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Residual Properties of Free Products of Infinitely Many Nilpotent Groups Amalgamating Cycles

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We prove that a free product of infinitely many finitely generated, torsion free, nilpotent groups amalgamating isolated cycles is residually finite-p for all primes p so long as the number of generators and nilpotency class of the factors are both bounded. Examples illustrate that if we remove any single hypothesis, the resulting group need not even be residually finite.

1. INTRODUCTION

Let **X** be a class of groups, e.g., the finite groups **F**, the finite p-groups (p prime) $\mathbf{F}_p$, and the finitely generated torsion free nilpotent groups **N**. A group G is *residually* **X** iff for each $g \neq 1$ in G there exists an image $G^* \in \mathbf{X}$ of G such that $g^* \neq 1$. Denote this new class **RX**. A subgroup, H, of a group, G, is *isolated* in G iff $g \in H$ whenever $g \in G$ and $g^i \in H$ for some i.

Let $\{A_i\}_{i \in I}$ be a (possibly infinite) collection from **N**, $H = \langle h \rangle$ an infinite cyclic group that embeds via an injection φ_i into each A_i with image $H_i = \langle h_i \rangle$. For the entire discussion, let us fix F to be the (restricted) *free product of the* A_i *amalgamating* H:

$$F = \prod\nolimits_H^* A_i = \langle A_i, i \in I : h\varphi_j = h\varphi_k, h \in H, j, k \in I \rangle.$$

If additionally the H_i are central in each A_i, we may also define the (restricted) *direct product of the* A_i *amalgamating* H:

$$\prod\nolimits_H^\times A_i = \langle A_i, i \in I : h\varphi_j = h\varphi_k, [A_j, A_k] = 1, h \in H, j, k \in I \rangle.$$

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The central thread of this paper is the question: when is $F = \prod_H^* A_i$ residually finite-p for all primes p?

In 1964, Higman published [6], an investigation of when an amalgam of two finite p-groups embeds into a finite p-group. Using this investigation, one can prove that a free product of finitely many finite p-groups amalgamating a cycle is $\mathbf{RF}_p$. Kim and McCarron established more generally in Theorem 4.3 of [7] that a free product of finitely many $\mathbf{RF}_p$ groups amalgamating a finite cycle is again $\mathbf{RF}_p$. Since $\mathbf{N}$ groups are $\mathbf{RF}_p$ for any prime p [5], Kim and McCarron continue by considering when a free product of $\mathbf{N}$ groups amalgamating an isolated infinite cycle is $\mathbf{RF}_p$. Theorem 4.4 of [7] shows it is if there are finitely many groups, while Theorem 3.8 of [7] proves it for infinitely many *isomorphic* groups such that *the isomorphisms of the factors extend the amalgamation isomorphisms* (though the amalgamated subgroup need not be cyclic in Theorem 3.8). A much stronger, common generalization of these two results is true. We obtain the following interesting and complete result.

2. THE MAIN THEOREM

THEOREM 1. *Suppose* (i) $\forall i \in I$, H_i *is isolated cyclic in* A_i; (ii) *the* A_i *are of bounded class*; *and* (iii) *the* A_i *are of bounded rank. Then* $F \in \mathbf{RF}_p$ *for all primes* p.

Furthermore, if any of these three conditions is omitted, F *need not be* $\mathbf{RF}$.

We shall need the result that *there are finitely many finite* p*-groups of exponent less than* q, *class less than* c, *and generated by less than* n *generators*. This follows from:

LEMMA 2. *Let* A *be an* n *generator nilpotent group of class* c *and exponent* q. *Then* $o(A) \le \prod_{i=1}^c n^{iq}$, *where* $o(A)$ *is the order of* A.

Proof. Each lower central factor Q_i, say, of A is a finitely generated abelian group of exponent q, generated by at most n^i basic commutators of weight i. Thus $o(Q_i) \le n^{iq}$ and $o(A) = \prod_{i=1}^c o(Q_i) \le \prod_{i=1}^c n^{iq}$. ∎

Proof of Theorem 1. Let c be the class bound, p a prime, and $g \in F$, $g \ne 1$. Let us assume g has length $d > 0$ (the case $d = 0$ is similar). So $g = \prod_{j=1}^d a_j$ where $a_j \in A_{i_j} \setminus H_{i_j}$ and $i_j \ne i_{j+1}$ for all j. Note that only finitely many of the A_i are called upon here. Since H_{i_j} is isolated in each A_{i_j}, there exist finite p-groups A_{i_j}/M_{i_j} such that $a_j M_{i_j} \notin \langle h_{i_j} M_{i_j} \rangle$ [2, Theorem 2.5]. Unfortunately the $h_{i_j} M_{i_j}$ may not be of the same order so we cannot amalgamate them. We amend the M_{i_j} to ameliorate this difficulty.

Let $p^n = \max o(h_{i_j} M_{i_j})$, $1 \le j \le d$. Now [4] tells us that there exists an integer $f(p, c)$ such that an element which is a product of $p^{n+f(p,c)}$ powers is itself a p^n power. So in $A_i/A_i^{p^{n+f(p,c)}}$, where $i \in I$, the order of the image of h_i is a multiple of p^n. By successively factoring out p-cycles, it is then easy to find $N_i \supseteq A_i^{p^{n+f(p,c)}}$ such that $o(h_i N_i) = p^n$ in A_i/N_i. For each i_j $(1 \le j \le d)$ we take $A_{i_j}^* = A_{i_j}/M_{i_j} \cap N_{i_j}$, and for all other $i \in I$ we take $A_i^* = A_i/N_i$. Then for all $i \in I$, H_i^* is cyclic or order p^n.

Now the key is that, by Lemma 2, there are only finitely many isomorphism classes of A_i^*, and only finitely many ways in which H^* may embed into A_i^* (here $i \in I \setminus \{i_j\colon 1 \le \mathrm{j} \le d\}$). Say we select as representatives $A_{k_m}^*$ (along with appropriate embeddings of H^*) where $k_m \in I \setminus \{i_j\colon 1 \le j \le d\}$ and $1 \le m \le e$. Then F has image F^*, the free product of the $A_{i_j}^*$ and $A_{k_m}^*$ amalgamating H^*. Further, g^* has the same syllable length as g, so $g^* \neq 1$. We have thus moved from the difficult situation of having to contend with infinitely many groups to finitely many by making the critical step of passing to finite p-groups first. This is enough to complete our proof since, by [5], a free product of finitely many finite p-groups with cyclic amalgamation is residually finite-p. ∎

Note that the A_i themselves may not fall into finitely many isomorphism classes, as there are, for example, infinitely many 3 generator, torsion-free nilpotent groups of class 2. Indeed, let B be the free nilpotent group of class 2 on $\{a, b\}$. For each integer $i > 1$, let A_i be the direct product of B with an infinite cycle, $\langle c \rangle$, amalgamating $[a, b] = c^i$. No two of the A_i are isomorphic because no two of the $A_i/A_i' = C_\infty \times C_\infty \times C_i$ are isomorphic.

3. THE COUNTEREXAMPLES

We now show three examples of amalgamated free products which are not even residually finite. In each case, exactly one of the hypotheses, (i), (ii), or (iii), in Theorem 1 fails.

(i) For each positive integer, i, let A_i be the infinite cycle $\langle a_i \rangle$ and $H_i = \langle a_i^i \rangle$. Let F^* be any finite image of F, and let $n = o(F^*)$. Then $a_1^* = a_n^{*\,n} = 1$, so F is not residually finite.

(ii) For each positive integer, i, let A_i be the free nilpotent group of class i on $\{a_i, b_i\}$ and $H_i = \langle [a_i, b_i, \ldots, b_i] \rangle$ where the commutator has length i. Let F^* be any finite image of F, and let $n = o(F^*)$. Since A_{n+1}^* has order at most n, it has class at most n. Thus $a_1^* = [a_{n+1}, b_{n+1}, \ldots, b_{n+1}]^* = 1$ since the commutator has length $n+1$, so F is not residually finite.

(iii) For each positive integer, i, let A_i be the direct product of i copies of B, the free nilpotent group of class 2 and rank 2, amalgamating the centre $Z = \langle z \rangle$ ($z = [a_{ij}, b_{ij}]$ where a_{ij} and b_{ij} are the free generators in the jth copy of B in A_i). Let F^* be any finite image of F, and let $n = o(F^*)$. Now $a^*_{n+1,j} = a^*_{n+1,j'}$ for some $j \neq j'$ since F^* has only n members. Hence $z^* = [a_{n+1,j}, b_{n+1,j}]^* = [a_{n+1,j'}, b_{n+1,j}]^* = 1$ since elements in different factors of an amalgamated direct product commute. So F is not residually finite.

4. EXTENSIONS

The above examples delineate Theorem 1, but it can still be extended slightly to the following:

THEOREM 3. *Suppose H_i is isolated cyclic in A_i for all $i \in I$. If $I = I' \cup I''$ for some I' and I'' such that* (i) *for all $i' \in I'$, $H_{i'}$ is a retract of $A_{i'}$ and* (ii) *the $A_{i''}$, where $i'' \in I''$, are bounded in rank and class, then $F \in \mathbf{RF}_p$ for all primes p.*

Proof. The proof is the same as the Proof of Theorem 1 except that we take $A^*_{i'} = H_{i'}$ for all $i' \in I' \setminus \{i_j\colon 1 \le j \le d\}$. ∎

It does not seem unreasonable to conjecture that the converse is true as well, hence giving a characterization of those F which are $\mathbf{RF}_p$ for all primes p.

Since free groups are residually $\mathbf{N}$ [8], one might expect, for the same reasons as those stated in [1], an analogue of Theorem 1, such as—a free product of *free groups* bounded in rank and amalgamating an isolated cycle is $\mathbf{RF}_p$ for all primes p. The following is a counterexample.

For each positive integer, i, let B_i be the free group on $\{a_i, b_i\}$, $K_i = \langle a_i^i b_i^i \rangle$, and $P = \prod^*_{K_i} B_i$. Let P^* be any finite image of P, and let $n = o(P^*)$. Then $a_1^* b_1^* = a_n^{*n} b_n^{*n} = 1$, so P cannot be residually finite. Incidentally, this underlines the importance of the implied isomorphisms in Theorem 3.9 of [7] respecting the amalgamation. Here the isomorphism between $\langle a_i^i b_i^i \rangle$ and $\langle a_{i+1}^{i+1} b_{i+1}^{i+1} \rangle$ is *not* extendable to an isomorphism between B_i and B_{i+1}.

Next, we present an improvement to Lemma 4.5 of [7] that eliminates the constraint in Theorem 4.6 of [7] that p be a prime greater than the nilpotence class of the groups in the theorem. We again take the opportunity to generalize to the case of infinitely many A_i.

LEMMA 4. *Let $A \in \mathbf{N}$, $\langle w \rangle$ isolated in A, $g \in A$ such that $g \notin \langle w^{p^n} \rangle$ where p is any prime and $n \ge 0$. Then A has a homomorphic image $A^* \in \mathbf{F}_p$ with $g^* \notin \langle w^{p^n} \rangle^*$.*

Proof. We may use the same argument as that in Lemma 4.5 of [7], except that we replace the last paragraph of that lemma with the following:

(2) If $w \in Z(A)$ then, by Subcase 1, (of Lemma 4.5), $g \notin Z(A)^{p^t}\langle w^{p^n}\rangle(*)$, for some t. Now by [4], there is an integer, $f(p,c)$ say (depending only on p and c, the class of A), such that an element which is a product of $p^{t'}$ powers is itself a p^t power where $t' = t + f(p,c)$. Thus $A^{p^{t'}} \cap Z(A) \subseteq Z(A)^{p^t}$ since $Z(A)$ is isolated in A. If $g \in A^{p^{t'}}\langle w^{p^n}\rangle$, then for some k, $gw^{p^n k} \in A^{p^{t'}} \cap Z(A) \subseteq Z(A)^{p^t}$, contradicting $(*)$. Therefore $g \notin A^{p^{t'}}\langle w^{p^n}\rangle$, i.e., $g^* \notin \langle w^{p^n}\rangle^*$ in $A^* = A/A^{p^{t'}}$. ∎

Theorem 4.6 of [7] can now be improved to:

THEOREM 5. *Fix any prime p. Suppose* (i) $\forall i \in I$, $h_i = w_i^{p^{n_i}}$ *where $\langle w_i\rangle$ is isolated cyclic in A_i and the n_i are bounded*, (ii) *the A_i are of bounded class, and* (iii) *the A_i are of bounded rank. Then $F \in$* $\mathbf{RF}_p$.

Proof. The proof is a variation of the Proof of Theorem 1. Let b be the n_i bound, c the class bound, and $g \in F$, $g \neq 1$. As before, write $g = \prod_{j=1}^{d} a_j$ where $a_j \in A_{i_j} \setminus H_{i_j}$ and $i_j \neq i_{j+1}$ for all j. By Lemma 4 there exist finite p-groups A_{i_j}/M_{i_j} such that $a_j M_{i_j} \notin \langle h_{i_j} M_{i_j}\rangle$.

Let $p^n = \max o(h_{i_j} M_{i_j})$, $1 \le j \le d$. Now [4] tells us there exists an integer $f(p,c)$ such that an element which is a product of $p^{n+n_i+f(p,c)}$ powers is itself a p^{n+n_i} power. So in $A_i/A_i^{p^{n+n_i+f(p,c)}}$, where $i \in I$, the order of the image of w_i is a multiple of p^{n+n_i}. It is then easy to find $N_i \supseteq A_i^{p^{n+n_i+f(p,c)}}$ such that $o(w_i N_i) = p^{n+n_i}$ in A_i/N_i. For each i_j ($1 \le j \le d$) we take $A_{i_j}^* = A_{i_j}/M_{i_j} \cap N_{i_j}$, and for all other $i \in I$ we take $A_i^* = A_i/N_i$. Then for all $i \in I$, H_i^* is cyclic of order p^n.

The exponents of the A_i^* are bounded by the greater of $p^{n+b+f(p,c)}$ and the exponents of the $A_{i_j}^*$, $1 \le j \le d$, so we can finish as before in Theorem 1. ∎

To see that the n_i bound is necessary, take $A_i = \langle w_i\rangle$ and $h_i = w_i^{2^i}$. It is easy to prove that F is not $\mathbf{RF_p}$ for any prime p.

Lastly, again using the same method as that in Theorem 1, we generalize to the case of infinitely many A_i a result of Baumslag [3, Theorem 5].

THEOREM 6. *Suppose* (i) $\forall i \in I$, *H_i is isolated* (*but not necessarily cyclic*) *in A_i*; (ii) *the A_i are of bounded class*; *and* (iii) *the A_i are of bounded rank. Then $F \in$* $\mathbf{RF}$.

Proof. Let c be the class bound, p a fixed prime greater than c, and $g \in F$, $g \neq 1$. Write $g = \prod_{j=1}^{d} a_j$ where $a_j \in A_{i_j} \setminus H_{i_j}$ and $i_j \neq i_{j+1}$ for all j. By (i) there exists n such that for all j, $a_j^* \notin H_{i_j}^*$ in $A_{i_j}^* = A_{i_j}/A_{i_j}^{p^n}$ [2, Theorem 2.5]. Indeed, let $A_i^* = A_i/A_i^{p^n}$ for all $i \in I$. Now by [2, Corollary 2.31] $A_i^{p^n} \cap H_i = H_i^{p^n}$ for all i. Therefore the H_i^* are all isomorphic.

There are only finitely many isomorphism classes of A_i^*, and only finitely many ways in which H^* may embed into A_i^* (here $i \in I \setminus \{i_j: 1 \le j \le d\}$). Say we select as representatives $A_{k_m}^*$ (along with appropriate embeddings of H^*) where $k_m \in I \setminus \{i_j: 1 \le j \le d\}$ and $1 \le m \le e$. Then F has image F^*, the free product of the $A_{i_j}^*$ and $A_{k_m}^*$ amalgamating H^* as a homomorphic image. Further, g^* has the same syllable length as g, so $g^* \ne 1$. This is enough to complete our proof since a free product of finitely many finite groups is residually finite [3, Theorem 2]. ∎

The above conclusion cannot be strengthened to $\mathbf{RF_p}$ for even a single prime p as the following example demonstrates. See also Theorem 3.8 of [7] in relation to this example. Let F be the free product of the free nilpotent group of class 2 on $\{a, b\}$ with another free nilpotent group of class 2 on $\{c, d\}$ amalgamating $a = [c, d]$ and $[a, b] = c$. Then $a = [c, d] = [a, b, d] = [c, d, b, d] = \dots$, so if F^* is any nilpotent image of F, we get $a^* = 1$. Thus F is not $\mathbf{RF_p}$ for any prime p.

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