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THE FREE PRODUCT OF RESIDUALLY FINITE GROUPS AMALGAMATED ALONG RETRACTS IS RESIDUALLY FINITE

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ABSTRACT. It is shown that residual finiteness is preserved by the generalized free product provided that the amalgamated subgroups are retracts of their respective factors. This result is applied to knot groups. The outcome is that the question of residual finiteness for knot groups need only be answered for prime knots.

1. **Introduction.** A group G is said to be residually finite if the intersection of the collection of all subgroups of finite index in G is the trivial group. Under certain conditions this property may be preserved by the generalized free product ([**1**], [**2**], [**3**], [**5**]). In particular, we show that the generalized free product of residually finite groups is residually finite if the amalgamated subgroups are retracts of the factors.

L. Neuwirth [**8**] has asked whether knot groups are residually finite. This question has remained unanswered except in special cases ([**6**], [**9**]). As a consequence of the theorem mentioned above, it is shown that this question need only be considered for prime knots.

2. **Proof of the theorem.** We shall find the following lemma useful. A proof is given in Chapter 3 of [**7**].

LEMMA 1. *Each split extension of a finitely generated residually finite group by a residually finite group is residually finite.*

THEOREM 1. *The free product of residually finite groups amalgamated along retracts is a residually finite group.*

PROOF. Let $\{G_i | i \in I\}$ be a collection of residually finite groups, and for each i in I, let K_i be a retract of G_i. Further suppose that 0 is a fixed element of I and for each $i \neq 0$, φ_i is an isomorphism of K_i onto K_0. Let G denote the generalized free product of the groups $\{G_i\}$ with the subgroups $\{K_i\}$ amalgamated according to the isomorphisms $\{\varphi_i\}$. Observe first that G is isomorphic to a group $\bar{G}$ obtained as follows. Since K_i is a retract of G_i for each i, there is a normal subgroup H_i of G_i with G_i a split extension

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of H_i by K_i. Thus G_i is isomorphic to an extension of H_i by K_i which is split according to a homomorphism ψ_i from K_i into the automorphism group of H_i. Let H be the free product of the subgroups H_i. There is a homomorphism ψ of K_0 into the automorphism group of H defined as follows. If $k \in K_0$ and $h_1 h_2 \cdots h_r \in H$ with $h_j \in H_{i_j}$ $(j=1, \cdots, r)$, then

$$(h_1 h_2 \cdots h_r)(k\psi) = [h_1(k\varphi_{i_1}^{-1}\psi_{i_1})][h_2(k\varphi_{i_2}^{-1}\psi_{i_2})] \cdots [h_r(k\varphi_{i_r}^{-1}\psi_{i_r})].$$

Let $\bar{G}$ be the extension of H by K_0 that is split via the homomorphism ψ. That G is isomorphic with $\bar{G}$ follows from the observation that the two groups can be defined by identical presentations.

Since K_0 is residually finite, we need only consider elements of $\bar{G}$ that lie in the subgroup H in order to prove the residual finiteness of $\bar{G}$. Let h be a nontrivial element of H. Then using the free product structure of H, we can write h as $h=h_1h_2 \cdots h_r$, where $h_j \neq 1$, $h_j \in H_{i_j}$ and $i_j \neq i_{j+1}$ $(j=1, \cdots, r)$. We now select a normal subgroup N_i of G_i for each i in I as follows. If i does not occur in the set of indices $\{i_1, \cdots, i_r\}$, then we put $N_i = G_i$. Otherwise, we choose N_i to be a normal subgroup of finite index in G_i such that none of the elements $h_1, \cdots, h_r$ belong to N_i. Put $\bar{N}_i = N_i \cap H_i$. For each i, $\bar{N}_i$ is a normal subgroup of G_i, is of finite index in H_i, and contains none of the elements $h_1, \cdots, h_r$.

Let X be the normal subgroup of H generated by the union of the subgroups $\bar{N}_i$. By definition, X is a normal subgroup of H. Furthermore if $k \in K_0$, then $kXk^{-1} \subset X$. This follows from the fact that each subgroup $\bar{N}_i$ of $\bar{G}$ is invariant under conjugation by elements of K_0. (Recall that $\bar{N}_i$ is normal in G_i and that conjugation of $\bar{N}_i$ by K_0 is identical with conjugation of $\bar{N}_i$ by K_i.) Thus X is a normal subgroup of $\bar{G}$, and the factor group $\bar{G}/X$ is a split extension of the group H/X by K_0.

Observe that H/X is isomorphic with the free product of the finite groups $H_i/\bar{N}_i$. Furthermore, the image of the reduced word $h=h_1h_2 \cdots h_r$ under the projection of H onto H/X corresponds to the reduced word

$$(h_1\bar{N}_{i_1})(h_2\bar{N}_{i_2}) \cdots (h_r\bar{N}_{i_r})$$

in the free product of the groups $H_i/\bar{N}_i$. In particular, hX is not the identity element of $\bar{G}/X$.

Finally we observe that since only finitely many of the groups $\bar{N}_i$ are not equal to the groups H_i, H/X is isomorphic to the free product of finitely many finite groups. Hence, by a theorem of K. W. Gruenberg **[4]**, H/X is a finitely generated residually finite group. By Lemma 1, $\bar{G}/X$ is residually finite. Thus G is residually a residually finite group. Hence G is itself residually finite. This completes the proof of Theorem 1.

3. **Application to knot groups.** By a *knot space*, we shall mean the complement of the interior of a regular neighborhood of a simple closed

curve tamely embedded in the three-sphere. To avoid difficulties, we assume that each knot space is given with a fixed embedding into the three-sphere. If K_1 and K_2 are knot spaces, we use the notation K_1+K_2 to denote the composition of K_1 and K_2 as defined by L. Neuwirth in Chapter VIII of [**8**]. We say that a knot space K is *prime* if whenever $K=K_1+K_2$ is written as a composition of knot spaces, then either K_1 or K_2 is the trivial knot space. As noted by Neuwirth, $\pi_1(K_1+K_2)$ is the generalized free product of $\pi_1(K_1)$ and $\pi_1(K_2)$ amalgamated along infinite cyclic subgroups A_1 and A_2 generated by meridional curves in K_1 and K_2. A standard topological construction shows that A_1 is a retract of $\pi_1(K_1)$ and A_2 is a retract of $\pi_1(K_2)$. This together with Theorem 1 proves the following.

THEOREM 2. *The fundamental group of a knot space is a residually finite group if and only if the fundamental group of each of the prime components of the knot space is residually finite.*

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