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CONJUGACY SEPARABILITY OF A CLASS OF 1-RELATOR PRODUCTS

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ABSTRACT. We prove the conjugacy separability of groups of the form $G = \langle a_1, \dots a_r, b_1, \dots, b_s : a_i^{e_i} = b_j^{f_j} = (U(a_1, \dots, a_r)V(b_1, \dots, b_s))^m = 1\rangle$, where $m > 1$.

INTRODUCTION

In the past decade much interest has been shown in so-called 1-relator products of groups, that is (factor) groups of the form A/N where $A = A_1 * A_2 * \cdots * A_n$ is the free product of the groups A_i $(1 \leq i \leq n)$ and N is the normal closure, in A, of a single element of A. In particular, all 1-relator groups, finitely generated Fuchsian groups, and triangle groups are of this form. Recently, attempts have been made to extend to 1-relator products the more familiar results, for example, Magnus's Freiheitssatz, already known in 1-relator case (see, e.g., [1, 3, 4, 5, 8, 14]).

In this paper we shall prove, in answer to a question posed by Fine and Rosenberger in [6], the following

Theorem. *Let* $G = \langle a_1, \dots, a_r, b_1, \dots, b_s : a_i^{e_i} = b_j^{f_j} = (UV)^m = 1\rangle$ *where* $1 \leq r$, $1 \leq s$, $e_j = 0$ *or* ≥ 2, $f_j = 0$ *or* ≥ 2, $m \geq 2$ *(for all* i, j *such that* $1 \leq i \leq r$, $1 \leq j \leq s$*)*,[1] $U = U(a_1, \dots, a_r)$, $V = V(b_1, \dots, b_s)$. *Then* G *is conjugacy separable.*

Recall that G is *conjugacy separable* iff to each pair $g_1, g_2 \in G$ *either* g_1 is conjugate to g_2 in G $(g_1 \sim_G g_2)$ *or* there exists a finite homomorphic image $\overline{G}$ of G in which $\overline{g}_1 \nsim_{\overline{G}} \overline{g}_2$. Interest in conjugacy separable groups in general stems from the fact that such groups, if finitely generated, have a solvable conjugacy problem.

The groups of the theorem may be regarded as generalisations of triangle groups. Triangle groups were proved conjugacy separable by Fine and Rosenberger in [6] as a preliminary to their proving the conjugacy separability of all Fuchsian groups.

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[1] These conditions will not be stated explicitly in the future.

THE PROOF OF THE THEOREM

The following proof is quite involved in detail. Accordingly we have split the proof into several lemmas in an attempt to make it easier to digest.

First note that, since free groups and free products of conjugacy separable groups are again conjugacy separable [12], we may assume that, between them, U, V involve all the generators a_i, b_j.

Next, some observations and a preliminary part of the proof. Let us write

$$(*) \qquad G = \langle a_1, \dots, a_r, a, b_1, \dots, b_s : a_i^{e_i} = a^m = b_j^{f_j} = 1, \ U^{-1}a = V \rangle .$$

Then, provided V has infinite order, G is the generalised free product of the free product of cycles $A = \langle a_1, \dots, a_r, a : a_i^{e_i} = a^m = 1 \rangle$ and the free product of cycles $B = \langle b_1, \dots, b_s : b_j^{f_j} = 1 \rangle$ with the cyclic subgroup $\langle h \rangle = \langle U^{-1}a \rangle = \langle V \rangle$ amalgamated. If V has finite order and U has infinite order, swap over the roles of U, V. If U, V both have finite order then $U = a_0^{-1} a_k^{\alpha} a_0$, $V = b_0^{-1} b_l^{\beta} b_0$ for suitable elements $a_0 \in A$, $b_0 \in B$ and for suitable generators of finite order $a_k \in A$, $b_l \in B$. In this case conjugate A and B by a_0, b_0, respectively, before going further so that G may be assumed to take the form $\langle a_1, \dots, a_r, a, b_1, \dots, b_s : a_i^{e_i} = b_j^{f_j} = (a_k^{\alpha} b_l^{\beta})^m = 1 \rangle$. Thus $G = F * R$ where F is a free product of cycles and $R = \langle a_k, b_l : a_k^{e_k} = b_l^{f_l} = (a_k^{\alpha} b_l^{\beta})^m = 1 \rangle$.

Now F is conjugacy separable and so is R. For: Let $K = e_k/(e_k, \alpha)$, $\gamma = \alpha/(e_k, \alpha)$, $L = f_l/(f_l, \beta)$, $\delta = \beta/(f_l, \beta)$. Then R is obtained from $R_0 = \langle x, y : x^K = y^L = (x^{\gamma} y^{\delta})^m = 1 \rangle$ by two successive generalised free products (i) $R_1 = \langle a_k : a_k^{e_k} = 1 \rangle *_{a_k^{e_k/K} = x} R_0$ and (ii) $R_2 = R_1 *_{y = b_l^{f_l/L}} \langle b_l : b_l^{f_l} = 1 \rangle$. By Dyer [2, Theorem 4], these will be conjugacy separable if R_0 is. But $R_0 \cong \langle x, y : x^K = y^L = (xy)^m = 1 \rangle$ (since $(\gamma, K) = (\delta, L) = 1$) and R_0 is known to be conjugacy separable by [6, Theorem 1].

Thus, in the main part of the proof given later, we shall assume that h ($= U^{-1}a = V$) has infinite order. We shall make use of the following lemmas, some of which look interesting in their own right.

Lemma A. *Let G, h be as in $(*)$ and let μ be a positive integer. Then $h^{\mu} \sim_G h^{-\mu}$ iff $h \sim_G h^{-1}$.*

Proof. If $c = h^{\mu} \sim_G h^{-\mu} = d$, then [9, p. 212] there exists a sequence $h^{i_1}, \dots, h^{i_r}$ of elements of $\langle h \rangle$ such that

$$(**) \qquad c = h^{\mu} \sim_A h^{i_1} \sim_B h^{i_2} \sim_A \cdots \sim_B h^{i_r} = h^{-\mu} = d .$$

Now, inside A, B we can only have $h^{\mu} \sim h^{\mu}$ or $h^{\mu} \sim h^{-\mu}$ since (see Lemma B) A, B have finite homomorphic images[2] $\overline{A}$, $\overline{B}$ in which $\overline{h}$ has any prescribed order. Thus each i_k is a μ or a $-\mu$. Furthermore if, say, $h^{\mu} = b^{-1} h^{\pm\mu} b$ then h is conjugate to $h^{\pm 1}$ [9, Exercise 9, p. 194] *using the same element* b. Thus $(**)$ implies that $h \sim_G h^{-1}$.

[2]Throughout we use $\overline{A}$, $\overline{\overline{A}}$, etc., somewhat indiscriminately to denote homomorphic images of A.

Lemma B. *Let A be the free product of finitely many cycles, and let h be an element of infinite order in A. Then there exists a finite homomorphic image $\overline{A}$ of A in which $\overline{h}$ has prescribed order λ.*

Proof. Assume, without loss of generality, that all the generators of A are involved in h. By [4] we can map A onto $\overline{A} \subseteq \mathrm{PSL}_2(\mathbb{C})$ so that $\overline{h}$ has order λ. But finitely generated subgroups of $\mathrm{PSL}_2(\mathbb{C})$ are residually finite (see [10]). Hence there exists a map of A onto a finite group in which $\overline{h}$ has order λ.

In our main proof we shall apply the two parts of the proof of the following lemma separately. For tidyness we unite them here into

Lemma C. *Let A and h be as above, and let F be a normal free subgroup of finite index of A contained in the kernel of the natural map from A onto the direct product of its (finitely many) finite cycles. Let $h^w \in F$ $(w \geq 1)$ and suppose that $h \nsim_A h^{-1}$. Then there exists a finite homomorphic image $\overline{A}$ of A in which $\overline{h} \neq \overline{1}$ and $\{\overline{h}^w\}^{\overline{A}} \cap \langle \overline{h}^w \rangle = \{\overline{h}^w\} \neq \{\overline{h}^{-w}\}$.*[3]

Proof. Let $S = \{\alpha_1 = 1, \alpha_2, \dots, \alpha_r\}$ be coset representatives of $A \bmod D$ where $D = FC$, C being the centraliser of h^w in A. Then each element of A is of the form $yf\alpha$ where $y \in C$, $f \in F$, $\alpha \in S$. Hence $\{h^w\}^A = \{\alpha_k^{-1} f^{-1} h^w f \alpha_k \colon \alpha_k \in S \ (k \neq 1),\ f \in F\} \cup \{f^{-1} h^w f \colon f \in F\}$. Now $\{f^{-1}h^w f \colon f \in F\} \cap \langle h^w \rangle = \{h^w\}$. (This is a singleton since $h^w \sim h^{w'}$ is ruled out by hypothesis if $w' = -w$ and by order considerations if $|w'| \neq |w|$.) Similarly, for each k $(1 \leq k \leq r)$, $\{\alpha_k^{-1} f^{-1} h^w f \alpha_k \colon f \in F\} \cap \langle h^w \rangle \subseteq \{h^w\}$. But if $k \neq 1$ then $\alpha_k^{-1} f^{-1} h^w f \alpha_k = h^w \Rightarrow f\alpha_k \in C \subseteq D \Rightarrow \alpha_k \in D$, which is impossible by choice of S. Hence $\{\alpha_k^{-1} f^{-1} h^w f \alpha_k \colon f \in F\} \cap \langle h^w \rangle = \varnothing$ (if $k \neq 1$). Consequently $\{f^{-1} h^w f \colon f \in F\} \cap \langle \alpha_k h^w \alpha_k^{-1} \rangle = \varnothing$ (if $k \neq 1$)—an empty intersection holding in the free group F. By Dyer [2, Lemmas 8 and 6], there exists a finite (nilpotent) homomorphic image F/X of F in which $\{\overline{f}^{-1} \overline{h}^w \overline{f} \colon \overline{f} \in \overline{F}\} \cap \langle \overline{\alpha}_k \overline{h}^w \overline{\alpha}_k^{-1} \rangle = \varnothing$ for each and, hence for *all*, $\alpha_k \neq 1$. Since X has finite index in A, we can assume without loss of generality that, in fact, $X \triangleleft A$. To modify X further so that also $\{\overline{f}^{-1} \overline{h}^w \overline{f} \colon \overline{f} \in \overline{F}\} \cap \langle \overline{h}^w \rangle = \{\overline{h}^w\}$ in A/X, take $Y = \Gamma_s(F)$, the sth term of the lower central series of F where s is such that $h^w \in \Gamma_{s-1}(F) \backslash \Gamma_s(F)$. Suppose that $\overline{h}^w$ has order v in A/X.

Let E/Y be a characteristic subgroup of $\Gamma_{s-1}(F/Y)$ (and, hence, of F/Y) such that the image of h^w in $F/Y\big/E/Y$ $(= F/E)$ has order v exactly. Note that E is then characteristic in F and hence normal in A. Since A/E is finitely generated and nilpotent by finite it is residually finite and so A has a subgroup Z, say, such that A/Z is finite and hZ has order vw. Replace X by $X \cap Z = U$. Then $\bar{\bar{A}} = A/U$ satisfies the lemma's conclusion. For, if $\bar{\bar{f}}_0^{-1} \bar{\bar{h}}^w \bar{\bar{f}}_0 = \bar{\bar{h}}^{tw} \in \{\bar{\bar{f}}^{-1} \bar{\bar{h}}^w \bar{\bar{f}} \colon \bar{\bar{f}} \in \bar{\bar{F}}\} \cap \langle \bar{\bar{h}}^w \rangle$ in $\bar{\bar{A}}$ (with $1 < t < v$), then $\overline{f}_0^{-1} \overline{h}^w \overline{f}_0 = \overline{h}^{tw}$ in A/Z. Since $h^w \in \Gamma_{s-1}(F)$, $\overline{f}_0^{-1} \overline{h}^w \overline{f}_0 = \overline{h}^w$ in F/E. Hence a similar equality holds in F/Z and so in A/Z.

Thus $(\overline{h}^w)^{t-1} = 1$ in A/Z—a contradiction. Hence $\bar{\bar{f}}_0^{-1} \bar{\bar{h}}^w \bar{\bar{f}}_0 = \bar{\bar{h}}^w$ in $\bar{\bar{A}}$.

To prove Lemma E we need the preliminary

[3] $\{a\}^A$ denotes the set of conjugates of a in A.

Lemma D. *Let $\langle h\rangle$, $\langle k\rangle$ be cyclic subgroups of the free group F and let $s \in F\backslash\langle h\rangle\langle k\rangle$. Then there exists a finite homomorphic image $\overline{F}$ of F in which $\overline{s} \in \overline{F}\backslash\langle\overline{h}\rangle\langle\overline{k}\rangle$.*

Proof. Suppose $[h, k] \neq 1$ in F and suppose $\overline{s}_i = \overline{h}^{\alpha_i}\overline{k}^{\beta_i}$ expresses the image of s as an element of $\langle\overline{h}\rangle\langle\overline{k}\rangle \subseteq F/\Gamma_i(F)$ $(= F_i$ say). Suppose $[h, k] \in \Gamma_{t-1}(F)\backslash\Gamma_t(F)$. If for $i \geq t$ all the α_i are equal, then for some $j \geq t$ we have $\beta_j \neq \beta_t$ (since $\bigcap_{i=1}^{\infty}\Gamma_i(F) = \langle 1\rangle$). But this implies that $\overline{k}$ has order $|\beta_j - \beta_t|$ in F_t—a contradiction, since each F_t is aperiodic. Thus $\alpha_r \neq \alpha_t$ for some $r > t$ and then $\overline{h}^{\alpha_r-\alpha_t} = \overline{k}^{\beta}$ in F_t for some $\beta \geq 0$. $\beta = 0$ gives a similar contradiction to that just noted. If $\beta \geq 1$ then, since $\Gamma_{t-1}(F)/\Gamma_t(F)$ is central in F_t, we have $[\overline{h}, \overline{k}]^{\beta} = [\overline{h}, \overline{k}^{\beta}] = \overline{1}$ in F_t—contradicting $[\overline{h}, \overline{k}] \neq \overline{1}$ in F_t. Thus there is a finitely generated torsionfree nilpotent homomorphic image $F_s = \overline{N}$, say, of F in which $\overline{s} \notin \langle\overline{h}\rangle\langle\overline{k}\rangle$. So by Stebe [13] there exists a finite homomorphic image of F in which $\overline{s} \notin \langle\overline{h}\rangle\langle\overline{k}\rangle$. If $[h, k] = 1$ then $\langle h\rangle\langle k\rangle = \langle h, k\rangle = \langle\zeta\rangle$ with $h = \zeta^{\kappa}$, $k = \zeta^{\lambda}$ for some $\zeta \in F$ and $\kappa, \lambda \in \mathbb{Z}$, and so the desired result holds in this case since free groups are Π_c (Stebe [11]). (Recall that a group G is Π_c iff, to each cyclic subgroup H and to each $g \in G\backslash H$, there exists a finite homomorphic image $\overline{G}$ of G in which $\overline{g} \notin \overline{H}$.)

Lemma E. *Let A and h be as usual, and let $s, g \in A$ be such that $s \notin H \cdot H^g$ where $H = \langle h\rangle$ and $H^g = g^{-1}Hg$. Then there exists a finite homomorphic image $\overline{A}$ of A in which $s \notin \overline{H} \cdot \overline{H}^{\overline{g}}$.*

Proof. Let F be the kernel of the natural map from A onto the direct product of its (finitely many) finite cycles. If $s \notin \langle h\rangle F$ factor out (the normal subgroup) $\langle h\rangle F$. Otherwise, Suppose that $s \in \langle h\rangle F$. Write $H = H_0 \cup hH_0 \cup \cdots \cup h^{w-1}H_0$ where w is the least positive power of h to lie in F, $h_0 = h^w$, and $H_0 = \langle h_0\rangle$.

Then $HH^g = \{h^{\alpha}g^{-1}h^{\beta}g : \alpha, \beta \in \mathbb{Z}\} = \{h^u h_0^S (h_0^T h^v)^g : 0 \leq u, v < w;\ S, T \in \mathbb{Z}\}$. Thus, if $s \notin HH^g$ then $(h^{-u})s(h^{-v})^g \notin H_0H_0^g$ for $0 \leq u, v < w$. If, for any u, v, $(h^{-u})s(h^{-v})^g \notin F$, factor out F. If $(h^{-u})s(h^{-v})^g \in F$ find a finite homomorphic image of F (as in Lemma D) and then of A in which $(\overline{h}^{-u})\overline{s}(\overline{h}^{-v})^{\overline{g}} \notin \overline{H} \cdot \overline{H}^{\overline{g}}$. Now take the intersection N, say, of the finitely many kernels (each of finite index in A) thus arising and set $\overline{A} = A/N$.

Lemma F. *Let A and h be as usual. If u, v are elements of A such that $u = h^i v h^j$ then either* (i) *$v^{-1}hv = h$ or* (ii) *$v^{-1}hv = h^{-1}$ or* (iii) *i, j are uniquely determined.*

Proof. If $u = h^i v h^j = h^k v h^l$ then $v^{-1}h^{i-k}v = h^{l-j}$. Thus we have $i - k = \pm(l - j)$ by order considerations. If $i = k$ then $l = j$. Otherwise, by [9, Exercise 9, p. 194], we deduce $v^{-1}hv = h^{\pm 1}$.

Note G. In case (iii) we have $\langle v^{-1}hv\rangle \cap \langle h\rangle = \langle 1\rangle$. For if $v^{-1}h^{\alpha}v = h^{\beta}$ then $\alpha = \pm\beta$.

Lemma H. *Let A, h, and F be as in Lemma* E *and let h^w be the least power of h in F. Suppose for all r such that $1 \leq r \leq s$ that $u_r, v_r \in A$ and that $u_r = h^{i_r}v_r h^{j_r}$ where the i_r, j_r are unique. Then there is a finite homomorphic image $\overline{A}$ of A in which $\overline{h}$ has order wp (p some prime) and, if, for $1 \leq r \leq s$, $\overline{h}^{k_r}, \overline{h}^{l_r}$ is a solution of $\overline{u}_r = x\overline{v}_r y$, then $k_r \equiv i_r$ and $l_r \equiv j_r \pmod p$.*

Proof. First note that the uniqueness of the solution of $u_r = h^{i_r}v_r h^{j_r}$ implies that for each r, $H \cap H^{v_r} = \langle 1\rangle$ (see Note G). This, in turn, implies that $W_r = [v_r^{-1}h^w v_r, h^w] \neq 1$. For otherwise $v_r^{-1}h^w v_r, h^w$ generate an infinite cyclic subgroup of F, a contradiction. Note that each W_r lies in F, which is free of rank > 1. If $[W_1, W_2] = 1$ set $W_2' = [W_2, z]$ for suitable z so that $[W_1, W_2'] \neq 1$. If $[W_1, W_2] \neq 1$ set $W_2' = W_2$. Repeat this with $W_3, W_4, \ldots$ in turn and consider $Q = [W_1, W_2', \ldots, W_s']$ $(\neq 1$ in $F)$. Now choose a finite homomorphic image of F of exponent p. (This is possible since F is finitely generated residually torsionfree nilpotent and each finitely generated torsionfree nilpotent group has such a homomorphic image: one may even choose p arbitrarily large and coprime to $|A/F|$ (see [7]).) Now pass to a homomorphic image $\overline{A}$ of A in which $\overline{Q}$ has order p. Of course none of the W_r is trivial in $\overline{A}$ since $\overline{Q} \neq \overline{1}$. But this means that $\langle \overline{h}^w\rangle \cap \langle \overline{h}^w\rangle^{\overline{v}_r} = \langle \overline{1}\rangle$. For if $\overline{v}_r^{-1}\overline{h}^{w\gamma}\overline{v}_r = \overline{h}^{w\delta}$ then $\overline{v}_r^{-1}\overline{h}^{w}\overline{v}_r = \overline{h}^{w\eta}$ for some η (since $|\overline{h}^w| = p$), which contradicts $\overline{W}_r \neq \overline{1}$. So suppose that, in $\overline{A}$ (an extension of a finite group of exponent p by a group of (coprime) order $|A/F|$), we have $\overline{u}_r = \overline{h}^{k_r}\overline{v}_r\overline{h}^{l_r} = \overline{h}^{i_r}\overline{v}_r\overline{h}^{j_r}$, i.e., $\overline{v}_r^{-1}\overline{h}^{\rho_r}\overline{v}_r = \overline{h}^{\sigma_r}$, where $\rho_r = k_r - i_r$, $\sigma_r = j_r - l_r$. Assuming $p \nmid \rho_r$, we have $p \nmid \sigma_r$. Consequently $\overline{v}_r^{-1}\overline{h}^{\rho_r w}\overline{v}_r = \overline{h}^{\sigma_r w}$ and $\overline{v}_r^{-1}\overline{h}^{pw}\overline{v}_r = \overline{h}^{pw}$ $(= \overline{1})$, which together imply $\overline{v}_r^{-1}\overline{h}^w\overline{v}_r \in \langle \overline{h}^w\rangle$—a contradiction.

We now return to the proof of the theorem assuming V to have infinite order and setting $a = UV$ and $h = U^{-1}a = V$. Let c, d be elements of G (as in $(*)$) of minimum length in their conjugacy classes and such that $c \not\sim_G d$. As usual, let $\|x\|$ denote the (generalised free product) normal form length of x so that $\|x\| = 0$ iff $x \in \langle h\rangle$ and $\|x\| = 1$ iff $x \in A\backslash\langle h\rangle$ or $x \in B\backslash\langle h\rangle$.

Case 1. $\|c\| = \|d\| = 0$. Thus $c = h^i$, $d = h^j$ with $c \not\sim_G d$. In particular, $i \neq j$. Now, if $|i| \neq |j|$ we can, by Lemma B, find a generalised free product $\overline{G}$ of two finite groups with $|\overline{h}| = |i|\,|j|$ so that $|\overline{h}^i| \neq |\overline{h}^j|$ and hence $\overline{h}^i \not\sim_{\overline{G}} \overline{h}^j$. So we may suppose that $c = h^i$, $d = h^{-i}$ and that $h^i \not\sim_G h^{-i}$. In particular $h^i \not\sim_A h^{-i}$, $h^i \not\sim_B h^{-i}$.

Now A is a finite extension of a finitely generated normal free subgroup F_A, say, chosen as in Lemma C. Likewise choose in B a finitely generated normal free subgroup of finite index, say F_B. Let $\langle h^\lambda\rangle = F_A \cap F_B$. (Note that $F_A \cap \langle h\rangle$, $F_B \cap \langle h\rangle$ are not necessarily equal.) By Lemma B there exist normal subgroups X_1, Y_1 of finite index in A, B and such that $X_1 \cap \langle h\rangle = Y_1 \cap \langle h\rangle = \langle h^\lambda\rangle$. Set $A_1 = F_A \cap X_1$, $B_1 = F_B \cap Y_1$. Then A_1, B_1 are normal of finite index and hence finitely generated and free in A, B respectively.

Now let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_r$; $\beta_1 = 1, \beta_2, \ldots \beta_s$ be coset representatives in A, B, respectively, chosen, as in Lemma C, modulo subgroups $D_A = A_1C_A$ and $D_B = B_1C_B$ where C_A, C_B are the centralisers in A, B of $\langle h^{i\lambda}\rangle$. Use Lemma C to find normal free subgroups X_2, Y_2 of finite index such that, in A/X_2, $\{\overline{\alpha}_k^{-1}\overline{f}^{-1}\overline{h}^{i\lambda}\overline{f}\overline{\alpha}_k : \overline{f} \in \overline{A}_1\} \cap \langle \overline{h}^{i\lambda}\rangle = \varnothing$ and, in B/Y_2, $\{\overline{\beta}_l^{-1}\overline{g}^{-1}\overline{h}^{i\lambda}\overline{g}\overline{\beta}_l : \overline{g} \in \overline{B}_1\} \cap \langle \overline{h}^{i\lambda}\rangle = \varnothing$ where $1 < k \leq r$, $1 < l \leq s$. Clearly (using Lemma B) we can modify X_2, Y_2 to assume, without loss of generality, that $X_2 \cap \langle h\rangle = Y_2 \cap \langle h\rangle$ $(= \langle h^{zi\lambda}\rangle$, say). Now choose, as in the proof of Lemma C, normal subgroups E_A, E_B of finite index in A, B such that $E_A \cap \langle h\rangle = E_B \cap \langle h\rangle = \langle h^{zi\lambda}\rangle$ and such that $\{\tilde{f}^{-1}\tilde{h}^{i\lambda}\tilde{f} : \tilde{f} \in \tilde{A}_1\} \cap \langle \tilde{h}^{i\lambda}\rangle = \{\tilde{h}^{i\lambda}\} \neq \{\tilde{h}^{-i\lambda}\}$ and $\{\tilde{g}^{-1}\tilde{h}^{i\lambda}\tilde{g} : \tilde{g} \in \tilde{B}_1\} \cap \langle \tilde{h}^{i\lambda}\rangle =$

$\{\tilde{h}^{i\lambda}\} \neq \{\tilde{h}^{-i\lambda}\}$, respectively, in A/E_A, B/E_B. Finally set $A_3 = E_A \cap X_2$, $B_3 = E_B \cap Y_2$.

The claim is that, in $\widehat{G} = A/A_3 *_{\langle \hat{h} \rangle} B/B_3$, (a generalised free product of two finite groups), we have $\hat{h}^i \nsim_{\widehat{G}} \hat{h}^{-i}$. For *if* $\hat{h}^i \sim_{\widehat{G}} \hat{h}^{-i}$ then $\hat{h}^{i\lambda} \sim_{\widehat{G}} \hat{h}^{-i\lambda}$, and there exists a sequence of (conjugate) powers of $\hat{h}$ [cf. (**)]. But, by choice of A_3, B_3, the only conjugate of $\hat{h}^{i\lambda}$ in A/A_3 and B/B_3 is $\hat{h}^{i\lambda}$ itself. This precludes the possibility that $\hat{h}^{i\lambda} \sim_{\widehat{G}} \hat{h}^{-i\lambda}$ and hence $\hat{h}^i \nsim_{\widehat{G}} \hat{h}^{-i}$, as required.

Case 2(i). $\|c\| = 0$, $\|d\| = 1$. Thus $d \in B \backslash H$. Let $c = h^i$. Since d is assumed of minimal length in its conjugacy class, we have $\{d\}^B \cap \langle h \rangle = \varnothing$. Suppose w is such that $d^w, h^w \in F_B$. Then $\{d^w\}^B \cap \langle h^w \rangle = \varnothing$ or else $b^{-1} d^w b = h^{tw}$ from which $b^{-1} d b = h^t$ follows. Hence there exists $M \lhd B$ such that $\{\overline{d^w}\}^{\overline{B}} \cap \langle \overline{h^w} \rangle = \varnothing$ in $\overline{B} = B/M$. (Apply the proof of the first part of Lemma C to the equality $\{y^{-1} \beta_l^{-1} f^{-1} d^w f \beta_l y\} \cap \langle h^w \rangle = \varnothing$ and its consequence $\{f^{-1} d^w f\} \cap \langle \beta_l h^w \beta_l^{-1} \rangle = \varnothing$.)

Case 2(ii). $\|c\| = 1$, $\|d\| = 0$. Despite the asymmetry ($\langle h \rangle$ is its own centraliser in A—it may be a proper subgroup of its centraliser in B) this case can be dealt with as subcase 2(i).

Case 2(iii). All other cases where $\|c\| \neq \|d\|$ are dealt with by passing to a generalised free product $\overline{G} = \overline{A} *_{\langle \overline{h} \rangle} \overline{B}$ of finite groups in which $\|\overline{c}\| = \|c\| \neq \|d\| = \|\overline{d}\|$. This is easily achieved by using the Π_c property of the free products A and B [11] to keep images of elements of $A \backslash \langle h \rangle$ and $B \backslash \langle h \rangle$ out of $\langle \overline{h} \rangle$.

Case 3. $\|c\| = \|d\| = 1$. Here, using the fact that A, B are Π_c and conjugacy separable, we can find a homomorphic image $\overline{G} = \overline{A} *_{\langle h \rangle} \overline{B}$ of $\overline{G}$ in which (i) $\overline{A}$, $\overline{B}$ are finite; (ii) $\overline{c}, \overline{d} \notin \langle \overline{h} \rangle$; and (iii) $\overline{c} \nsim \overline{d}$ in $\overline{A}$ nor in $\overline{B}$.

Case 4. Here $c \nsim_G d$ with $\|c\| = \|d\| \geq 2$. Let $c = u_1 u_2 \cdots u_r$, $d = v_1 v_2 \cdots v_r$ where the u_i and v_j alternate from $A \backslash H$ and $B \backslash H$.

Consider the system of equations

$$\text{I(i)} \qquad \begin{aligned} u_{i+1} &= x_0^{-1} v_1 x_1, \\ u_{i+2} &= x_1^{-1} v_2 x_2, \\ &\ \vdots \\ u_{i+r} &= x_{r-1}^{-1} v_r x_0. \end{aligned}$$

A *solution* of I(i) is a set $h_0, h_1, \ldots, h_{r-1}$ of elements of $\langle h \rangle$ that, on substituting h_i for x_i $(0 \leq i \leq r-1)$, satisfy the equations I(i) simultaneously. According to Dyer [2, after Theorem 2], $c \sim_G d$ iff for some value i $(0 \leq i < r)$ the equations I(i) have a solution. Thus, since $c \nsim_G d$, we know that, for no i, does I(i) have a solution $h_0, h_1, \ldots, h_{r-1}$. We show that there is a homomorphic image $\overline{G} = \overline{A} *_{\langle \overline{h} \rangle} \overline{B}$ with $\overline{A}, \overline{B}$ finite in which, for no i has the corresponding system of equations a solution $\overline{k}_0, \overline{k}_1, \ldots, \overline{k}_{r-1}$. The required result will then follow from [2, Theorem 4].

Now if for each i (for which all pairs u_{i+j} and v_j lie in the same factor A or B) there exists some t, possibly depending on i, such that $u_{i+t} \notin H v_t H$, then by Lemma E there is a finite homomorphic image of A (or B) in which $\overline{u}_{i+t} \notin \overline{H}\overline{v}_t\overline{H}$. If X is the intersection of all the corresponding kernels in A and Y is obtained similarly in B, then X and Y are easily modified to produce a generalised free product $\overline{G}$ of finite groups in which $\overline{c} \not\sim_{\overline{G}} \overline{d}$.

Consequently we may assume that for at least one value of i the equations I(i) have no solution in H and yet each *individual* equation $u_{i+j} = x_{j-1}^{-1} v_j x_j$ $(1 \leq j \leq r)$ is soluble in H so that a solution of $\overline{\mathrm{I}}(\overline{\mathrm{i}})$ in some generalised free product $\overline{G} = \overline{A} *_{\langle\overline{h}\rangle} \overline{B}$ of finite groups cannot immediately be ruled out. We claim that, even here, a generalised free product $\overline{G} = \overline{A} *_{\langle\overline{h}\rangle} \overline{B}$ of finite groups, which is a homomorphic image of G, can be found in which $\overline{c} \not\sim_{\overline{G}} \overline{d}$. First suppose for each k $(1 \leq k \leq r)$ that $v_k^{-1} h v_k = h$ (or h^{-1}). Thus each of the equalities $u_{i+j} = h^\alpha v_j h^\beta$ may be rewritten $u_{i+j} = h^{\alpha+\delta} v_j h^{\beta-\delta}$ (or $h^{\alpha+\delta} v_j h^{\beta+\delta}$). Using such adjustments we can certainly find $\alpha_0, \alpha_1, \ldots, \alpha_r \in \mathbb{Z}$ such that

$$(***) \qquad \begin{aligned} u_{i+1} &= h^{-\alpha_0} v_1 h^{\alpha_1}, \\ u_{i+2} &= h^{-\alpha_1} v_2 h^{\alpha_2}, \\ &\vdots \\ u_{i+r} &= h^{-\alpha_{r-1}} v_r h^{\alpha_r}, \end{aligned}$$

where $\alpha_r - \alpha_0 \neq 0$ since $c \not\sim_G d$. We may further modify this solution by replacing $h^{-\alpha_0}$ by $h^{-\alpha_0+\delta} = h^{-\beta_0}$, say, and hence h^{α_1} by $(h^{\beta_1} =)$ $h^{\alpha_1+\delta}$ if $v_1^{-1} h v_1 = h^{-1}$, or by $h^{\alpha_1-\delta}$ if $v_1^{-1} h v_1 = h$. Then replace $h^{-\alpha_1}$ by the appropriate $h^{-\alpha_1 \pm \delta}$.

Continuing in this way we see that: (I) if in $(***)$ the number of v_k for which $v_k^{-1} h v_k = h^{-1}$ is even, then α_r will be changed to $\beta_r = \alpha_r - \delta$ whereas (II) if this number is odd then α_r is replaced by $\beta_r = \alpha_r + \delta$. In case (I) $\beta_r - \beta_0 = \alpha_r - \alpha_0 \neq 0$. In case (II) $\beta_r - \beta_0 = \alpha_r - \alpha_0 - 2\delta$. Since δ can be chosen arbitrarily and since $c \not\sim_G d$, we deduce that, in $(***)$, $\alpha_r - \alpha_0$ must be an odd integer and that β_0, β_r can be chosen so that $\beta_r - \beta_0$ is any odd integer.

In case (I) consider the maximum, J, of the various differences $|\alpha_r - \alpha_0|$ as in $(***)$ as i varies over all integers from 0 to $r-1$. If we take normal subgroups M_A, M_B of finite index in A and B, respectively, such that in addition to satisfying all the other sufficient conditions imposed, we also have $M_A \cap A = M_B \cap B = \langle h^\varepsilon \rangle$ where $\varepsilon > J$, then for no i can the equations $\overline{\mathrm{I}}(\overline{\mathrm{i}})$ be solved in $A/M_A *_{\langle\overline{h}\rangle} B/M_B$.

A similar result holds for case (II) if we assume that ε is even, which is possible by [4, Theorem 1]. Thus $\overline{h}^{\beta_r - \beta_0} \neq \overline{1}$ since $\beta_r - \beta_0$ is odd and $\overline{h}$ has even order.

Thus we may assume for at least one—and hence for all—i that the equations $u_{i+j} = x_{i,j-1}^{-1} v_j y_{i,j}$ $(1 \leq j \leq r)$ have solutions $x_{i,j-1} = h_{i,j-1}$, $y_{i,j} = h'_{i,j}$ yet no solution as in I(i) and that for each i at least one of the equations, say $u_{i+k(i)} = x_{i,k(i)-1}^{-1} v_{k(i)} y_{i,k(i)}$, is soluble with *unique* x, y. For each i select one such equation, the $k(i)$th say, as above. Fixing i, consider in turn the $(k(i)+1)$st, $(k(i)+2)$nd, etc. equations of the system arranging if possible that $h'_{i,k(i)} = h_{i,k(i)}$ etc. (i.e., one tries to match each solution for a y with the solution for the next x^{-1} as far as possible). Since $c \not\sim_G d$ this matching must eventually fail, at the equation $u_{i+l(i)} = x^{-1}_{i,l(i)-1} v_{l(i)} y_{i,l(i)}$, say. (This will also be an equation with unique solution and might be the equation we started with if we can arrange for $y_{i,r} = x_{i,0}$.) We now choose p, as in Lemma H so that p is larger than all $|\alpha_i - \beta_i|$ (as i runs over all integers from 0 to $r-1$) and where h^{α_i} is the unique solution for $x_{i,l(i)-1}$ in the above equation and h^{β_i} is the (forced) value taken by $y_{i,l(i)-1}$ in the preceding equation. This choice of p leads, as in Lemma H, to a generalised free product $\overline{G}$ of finite groups in which $\bar{c} \not\sim_G \bar{d}$, as required.

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