal cyclic subgroup of $A$ and $B$. Then* Out $G$ *is residually finite.*

Let $S_k$ be closed surfaces of genus $k$. If we use the algebraic definition of the mapping class group of $S_k$ to be Out $(\pi_1(S_k))$ (cf. [6, p. 175]), then we are in a position to show that mapping class groups of all closed surfaces are residually finite. We first prove the following theorem.

THEOREM 3.4. *Let $G = A *_H B$ where $A$ is a free group of rank $\geq 2$ and $H = \langle h \rangle$ is a maximal cyclic subgroup of $A$. If $h \in Z(B)$ then $G$ has Property $A$.*

*Proof.* If $B = H$ then $G$ is a free group and so $G$ has Property A by Grossman [4]. Hence we assume $B \neq H$. Since $A$ is not cyclic, there exists an integer $m$ such that $H \cap \Gamma_m(A) = 1$. Let $1 \neq a \in \Gamma_m(A)$. Then $a \in A \backslash H$. Let $\alpha$ be a conjugating automorphism of $G$ and let $\alpha(g) = k_g^{-1}gk_g$ for $g \in G$. Without loss of generality, we can assume $\alpha(a) = a$. We shall show that $\alpha$ is an inner automorphism of $G$ by the following three steps:

(I) *Claim.* For each $b \in B$, we can choose $k_b \in A$.

Let $b \in B$ and $k_b = u_1u_2 \cdots u_r$ be an alternating product of the shortest length from $G$ such that $\alpha(b) = k_b^{-1}bk_b$. Then $k_{ba}^{-1}(ba)k_{ba} = \alpha(ba) = \alpha(b)\alpha(a) = k_b^{-1}bk_b \cdot a = u_r^{-1} \cdots u_2^{-1} \cdot u_1^{-1}bu_1 \cdot u_2 \cdots u_{r-1} \cdot u_r \cdot a$. Thus,

$$ba \sim_G u_{r-1}^{-1} \cdots u_2^{-1} \cdot u_1^{-1}bu_1 \cdot u_2 \cdots u_{r-1} \cdot u_rau_r^{-1}. \tag{3.6}$$

Note that if $u_r \in A$ then, by the choice of $a$, $u_rau_r^{-1} \notin H$.

(a) $b \in B \backslash H$.

(i) Suppose $u_1 \in B$. Since $H \subseteq Z(B)$, $u_1^{-1} b u_1 \notin H$. If $u_r \in A \backslash H$ then the R.H.S. of (3.6) is cyclically reduced of length $2(r-1)$. Since the L.H.S. of (3.6) is cyclically reduced of length 2, we have $r = 2$ and $k_b = u_1 u_2 \in BA$. Therefore, from (3.6), $ba \sim_G u_1^{-1} b u_1 \cdot u_2 a u_2^{-1}$, where both sides are cyclically reduced of length 2. Thus, by [6, p. 212], $ba \sim_H u_1^{-1} b u_1 \cdot u_2 a u_2^{-1}$, which implies that $b = h^{-i}(u_1^{-1} b u_1) h^j$ and $a = h^{-j}(u_2 a u_2^{-1}) h^i$ for some $i, j$. Since $a \in \Gamma_m(A)$ and $H \cap \Gamma_m(A) = 1$, we have $\bar{h}^{i-j} = 1$ in $\bar{A} = A/\Gamma_m(A)$, where $\bar{A}$ is a finitely generated torsion-free nilpotent group. Hence $i = j$. This implies $u_1^{-1} b u_1 = h^i b h^{-i}$. Thus $\alpha(b) = k_b^{-1} b k_b = u_2^{-1} u_1^{-1} b u_1 u_2 = u_2^{-1}(h^i b h^{-i}) u_2$. This means that we can choose $k_b = h^{-i} u_2 \in A$.

If $u_r \in B \backslash H$, then the R.H.S. of (3.6) is cyclically reduced of length $2r$. Since the L.H.S. of (3.6) is of length 2, we have $r = 1$. Hence $k_b = u_1 \in B$. As above, we can show that $\alpha(b) = u_1^{-1} b u_1 = h^i b h^{-i}$ for some $i$. Hence we can choose $k_b = h^{-i} \in A$.

(ii) Suppose $u_1 \in A$. If $u_r \in A \backslash H$, then the R.H.S. of (3.6) is cyclically reduced of length $2r$. Since the L.H.S. of (3.6) is of length 2, we have $r = 1$. Hence $k_b = u_1 \in A$.

If $u_r \in B \backslash H$ then $r \geq 2$ and the R.H.S. of (3.6) is cyclically reduced of length $2(r+1) \geq 6$. Since the L.H.S. of (3.6) is of length 2, this case does not occur.

(b) $b = h^s$ for some $s$. Since $b = h^s \in Z(B)$, we may assume that $u_1 \in A$ in (3.6). Since $u_1 \in A \backslash H$ and $H$ is maximal cyclic in $A$, we have $u_1^{-1} b u_1 = u_1^{-1} h^s u_1 \in A \backslash H$. If $u_r \in B \backslash H$, then the R.H.S. of (3.6) is cyclically reduced of length $2r$. Since $\|ba\| = \|h^i a\| = 1$, this case does not occur. If $u_r \in A \backslash H$ and $r \geq 2$, then the R.H.S. of (3.6) is cyclically reduced of length $2(r-1)$. Since the L.H.S. of (3.6) is of length 1, we have $r = 1$ and $k_b = u_1 \in A$.

(II) *Claim.* There exists a fixed $u \in C_A(a)$ so that $k_y = u$ for all $y \in B$.

Let $b \in B \backslash H$ be fixed and let $y$ run over all elements of $B \backslash H$. Because of (I), let $k_b = w \in A$. Then $k_{by}^{-1}(by) k_{by} = \alpha(by) = \alpha(b)\alpha(y) = w^{-1} b w \cdot k_y^{-1} y k_y$, where $k_y \in A$ by (I). Hence,

$$by \sim_G b \cdot w k_y^{-1} \cdot y \cdot k_y w^{-1}. \tag{3.7}$$

Since $b, y \in B \backslash H$, if $w k_y^{-1} \in A \backslash H$ then the R.H.S. of (3.7) is cyclically reduced of length 4. Hence (3.7) does not hold. Thus $w k_y^{-1} \in H$. Let $k_y = h^{-s} w$, where $s$ depends on $y$. Then $k_{ya}^{-1}(ya) k_{ya} = \alpha(ya) = \alpha(y)\alpha(a) = k_y^{-1} y k_y \cdot a = w^{-1} h^s y h^{-s} w \cdot a = w^{-1} y w \cdot a$, since $h \in Z(B)$. Hence we have $ya \sim_G y \cdot w a w^{-1}$, where both sides are of length 2. This implies $ya \sim_H y \cdot w a w^{-1}$, whence $y = h^{-i} y h^j$ and

$a = h^{-j}waw^{-1}h^i$ for some $i, j$. Since $h \in Z(B)$ and $y \in B$, $i = j$. Hence $a = h^{-i}waw^{-1}h^i$. Take $h^{-i}w = u \in C_A(a)$. Then $k_y = h^{-s}w = h^{-s}h^iu = h^{i-s}u$. This implies $\alpha(y) = k_y^{-1}yk_y = u^{-1}h^{s-i}yh^{i-s}u = u^{-1}yu$. This shows that $\alpha(y) = u^{-1}yu$ for all $y \in B\backslash H$. Now $\alpha(h) = \alpha(hy \cdot y^{-1}) = \alpha(hy)\alpha(y^{-1}) = u^{-1}(hy)u \cdot u^{-1}y^{-1}u = u^{-1}hu$. Hence $\alpha(y) = u^{-1}yu$ for all $y \in B$.

(III) *Claim.* Let $\bar{\alpha} = \operatorname{Inn} u^{-1} \circ \alpha$. Then, by (II), $\bar{\alpha}(y) = y$ for all $y \in B$. Moreover, since $u \in C_A(a)$, $\bar{\alpha}(a) = u\alpha(a)u^{-1} = uau^{-1} = a$. We shall show that $\bar{\alpha}$ is the identity on $G$. It suffices to show that $\bar{\alpha}(x) = x$ for all $x \in A\backslash H$. For convenience, we again write $\bar{\alpha}(g) = k_g^{-1}gk_g$ for $g \in G$.

Let $x \in A\backslash H$ and $k_x = u_1u_2\cdots u_r$ be an alternating product of the shortest length from $G$ such that $\bar{\alpha}(x) = k_x^{-1}xk_x$. Then $k_{xa}^{-1}(xa)k_{xa} = \bar{\alpha}(xa) = k_x^{-1}xk_x \cdot a = u_r^{-1}\cdots u_1^{-1}xu_1\cdots u_r \cdot a$. Hence

$$xa \sim_G u_{r-1}^{-1}\cdots u_2^{-1} \cdot u_1^{-1}xu_1 \cdot u_2\cdots u_{r-1} \cdot u_rau_r^{-1}. \tag{3.8}$$

Suppose $u_1 \in B$. If $u_r \in A\backslash H$ so that $\|k_x\| \geq 2$, then the R.H.S. of (3.8) is cyclically reduced of length $2r$. If $u_r \in B\backslash H$, then the R.H.S. of (3.8) is cyclically reduced of length $2(r+1)$. Since the L.H.S. of (3.8) is $xa \in A$, both of these cases do not occur. Hence $u_1 \in A$.

We note that if $u_1 \in A$ and $r \geq 2$ then we may assume $u_1^{-1}xu_1 \notin H$. For, if $u_1^{-1}xu_1 = h^s$ then $u_2^{-1}u_1^{-1}xu_1u_2 = u_2^{-1}h^su_2 = h^s = u_1^{-1}xu_1$. This reduces the length of $k_x$. We also note that if $u_r \in A$ then $u_rau_r^{-1} \in A\backslash H$ by the choice of $a$.

So suppose $u_1 \in A$ and $r \geq 2$. If $u_r \in A\backslash H$, then the R.H.S. of (3.8) is cyclically reduced of length $2(r-1)$. If $u_r \in B\backslash H$ then the R.H.S. of (3.8) is cyclically reduced of length $2r$. As before, since the length of the L.H.S. of (3.8) is at most 1, both of these cases do not occur. Thus we must have $u_1 \in A$ and $r \leq 1$; i.e., $k_x \in A$ for each $x \in A\backslash H$. Also by (II), $\bar{\alpha}(h^i) = h^i$. Therefore, $\bar{\alpha}$ restricted to $A$ is a conjugating endomorphism of the free group $A$. Hence, by Theorem 2.5, $\bar{\alpha}$ restricted to $A$ is an inner automorphism of $A$. Thus there exists a fixed $g \in A$ such that $\bar{\alpha}(x) = g^{-1}xg$ for all $x \in A$. Since $h = \bar{\alpha}(h) = g^{-1}hg$, $g \in C_A(h) = H$. Let $g = h^s$ for some $s$. Then $a = \bar{\alpha}(a) = h^{-s}ah^s$. Thus, if $s \neq 0$, then $a \in H$, contradicting the choice of $a$. Hence, $s = 0$. This means that $\bar{\alpha}(x) = x$ for all $x \in A$. Hence $\bar{\alpha}$ is the identity on $G$, as required. This shows that $G$ has Property A.

By [5], if $B$ is finitely generated nilpotent, then the group $G$ in Theorem 3.4 is conjugacy separable. Applying Theorems 2.3 and 3.4, we immediately have:

THEOREM 3.5. *Let $G = A *_H B$, where $A$ is a free group of rank $\geq 2$ and $H = \langle h \rangle$ is a maximal cyclic subgroup of $A$. If $B$ is finitely generated nilpotent and $h \in Z(B)$ then* Out $G$ *is residually finite.*

It is now easy to show mapping class groups of all closed surfaces are residually finite.

THEOREM 3.6. *Mapping class groups of closed surfaces $S_k$ are residually finite.*

*Proof.* If $S_k$ is orientable, then its fundamental group is given by (1.1), which is a generalized free product of two free groups amalgamating a maximal cyclic subgroup except when $k = 1$, in which case the theorem is clearly true. Hence by Theorem 3.2, the outer automorphism group is residually finite for all $k$. If $S_k$ is nonorientable, its fundamental group $G_k$ is given by (1.2). If $k > 3$, then $G_k$ is a generalized free product of two free groups amalgamating a maximal cyclic subgroup, whence Out $G_k$ is residually finite. If $k = 1$, $G_1 = \langle a_1; a_1^2 \rangle$. Clearly, Out $G_1$ is finite. If $k = 2$, $G_2 = \langle a_1 \rangle \underset{a_1^2 = a_2^{-2}}{*} \langle a_2 \rangle$. By Theorem 2.4, Out $G_2$ is residually finite. If $k = 3$, $G_3 = \langle a_1, a_2 \rangle \underset{a_1^2 a_2^2 = a_3^{-2}}{*} \langle a_3 \rangle$. Therefore, by Theorem 3.5, Out $G_3$ is residually finite. Since by Corollary 11.7 in [7] mapping class groups of closed surfaces are isomorphic to the outer automorphism groups of the fundamental groups of $S_k$, it follows that mapping class groups of closed surfaces are residually finite. This completes the proof.

Theorem 3.6 gives a positive answer to a question raised by A. M. Gaglione at the Zassenhaus Group Theory Conference 2000, held at the Ohio State University: Is the mapping class group of a closed nonorientable surface residually finite?

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