ASCENDING HNN EXTENSIONS OF FINITELY GENERATED FREE GROUPS ARE HOPFIAN

ROSS GEOGHEGAN, MICHAEL L. MIHALIK, MARK SAPIR
AND DANIEL T. WISE

Abstract

It is shown that every ascending HNN extension of a finitely generated free group is Hopfian. An important ingredient in the proof is that under certain hypotheses on the group H, if G is an ascending HNN extension of H, then $\mathrm{cd}(G) = \mathrm{cd}(H) + 1$.

1. *Statement of results*

A group G is *Hopfian*, provided that every surjective endomorphism of G is an automorphism. This notion originated in connection with Hopf's question of whether a degree 1 map from a closed manifold to itself must be a homotopy equivalence. While it is easy to give examples of infinitely generated groups which are not Hopfian, for some time it was an open question as to whether every finitely generated group is indeed Hopfian. The first finitely generated example was given by B. Neumann in [**13**], and shortly thereafter the following finitely presented example was given by G. Higman in [**8**]:

$$\langle a, s, t \mid a^t = a^2, a^s = a^2 \rangle,$$

where x^y denotes $y^{-1}xy$. Another famous non-Hopfian example was given by Baumslag and Solitar [**2**]:

$$\langle b, s \mid (b^2)^s = b^3 \rangle.$$

The issue of which groups are Hopfian has received consistent attention over the last fifty years. One of the reasons for this is a surprising connection with residual finiteness. A group G is *residually finite* provided that $\{1_G\}$ is the intersection of the finite index subgroups of G. It was proved by Malćev that if G is finitely generated and residually finite, then G is Hopfian [**11**].

Let $\phi : H \to H$ be an injective endomorphism. The *ascending HNN extension* (or mapping torus) determined by ϕ is the group $H*_\phi$ presented by $\langle H, t \mid h^t = \phi(h), \forall h \in H \rangle$. Using this terminology, our main theorem is stated as follows, and proved, in Section 3.

Theorem 3.3. *Every ascending HNN extension of a finitely generated free group is Hopfian.*

Received 3 Jun 1999; revised 12 May 2000.

2000 *Mathematics Subject Classification* 20E26, 20E06, 20J06, 20F65.

Third author partially supported by NSF grant no. DMS-9623284; fourth author supported as an NSF Postdoctoral Fellow under grant no. DMS-9627506.

The special case of Theorem 3.3 when ϕ is an isomorphism was previously known. In that case, $H*_\phi$ is isomorphic to a semidirect product $H \rtimes_\phi \mathbb{Z}$, and is easily seen to be residually finite (and thus Hopfian) by Malćev's result. Theorem 3.3 adds some credence to the following conjecture.

Conjecture 1.1 (Residually finite groups conjecture). *Let G be an ascending HNN extension of a finitely generated free group. Then G is residually finite.*

We note that this conjecture is proved for certain subclasses in [**14**], and in [**20**].

As observed by D. Meier [**12**], the above non-Hopfian examples are readily generalized to certain multiple ascending HNN extensions, as well as to certain HNN extensions which are not ascending. For instance, if $\phi : F \to F$ is a proper monomorphism of the group F, then the group

$$G = \langle F, s, t \mid f^s = \phi(f), f^t = \phi(f), \forall f \in F\rangle$$

is not Hopfian. Indeed, the endomorphism of G induced by $f \mapsto \phi(f), s \mapsto s, t \mapsto t$ is surjective because $sfs^{-1} \mapsto s\phi(f)s^{-1} = f$. However, this endomorphism is not injective because if $x \notin \phi(F)$, then $sxs^{-1}tx^{-1}t^{-1} \neq 1_G$ by the normal form theorem for HNN extensions [**10**, Theorem IV.2.1], but $sxs^{-1}tx^{-1}t^{-1} \mapsto s\phi(x)s^{-1}t\phi(x)^{-1}t^{-1} = 1_G$.

These examples demonstrate that while Theorem 3.3 holds for a single ascending HNN extension of a finitely generated free group, a double ascending HNN extension can fail to be Hopfian. We note, however, that a semidirect product $F_n \rtimes F_m$ of two finitely generated free groups is easily seen to be residually finite, and thus Hopfian, as before.

In pursuit of the Hopf property, it is certainly helpful to assume that the HNN extension is ascending; this is not enough, however, and some restrictions on the base group are also necessary. Indeed, two examples are given in [**15**] of ascending HNN extensions of finitely generated residually finite groups which are not residually finite. One of these examples has very few finite quotients. The other example is actually not Hopfian.

Finitely generated free groups are Hopfian, and this plays a definite role in the proof of Theorem 3.3. This was first proved by Nielsen in 1929, using Nielsen transformations [**10**, I.3.5]. Note that since free groups are residually finite, one can also deduce this from Malćev's result.

Recently, Sela proved the remarkable result that torsion-free word-hyperbolic groups are Hopfian [**17**]. Note that it is still unknown whether every word-hyperbolic group is residually finite. There are numerous examples of ascending HNN extensions which are not word-hyperbolic. For instance, any such group containing $BS(1, n) = \langle a, t \mid a^t = a^n\rangle$ as a subgroup cannot be word-hyperbolic. In fact, it is still an open question as to whether an ascending HNN extension of a finitely generated free group is word-hyperbolic if and only if it does not contain $BS(1, n)$ as a subgroup. (See [**9**] for some recent progress on this problem.) While Theorem 3.3 cannot be deduced from Sela's result, one might hope for the following attractive common generalization.

Conjecture 1.2 (Word-hyperbolic groups conjecture). *Let G be an ascending HNN extension of a word hyperbolic group H; then G is Hopfian.*

Because one-ended word-hyperbolic groups do not admit proper monomorphisms into themselves [**16**], the most important case to consider in Conjecture 1.2 is the case when G is a semidirect product $G = H \rtimes \mathbb{Z}$.

There are two possible direct generalizations of Theorem 3.3.

CONJECTURE 1.3 (Iterated ascending HNN extensions conjecture). *Let $G_1, \ldots, G_n$ be groups such that G_1 is a finitely generated free group, and each G_{k+1} is an ascending HNN extension of G_k. Then G_n is Hopfian.*

CONJECTURE 1.4. *Let G be an ascending HNN extension of a free group of infinite rank. If G is finitely generated, then G is Hopfian.*

We note that Conjecture 1.4 is known to be true in the special case where G is a free by cyclic group which is finitely generated. Indeed, Baumslag showed [**1**] that such groups are residually finite.

A crucial ingredient in the proof of Theorem 3.3 is the following theorem, stated and proved in Section 2.

THEOREM 2.6. *Suppose that F is a finitely generated free group, that $\phi : F \to F$ is a monomorphism, and that $G = F*_\phi$. Assume that G is isomorphic to $H*_\tau$, where H is of type FP_2 and $\tau : H \to H$ is a monomorphism. Then H is free.*

In particular, Theorem 2.6 holds when H is finitely presented, or even finitely generated, by Remark 2.7.

Theorem 2.6 is the case $n = 2$ of Theorem 2.1, which we will state at the start of the next section, after recalling various homological definitions. A group G is of type FP_n if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution which is finitely generated in dimensions less than or equal to n. The *cohomological dimension* of G, denoted by $\mathsf{cd}(G)$, is defined by $\mathsf{cd}(G) \leqslant n$ if and only if there is a projective resolution of $\mathbb{Z}$ which vanishes above dimension n. General references for this subject are [**5**] and [**4**]. Some facts worth recalling are as follows.

(i) $\mathsf{cd}(G) = 1$ if and only if G is free [**18**, **19**].

(ii) A group of type FP_1 is finitely generated, and vice versa; a finitely presented group is of type FP_2.

(iii) A group G is defined to be of type FP if and only if there is a finitely generated projective resolution of $\mathbb{Z}$. It is a fact that G is of type FP if and only if, for some n, G is of type FP_n and $\mathsf{cd}(G) \leqslant n$; see [**5**, VIII.6.1].

(iv) If $H \subset G$, then $\mathsf{cd}(H) \leqslant \mathsf{cd}(G)$; see [**5**, Proposition 8.2.4].

(v) If G is of type FP_n, then for $k \leqslant n$, $\mathsf{H}^k(G, -)$ commutes with direct limits (see [**4**, 2.4] or [**5**, VIII.4.6]). In particular, for $k \leqslant n$, we have $\mathsf{H}^k(G, \oplus_i M_i) \cong \oplus_i \mathsf{H}^k(G, M_i)$.

(vi) If G is an HNN extension with base group H, then $\mathsf{cd}(G) \leqslant \mathsf{cd}(H) + 1$; see [**4**, Proposition 6.12].

Theorem 2.1 below partially generalizes Bieri's result [**4**] that if H is of type FP and if $G = H \rtimes \mathbb{Z}$, then $\mathsf{cd}(G) = \mathsf{cd}(H) + 1$.

2. *Proof of ascending dimension*

THEOREM 2.1. *Let H be a group of type FP_n, where $0 \leqslant n < \infty$. Suppose that $\mathsf{cd}(H) = n$ and $\mathsf{H}^{n-1}(H, \mathbb{Z}H) = 0$. Then if $\tau : H \to H$ is a monomorphism, then $\mathsf{cd}(H*_\tau) = n+1$.*

REMARK 2.2. Note that the hypotheses of the theorem are satisfied when H is an n-dimensional orientable Poincaré-duality group.

Proof of Theorem 2.1. Let $G = H*_\tau$. If $n = 0$, then $\mathsf{H}^{-1}(H, \mathbb{Z}H)$ is undefined, but the conclusion of the theorem is obvious because $H = \{1\}$ and $G \cong \mathbb{Z}$. For the remainder of the proof, we assume that $n \geqslant 1$.

Since $\mathsf{cd}(H) = n$ and G is an HNN extension of H, Fact (vi) implies that $\mathsf{cd}(G) \leqslant n+1$. Since $G \supset H$, Fact (iv) implies that $\mathsf{cd}(G) \geqslant \mathsf{cd}(H) = n$. Consequently, $\mathsf{cd}(G)$ is either n or $n+1$.

By [**5**, Proposition 8.2.3], if $\mathsf{cd}(G) = n$, then $\mathsf{H}^n(G, \mathbb{Z}G) \neq 0$. We will show below that $\mathsf{H}^n(G, \mathbb{Z}G) = 0$, and we can therefore conclude that $\mathsf{cd}(G) = n+1$.

The Mayer–Vietoris sequence [**4**, Theorem 2.12] for the ascending HNN extension $G = H*_\tau$ gives

$$\mathsf{H}^{n-1}(H, \mathbb{Z}G) \to \mathsf{H}^n(G, \mathbb{Z}G) \to \mathsf{H}^n(H, \mathbb{Z}G) \xrightarrow{\alpha} \mathsf{H}^n(H, \mathbb{Z}G).$$

Since H is of type FP_n, [**6**, Theorem 0.1] implies that α is a monomorphism. Since H is of type FP_{n-1}, by Fact (v) we have $\mathsf{H}^{n-1}(H, \mathbb{Z}G) \cong \oplus_{G/H} \mathsf{H}^{n-1}(H, \mathbb{Z}H) = 0$. By combining the facts that α is a monomorphism and that $\mathsf{H}^{n-1}(H, \mathbb{Z}G) = 0$, we deduce that $\mathsf{H}^n(G, \mathbb{Z}G) = 0$, as claimed.

REMARK 2.3. Theorem 2.1 holds if we replace the requirement that G be an ascending HNN extension with the following, less restrictive, requirement: G is an HNN extension with base group H and with edge group E, such that at least one of the monomorphisms of E to H induces a monomorphism $\mathsf{H}^*(H, \mathbb{Z}H) \to \mathsf{H}^*(E, \mathbb{Z}H)$. This holds, for instance, when the image of one of the monomorphisms has finite index in H.

LEMMA 2.4. *Let H be finitely generated and torsion-free, but not free, and consider an ascending HNN extension $H*_\tau$ of H. Then there is a subgroup $K*_\sigma$ of $H*_\tau$, such that K is a one-ended free factor of H.*

Proof. Since H is torsion-free and finitely generated, [**18**] and [**3**] give a decomposition $H = F * A_1 * \ldots * A_m$, where each A_i is finitely generated and one-ended, and F is finitely generated and free. Furthermore, $m \geqslant 1$ because H is not free.

As A_i is one-ended for each i, the Kuroš subgroup theorem [**10**, Theorem IV.1.10] implies that $\tau(A_i)$ is a subgroup of hA_jh^{-1} for some j and some $h \in H$. Hence there exists $n > 0$ such that $\tau^n(A_i) \subset \hat{h}A_i\hat{h}^{-1}$ for some i and some $\hat{h} \in H$.

Let $H*_\tau$ be presented by $\langle H, s \mid s^{-1}hs = \tau(h), \forall h \in H\rangle$. Define $\sigma : A_i \to A_i$ by $\sigma(a) = \hat{h}^{-1}\tau^n(a)\hat{h}$. Consider the HNN extension A_i*_σ presented by $\langle u, A_i \mid u^{-1}au = \sigma(a), \forall a \in A\rangle$. We will show that there is a monomorphism $f : A_i*_\sigma \to H*_\tau$, extending the inclusion of A_i into H. Define $f(u) = s^n\hat{h}$ and $f(a) = a$ for all $a \in A_i$. Now $f(u^{-1})f(a)f(u) = \hat{h}^{-1}s^{-n}as^n\hat{h} = \hat{h}^{-1}\tau^n(a)\hat{h} = \sigma(a) = f(\sigma(a))$. Hence f extends to a homomorphism $f : A_i*_\sigma \to H*_\tau$. Any element $g \in A_i*_\sigma$ has the form $g = u^jau^{-k}$

for $j,k \geqslant 0$ and $a \in A_i$. If $g \in \ker(f)$, then clearly $j = k$, but then $a \in \ker(f)$. As $f(a) = a$, $a = 1$ and we see that f is a monomorphism. The result follows with $K = A_i$.

By applying Lemma 2.4, we can obtain the following strengthening of the case $n = 2$ of Theorem 2.1.

COROLLARY 2.5. *Let H be of type FP_2, and suppose that* $\mathsf{cd}(H) = 2$. *Let $\tau : H \to H$ be a monomorphism, and let G denote $H*_\tau$. Then* $\mathsf{cd}(G) = 3$.

Proof. Since $\mathsf{cd}(H) = 2$, we see that H is not free, and so by Lemma 2.4, there is a monomorphism $E \hookrightarrow G$, where $E = K*_\sigma$ and K is a one-ended free factor of $H = K * L$.

The group K is also of type FP_2, by [**4**, Proposition 2.13(a)]. To see that $\mathsf{cd}(K) = 2$, first observe that $\mathsf{cd}(K) \geqslant 2$ because K is one-ended, and hence is not free. Second, observe that Fact (iv) implies that $\mathsf{cd}(K) \leqslant \mathsf{cd}(H) = 2$.

By Theorem 2.1 we see that $\mathsf{cd}(E) = 3$. Since $G \supset E$, Fact (iv) implies that $\mathsf{cd}(G) \geqslant 3$. Since $\mathsf{cd}(H) = 2$ and G is an HNN extension of H, Fact (vi) implies that $\mathsf{cd}(G) \leqslant 3$. Consequently, $\mathsf{cd}(G) = 3$.

THEOREM 2.6. *Suppose that F is a finitely generated free group, that $\phi : F \to F$ is a monomorphism, and that $G = F*_\phi$. Assume that G is isomorphic to $H*_\tau$, where H is of type FP_2 and $\tau : H \to H$ is a monomorphism. Then H is free.*

Proof. Suppose that H is not free. Then $\mathsf{cd}(H) \geqslant 2$. But $H \leqslant G$ and $\mathsf{cd}(G) \leqslant 2$. Thus $\mathsf{cd}(H) = 2$. As H is a subgroup of $F*_\phi$, it is torsion-free. By Corollary 2.5, $\mathsf{cd}(G) = 3$, which is a contradiction.

REMARK 2.7. A group G is said to be *coherent*, provided that every finitely generated subgroup of G is finitely presented. Feighn and Handel proved that ascending HNN extensions of free groups are coherent [**7**]. Consequently, it is sufficient to assume in the statement of Theorem 2.6 that H is finitely generated.

3. *Proof of the Hopf property*

The following lemma characterizes ascending HNN extensions.

LEMMA 3.1. *Let H be a subgroup of G, and let $t \in G$. Suppose that*

(1) $t^{-1}Ht \subseteq H$;
(2) $G = \langle H, t\rangle$;
(3) $t^n \notin H$ *for any* $n \neq 0$.

If ϕ is the endomorphism of H induced by conjugating by t, then G is isomorphic to the ascending HNN extension $H*_\phi$ presented by $\langle H, s \mid s^{-1}hs = \phi(h), \forall h \in H\rangle$.

Proof. There is a natural homomorphism from $H*_\phi$ onto G, which is induced by the identity map on H and the map $s \mapsto t$. Every element of G can be represented in the form $s^m w s^{-n}$, where $m,n \geqslant 0$ and $w \in H$. We need to show that our homomorphism has trivial kernel.

Suppose that $a = s^m w s^{-n}$ is in the kernel (here, $w \in H$). This means that $t^m w t^{-n} = 1_G$, and so $w = t^{n-m}$. By Condition (3), $m = n$, and $w = 1$. But then $a = 1_G$, and so the kernel is trivial, as required.

LEMMA 3.2. *Let $G = H*_\phi$ be an ascending HNN extension.*

(1) *If G has finitely generated abelianization and ρ is a surjective endomorphism of G, then $G = \rho(H)*_\gamma$ for some endomorphism γ of $\rho(H)$.*

(2) *Moreover, if ρ is injective on H, then ρ is an isomorphism.*

Proof. Let t be the stable letter of $H*_\phi$. Conjugation by $\rho(t)$ induces an endomorphism γ of $\rho(H)$, and we form the ascending HNN extension $\rho(H)*_\gamma$. Observe that ρ factors as $H*_\phi \to \rho(H)*_\gamma \to G$. The first map is clearly surjective and an isomorphism when ρ is an injection on H. We will apply Lemma 3.1 to see that the second map is an isomorphism.

It is clear that $\rho(t)^{-1}\rho(H)\rho(t) \subseteq \rho(H)$ and $G = \rho(G) = \langle \rho(H), \rho(t) \rangle$. So, in order to apply Lemma 3.1, we must show that $\rho(t)^n \notin \rho(H)$ for any $n \neq 0$.

Let G_{ab} denote the abelianization $G/[G, G]$ of G, and let $\mu : G \to G_{ab}$ denote the abelianization map. Because the quotient of G by the normal closure of H is infinite cyclic, $\mu(H)$ is of infinite index in G_{ab}.

Finitely generated abelian groups are Hopfian, and so the surjective endomorphism $\rho : G \to G$ projects to an isomorphism $\rho_{ab} : G_{ab} \to G_{ab}$.

Arguing by contradiction, suppose that $\rho(t)^n = \rho(w)$ for some $n \neq 0$ and $w \in H$. Then $\mu(\rho(H))$ is of finite index in G_{ab}. But then the isomorphism ρ_{ab} takes the infinite index subgroup $\mu(H)$ to the finite index subgroup $\rho_{ab}(\mu(H)) = \mu(\rho(H))$.

THEOREM 3.3. *Every ascending HNN extension of a finitely generated free group is Hopfian.*

Proof. Consider an ascending HNN extension $G = F*_\phi$ of a finitely generated free group F. Let t be the stable letter of this extension. We can assume that rank(F) is minimal, in the sense that G cannot be represented as an ascending HNN extension of a free group of smaller rank. Let $\rho : G \to G$ be a surjective endomorphism. By Lemma 3.2(1), $G = \rho(F)*_\gamma$ for some endomorphism γ. By Theorem 2.6 and Remark 2.7, $\rho(F)$ is free. Since rank(F) is minimal, the rank of $\rho(F)$ must equal rank(F), and so the restriction of ρ to F is an injection (because finitely generated free groups are Hopfian [**10**]). The theorem now follows from Lemma 3.2(2).

REMARK 3.4. In the proof of Theorem 3.3, we represent G as an ascending HNN extension of a free group F which is of minimal rank. We note that the minimal number of generators for G may actually be smaller than rank$(F) + 1$. For example, consider the ascending HNN extension

$$G = \langle a_1, \ldots, a_r, t \mid a_i^t = a_{i+1} \text{ (subscripts mod } r) \rangle.$$

Then G is generated by a_1 and t, but G cannot be expressed as an ascending HNN extension of a free group of rank 1. Indeed, the groups $BS(1, n) = \langle a, t \mid a^t = a^n \rangle$ are solvable, and hence do not contain a free subgroup of rank $\geqslant 2$.

References

1. G. Baumslag, 'A non-cyclic, locally free, free-by-cyclic group all of whose finite factor groups are cyclic', *Bull. Austral. Math. Soc.* 6 (1972) 313–314.

2. G. Baumslag and D. Solitar, 'Some two-generator one-relator non-Hopfian groups', *Bull. Amer. Math. Soc.* 68 (1962) 199–201.

3. G. M. Bergman, 'On groups acting on locally finite graphs', *Ann. of Math.* (2) 88 (1968) 335–340.

4. R. Bieri, *Homological dimension of discrete groups*, 2nd edn (Queen Mary College, Department of Pure Mathematics, London, 1981).

5. K. S. Brown, *Cohomology of groups*, corrected reprint of the 1982 original (Springer, New York, 1994).

6. K. S. Brown and R. Geoghegan, 'Cohomology with free coefficients of the fundamental group of a graph of groups', *Comment. Math. Helv.* 60 (1985) 31–45.

7. M. Feighn and M. Handel, 'Mapping tori of free group automorphisms are coherent', *Ann. of Math.* (2) 149 (1999) 1061–1077.

8. G. Higman, 'A finitely related group with an isomorphic proper factor group', *J. London Math. Soc.* 26 (1951) 59–61.

9. I. Kapovich, 'On mapping tori of endomorphisms of free groups', *Comm. Algebra* 28 (2000), 2895–2917.

10. R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Ergeb. Math. Grenzgeb. 89 (Springer, Berlin, 1977).

11. A. I. Malcév, 'On homomorphisms onto finite groups' (in Russian), *Uchen. Zap. Ivanov Gos. Ped. Inst.* 18 (1956) 49–60; English translation in: Amer. Math. Soc. Transl. Ser. 2, 119 (1983) 67–79.

12. D. Meier, 'Non-Hopfian groups', *J. London Math. Soc.* (2) 26 (1982) 265–270.

13. B. H. Neumann, 'A two-generator group isomorphic to a proper factor group', *J. London Math. Soc.* 25 (1950) 247–248.

14. M. Sapir, 'Topological dynamics of polynomial maps over p-adic numbers and residual finiteness of ascending HNN extensions', Preprint, 1999.

15. M. Sapir and D. T. Wise, 'Non-residually finite ascending HNN extensions', *J. Pure Appl. Algebra*, to appear.

16. Z. Sela, 'Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups, II', *Geom. Funct. Anal.* 7 (1997) 561–593.

17. Z. Sela, 'Endomorphisms of hyperbolic groups. I. The Hopf property', *Topology* 38 (1999) 301–321.

18. J. R. Stallings, 'On torsion-free groups with infinitely many ends', *Ann. of Math.* (2) 88 (1968) 312–334.

19. R. G. Swan, 'Groups of cohomological dimension one', *J. Algebra* 12 (1969) 585–610.

20. D. T. Wise, 'Residual finiteness of positive one-relator groups', *Comment. Math. Helv.*, to appear.

Geoghegan
Dept. of Math.
SUNY
Binghamton, NY 13902
USA

ross@math.binghamton.edu

Mihalik & Sapir
Dept. of Math.
Vanderbilt University
Nashville, TN 37240
USA

mihalik@math.vanderbilt.edu
sapir@math.vanderbilt.edu

Wise
Dept. of Math.
Cornell University
Ithaca, NY 14853
USA

daniwise@math.cornell.edu