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NOTES ON THE BAUMSLAG-SOLITAR NONRESIDUALLY FINITE EXAMPLES

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ABSTRACT. We examine the abelianization of G. Baumslag and Solitar's example of a one-generator group that is not residually finite. In particular, the nonfinitely-generated commutator subgroup is shown to be not residually finite. We also review a specific example of a cyclic extension of a residually finite group that is not residually finite.

Theorem 1. *If the sequence* $1 \longrightarrow N \longrightarrow E \longrightarrow \mathbb{Z} \longrightarrow 1$ *is exact, where N is finitely generated and residually finite, then E is residually finite.*

A proof of this result may be found in Hempel [Hem, Corollary 15.21, p. 180]. More general forms of this result, replacing $\mathbb{Z}$ with any residually finite group and requiring that the sequence split, were proved by Mal'cev [Mal] and Miller [Mil, Theorem III.7, p. 29].

I originally conjectured that this result is still true even if we drop the condition that N is finitely generated. The first example in this paper is part of a proposed counterexample to this conjecture, and the second example is a simple counterexample which was pointed out to me by Geoff Mess. This second example also follows from work done by Gruenberg [Gruen].

Consider the example given by G. Baumslag and Solitar [BS] of a one relator group that is not residually finite: $\langle a, b \mid a^{-1}b^2a = b^3\rangle$. Abelianizing this group maps $b \mapsto 1$ and yields $\mathbb{Z}$, the free group with a single generator a. We will refer to the kernel of this abelianization as N. If we define $b_i \equiv a^i b a^{-i}$ then N has the following explicit (neither finitely presented nor finitely generated) presentation:

$$N = \langle \dots b_{-1}, b_0, b_1, b_2, \dots \mid b_i^2 = b_{i+1}^3\rangle .$$

This group fits into the exact sequence

$$1 \longrightarrow N \longrightarrow BS\ Group \longrightarrow \mathbb{Z} \longrightarrow 1 .$$

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If N is residually finite, this will be an example of a semidirect product of a residually finite group by $\mathbb{Z}$ which is not itself residually finite. We prove that this is not the case:

Theorem 2. *N is not residually finite.*

In fact, we show that the only finite quotients of N are cyclic. Thus, if $g \in [N, N]$, then for any finite representation, $N \xrightarrow{\alpha} \Gamma$, we get $\alpha(g) = 1$. We will need a technical lemma, whose proof we defer until after the proof of the theorem.

Lemma. *Let $N \xrightarrow{\alpha} \Gamma$, where Γ is a finite group, and let $\gamma_i \equiv \alpha(b_i)$. Then for all i, we get that 3 does not divide the order of γ_i and 2 does not divide the order of γ_i.*

Proof of Theorem 2. Assume that there is a map $N \xrightarrow{\alpha} \Gamma$ which maps N to a finite group. From Lemma 1 we learn that for all i the order of γ_i is some m where 3 does not divide m and 2 does not divide m. As $3 \nmid m$ we see that γ_i^3 is a generator of $\langle \gamma_i \rangle \cong \mathbb{Z}_m$. Similarly, γ_{i+1}^2 is a generator of $\langle \gamma_{i+1} \rangle \cong \mathbb{Z}_m$. Thus, $\gamma_i^3 = \gamma_{i+1}^2$ and we see that $\langle \gamma_i \rangle = \langle \gamma_{i+1} \rangle$. By continuing this process, we see that $N = \langle \gamma_0 \rangle \cong \mathbb{Z}_m$, the cyclic group of order m. As this group is abelian, the kernel of the projection of N onto this quotient includes the commutator subgroup of N. Thus, all finite index normal subgroups of N include the commutator subgroup, and, in particular, their intersection is not empty. Hence N is not residually finite.

We now prove the lemma:

Proof of lemma. Recall that $\gamma_i \equiv \alpha(b_i)$. Let $o(\Gamma) = 3^K 2^L M$, where $3 \nmid M$ and $2 \nmid M$. As $o(\gamma_i) \mid o(\Gamma)$, then if $o(\gamma_i) = 3^{k_i} 2^{l_i} m_i$ where $3 \nmid m_i$ and $2 \nmid m_i$, we find that $k_i \leq K$ and $l_i \leq L$.
Claim. $\forall i \; 3 \nmid o(\gamma_i)$.
Assume the opposite, so for some i, $3^{k_i} \mid o(\gamma_i)$ where $k_i > 0$. We now show that for any $j \leq i$, we have $k_{j-1} = k_j + 1$ and hence, by induction, $k_j = k_i + (i - j)$.
Case **I.** $2 \mid o(\gamma_j)$.
$\Rightarrow o(\gamma_{j-1}) = 2 \cdot o(\gamma_j)/2$.
This holds as $1 = \gamma_j^{3^{k_j} 2^{l_j} m_j} = (\gamma_j^2)^{3^{k_j} 2^{l_j - 1} m_j} = (\gamma_{j-1}^3)^{3^{k_j+1} 2^{l_j - 1} m_j} = \gamma_{j-1}^{3^{k_j+1} 2^{l_j - 1} m_j}$, thus $o(\gamma_{j-1}) \mid 3^{k_j+1} 2^{l_j - 1} m_j$. However, if $o(\gamma_{j-1}) = 3^{k_j+1} 2^{l_j-1} m_j/\mu$, then we have $1 = \gamma_{j-1}^{3^{k_j+1} 2^{l_j-1} m_j/\mu} = \gamma_j^{3^{k_j} 2^{l_j} m_j/\mu}$, which is a contradiction.
Case **II.** $2 \nmid o(\gamma_j)$, so $o(\gamma_j^2) = o(\gamma_j)$. However, since $\gamma_{j-1}^3 = \gamma_j^2$, we have $o(\gamma_{j-1}^3) = o(\gamma_j^2) = o(\gamma_j) = 3^{k_j} m_j \;\Rightarrow o(\gamma_{j-1}) = 3 \cdot o(\gamma_{j-1}) = 3^{k_j+1} m_j$. Again, this holds for all $k_j > 0$.

But $k_j \leq K$ is a finite bound for k_j. This contradiction yields that $\forall i \; 3 \nmid o(\gamma_i)$, proving the above claim. Similarly, one may show that $\forall i \; 2 \nmid o(\gamma_i)$.

Now, if both $2 \nmid o(\gamma_1)$ and $3 \nmid o(\gamma_i)$ we have $o(\gamma_{i-1}) = o(\gamma_{i-1}^3) = o(\gamma_i^2) = o(\gamma_i) = m_i = o(\gamma_i) = o(\gamma_i^3) = o(\gamma_{i+1}^2) = o(\gamma_{i+1})$. $\forall j\ o(\gamma_j) = m_i$ so we may drop the subscript i to get $\forall j\ o(\gamma_j) = m$, where $3 \nmid m$, $2 \nmid m$.

More generally, Baumslag and Solitar [BS] produced an entire class of one-relator groups which are not residually finite. The nonzero integers p and q are said to be *meshed* if either p or q divides the other or if p and q have precisely the same set of prime divisors.

Theorem 3 (Baumslag–Solitar). *Let p and q be nonzero integers. Then*

$$G_{p,q} \equiv \langle a, b \mid a^{-1} b^p a = b^q \rangle$$

is Hopfian if and only if p and q are meshed.

Note that a result of Mal'cev [MKS, p. 415] is that any finitely generated residually finite group is Hopfian. Thus, for p and q not meshed, $G_{p,q}$ is not Hopfian and, as it is finitely generated, $G_{p,q}$ is not residually finite. We further note that if we denote the commutator of $G_{p,q}$ by $N_{p,q}$ these groups fall into the exact sequence

$$1 \longrightarrow N_{p,q} \longrightarrow G_{p,q} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_{|p-q|} \longrightarrow 1.$$

$N_{p,q}$ has the presentation

$$N = \langle \dots b_{-1}, b_0, b_1, b_2, \dots \mid b_i^q = b_{i+1}^p \rangle$$

where $b_i \equiv a^i b^{p-q} a^{-i}$. The following theorem may be proven by a simple rewrite of the proof of Theorem 2:

Theorem 4. *If p and q are mutually prime, then $N_{p,q}$ is not residually finite.*

We can now handle the more general case of p and q not meshed by reducing it to the case of p and q mutually prime, as shown previously. The method used to do this was suggested by G. Baumslag.

Theorem 5. *If p and q are not meshed then the group $N_{p,q}$ is not residually finite.*

Proof. If p and q are mutually prime, then we have the case dealt with in Theorem 4, so we will assume that $gcd(p, q) = r \neq 1$. Define P as p/r and Q as q/r. Consider the subgroup of $N_{p,q}$ generated by $\{ b_i^r \}$, which is isomorphic to $N_{P,Q}$. As $\gcd(P, Q) = 1$ we have reduced the problem to that dealt with in Theorem 4 above, so $N_{P,Q}$ is not residually finite. Any subgroup of a residually finite group is itself residually finite, so $N_{p,q}$ is not residually finite.

The following example of a non-residually finite cyclic extension of a residually finite group was suggested by Geoff Mess: Consider the wreath product of the alternating group A_5 by $\mathbb{Z}$. This may be considered as an extension:

$$1 \longrightarrow \oplus_{i \in \mathbb{Z}} A_5 \longrightarrow A_5 \wr \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 1.$$

We note that while $\oplus_{i\in\mathbb{Z}} A_5$ is obviously residually finite, we can prove that $A_5 \wr \mathbb{Z}$ is not.

Theorem 6. *$A_5 \wr \mathbb{Z}$ is not residually finite.*

Proof. We use the notation $(A_5)_{i+1} = t(A_5)_i t^{-1}$ to describe the action of $\mathbb{Z}$ on our wreath product. Assume that there is some homomorphism $\alpha : A_5 \wr \mathbb{Z} \to \Gamma$, where Γ is finite and nontrivial. If we look at the image of the generator of $\mathbb{Z}$ we see that there is some least integer n, such that $\alpha(t^n) = 1$. As A_5 is simple, $\alpha((A_5)_i)$ must be either trivial or isomorphic to A_5. Now find an integer i such that the image of A_5 is nontrivial. (If none exists then the image of α is cyclic, hence abelian, and we are done.) We get $\alpha((A_5)_{i+n}) = \alpha((A_5)_i)$. We note that from our construction of the wreath product that $(A_5)_{i+n}$ must be in the centralizer of $(A_5)_i$, but the center of A_5 is trivial. Thus $\alpha((A_5)_{i+n})$ must be trivial, contradicting our assumption that we could find i such that $\alpha((A_5)_i)$ is not trivial. Thus Γ must be cyclic, and the wreath product is not residually finite (in fact, having only cyclic quotients).

This result is also a consequence of work by Gruenberg [Gruen, Theorem 3.1].

Theorem 7. *Let $\mathscr{P}$ be any property satisfying the condition that whenever a group has $\mathscr{P}$, then all its subgroups also have $\mathscr{P}$. If $W = G \wr \Gamma$ is residually $\mathscr{P}$ where Γ is transitive, then either Γ is $\mathscr{P}$ or G is abelian.*

References

[BS] G. Baumslag and D. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **68** (1962), 199–201.

[Gruen] K. W. Gruenberg, *Residual properties of infinite groups*, Proc. London Math. Soc. (3) **7** (1957), 29–62.

[Hem] John Hempel, *3-Manifolds*, Princeton, 1976.

[MKS] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory*, Dover, New York, 1976.

[Mal] Anatoly I. Mal'cev, *O gomomorfizmax na konechnie gruppi*, Uchen. Zap. Ivanovsk. Ped. Inst. **18** (1958), 49–60; (English Trans. *On homomorphisms onto finite groups*, by J. C. Lennox).

[Mil] Charles F. Miller, *Group theoretic decision problems*, Princeton, 1978.

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