On the Profinite Topology on Negatively Curved Groups

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Let $H$ and $K$ be quasiconvex subgroups of a negatively curved locally extended residually finite (LERF) group $G$. It is shown that if $H$ is malnormal in $G$, then the double coset $KH$ is closed in the profinite topology of $G$. In particular, this is true if $G$ is the fundamental group of an atoroidal LERF hyperbolic 3-manifold, and $H$ is the fundamental group of a totally geodesic boundary component of such manifold.

INTRODUCTION

The profinite topology on a group $G$ is defined by proclaiming all finite index subgroups of $G$ to be the base open neighborhoods of the identity in $G$. We denote it by $PT(G)$. A group $G$ is residually finite (RF) if the trivial subgroup is closed in $PT(G)$ and a group $G$ is locally extended residually finite (LERF) if any finitely generated subgroup of $G$ is closed in $PT(G)$.

E. Rips and the author showed in [G-R] that for any finitely generated subgroups $H$ and $K$ of a free group $F$ the double coset $KH$ is closed in $PT(F)$. G. A. Niblo generalised that result in [Niblo] showing that finitely generated Fuchsian groups have this property.

In this paper we prove the following:

**Theorem 1.** Let $H$ and $K$ be quasiconvex subgroups of a negatively curved LERF group $G$. If $H$ is malnormal in $G$ then the double coset $KH$ is closed in the profinite topology on $G$.

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Recall that a subgroup $H$ is malnormal in $G$ if for any $g \notin H$ the intersection of $H$ and $gHg^{-1}$ is trivial, and a group $G$ is locally quasiconvex if all finitely generated subgroups of $G$ are quasiconvex in $G$.

Remarks. (1) Let $G_1$ be a finite index subgroup of a negatively curved group $G$ containing $H$, and let $K$ be a quasiconvex subgroup of $G$. Then the intersection $K_1$ of $K$ and $G_1$ is quasiconvex in $G_1$. As $KH$ is a finite union of cosets $k_iK_iH$, it follows that if $K_1H$ is closed in $PT(G_1)$ (hence $K_1H$ is also closed in $PT(G)$), then $KH$ is closed in $PT(G)$.

(2) A theorem of M. Hall in [Hall] implies that any finitely generated subgroup $H$ of a free group $F$ is a free factor of some finite index subgroup $F_1$ of $F$. As a free factor is malnormal and as free groups are locally quasiconvex, Theorem 1 and Remark 1 imply the result in [G-R].

(3) A theorem of P. Scott in [Scott] implies that given a finitely generated subgroup $H$ of a surface group $S$ there exists a finite index subgroup $S_1$ of $S$ and a graph of groups decomposition of $S_1$ such that $H$ is a vertex group and all edge groups are infinite cyclic groups malnormal in $S_1$. Then $H$ is malnormal in $S_1$. As surface groups are locally quasiconvex, negatively curved, and LERF, Theorem 1 and Remark 1 imply the result in [Niblo].

(4) If $G$ is the fundamental group of an atoroidal hyperbolic 3-manifold then $G$ is negatively curved. If $H$ is the fundamental group of a totally geodesic boundary component of such manifold, then $H$ is malnormal in $G$. Moreover, the result of Thurston [Thu] states that such $G$ is locally quasiconvex; hence Theorem 1 implies that for any finitely generated subgroup $K$ of $G$, the double coset $KH$ is closed in $PT(G)$.

(5) Denote the minimal subgroup of $G$ containing $H$ and $K$ by $\langle H, K \rangle$.

In order to show that $KH$ is closed in $PT(G)$, it is enough to show that for any $g \in G$ such that $g \notin KH$ there exists a set $S$ closed in $PT(G)$ which contains $KH$, but does not contain $g$.

Assume that $G$ is LERF and $H$ and $K$ are finitely generated. Fix $g \notin KH$. We can easily find the required closed set $S$, if we can find a finite index subgroup $H_g$ of $H$ and a set $\{a_i, 1 \leq i \leq m\}$ of right coset representatives of $H_g$ in $H$ such that $ga_i^{-1} \notin \langle H_g, K \rangle$ for all $a_i$. Indeed, as $G$ is LERF, there exists a finite index subgroup $G_1$ of $G$ which contains the (finitely generated) subgroup $\langle H_g, K \rangle$, but does not contain the elements $ga_i^{-1}$ for all $a_i$. Then the coset $G_1a_i$ contains $KH_ga_i$, but does not contain $g$, so the closed set $S = \bigcup G_1a_i$ contains the set $KH = \bigcup KH_ga_i$, but does not contain $g$.

If $H \cap K$ has finite index in $H$, we can take $H_g$ to be $H \cap K$. The general case is treated below.
Let $X$ be a set, let $X^* = \{x, x^{-1} \mid x \in X\}$, and for $x \in X$ define $(x^{-1})^{-1} = x$. Denote the set of all words in $X$ by $W(X)$, and denote the equality of two words by "$\equiv$". Let $G$ be a group generated by the set $X^*$, and let Cayley$(G)$ be the Cayley graph of $G$ with respect to the generating set $X^*$. The set of vertices of Cayley$(G)$ is $G$, the set of edges of Cayley$(G)$ is $G \times X^*$, and the edge $(g, x)$ joins the vertex $g$ to $gx$.

**Definition 1.** The label of the path $p = (g, x_1)(gx_1, x_2) \cdots (gx_{n-1}, x_n)$ in Cayley$(G)$ is the word Lab$(p) \equiv x_1 \cdots x_n \in W(X^*)$. As usual, whenever it is convenient, we identify the word Lab$(p)$ with the corresponding element in $G$. Denote the length of the path $p$ by $|p|$, where $|(g, x_1)(gx_1, x_2) \cdots (gx_{n-1}, x_n)| = n$.

A geodesic in the Cayley graph is a shortest path joining two vertices. A group $G$ is $\delta$-negatively curved if any side of any geodesic triangle in Cayley$(G)$ belongs to the $\delta$-neighborhood of the union of two other sides (see [Gr] and [C-D-P]). Let $\delta \leq 1$, $L > 0$ and let $\epsilon > 0$. A path $p$ in Cayley$(G)$ is a $(\lambda, \epsilon)$-quasigeodesic if for any subpath $p'$ of $p$ and for any geodesic $\gamma$ with the same endpoints as $p'$, $|\gamma| > \lambda |p'| - \epsilon$. A path $p$ is a local $(\lambda, \epsilon, L)$-quasigeodesic in Cayley$(G)$ if any subpath of $p$ which is shorter than $L$ is a $(\lambda, \epsilon)$-quasigeodesic (cf. [C-D-P, p. 24]).

Theorem 1.4 [p. 25 of C-D-P] (see also Gr, p. 187) states that for any $\lambda_0 \leq 1$ and for any $\epsilon_0 > 0$ there exist constants $(L, \lambda, \epsilon)$ which depend only on $(\lambda_0, \epsilon_0)$ and on $\delta$, such that any local $(\lambda_0, \epsilon_0, L)$-quasigeodesic in Cayley$(G)$ is a global $(\lambda, \epsilon)$-quasigeodesic in Cayley$(G)$.

Recall that $H$ is a $\mu$-quasiconvex subgroup of $G$ if any geodesic in Cayley$(G)$ which has its endpoints in $H$ belongs to the $\mu$-neighborhood of $H$.

We use the following property of malnormal quasiconvex subgroups of finitely generated groups proven in [Gi 1].

**Lemma 2.** Let $H$ be a malnormal $\mu$-quasiconvex subgroup of a finitely generated group $G$. Let $\gamma_1, \gamma_2$ be a path in Cayley$(G)$ such that $\gamma_1$ and $\gamma_2$ are geodesics in Cayley$(G)$, Lab$(\gamma_1) \in H$, Lab$(\gamma_2) \in H$, and Lab$(\gamma) \not\in H$. Then for any $\rho \geq 0$, there exists a positive constant $N(\rho)$ which depends only on $\rho$, on $G$, and on $\mu$ such that any subpath of $\gamma_1$ which belongs to the $\rho$-neighborhood of $\gamma_2$ is shorter than $N(\rho)$.

**Definition 3.** Let $H$ and $K$ be subgroups of $G$. Consider an element $l$ of $\langle H, K \rangle$ such that $l \not\in H \cap K$. There exists a word $w \in W(X)$ representing $l$ of the following form: $w = h_1 k_1 \cdots k_{m-1} h_m$, where $h_i$ represents an element in $H$, but not in $H \cap K$, $k_j$ represents an element in $K$,
but not in $H \cap K$, $k_i$ and $h_i$ are geodesics in $G$, $h_i$ is a shortest representative of the coset $h_i(H \cap K)$, $h_m$ is a shortest representative of the coset $(H \cap K)h_m$, and for $1 < i < m$, $h_i$ is a shortest representative of the double coset $(H \cap K)h_i(H \cap K)$. (The words $h_1$ or $h_m$ might be trivial.) We call such $w_i$ a good word representative of $l$. Let $p$ be a path in Cayley$(G)$ with the decomposition of the following form: $p = \eta_1 \kappa_1 \cdots \kappa_{m-1} \eta_m$, where $\text{Lab}(\eta_i) = h_i$ and $\text{Lab}(\kappa_i) = k_i$. If $\text{Lab}(p) = h_1 \kappa_1 \cdots \kappa_m = l$ is a good word representative of $l$, we will call such $p$ a good path representative of $l$.

We need the following result, which was proven, but not explicitly stated in [Gi 2]. We use the notation of Definition 3.

**Lemma 4.** Let $H$ and $K$ be $\mu$-quasiconvex subgroups of a $\delta$-negatively curved group $G$, and let $H$ be malnormal in $G$. There exist constants $C$, $\lambda$, and $\epsilon$ which depend only on $G$, $\delta$, and $\mu$ such that if $p = \eta_1 \kappa_1 \cdots \kappa_{m-1} \eta_m$ is a good path representative of $l \in \langle H, K \rangle$ and all subpaths $\eta_i$ are longer than $C$, then $p$ is a $(\lambda, \epsilon)$-quasigeodesic in Cayley$(G)$.

**Proof.** We choose the constants $C$, $\lambda$, and $\epsilon$, as follows. Let $A$ be the number of words in $G$ which are shorter than $2\mu + \delta$, and let $N(2\delta)$ be the constant defined in Lemma 2 for $H$ in $G$ with $\rho = 2\delta$. Let $\lambda_0 = \frac{1}{2}$, and let $\epsilon_0 = 4\mu \cdot A + N(2\delta)$. As mentioned above, there exist constants $(L, \lambda, \epsilon)$ which depend only on $(\lambda_0, \epsilon_0)$ and on $\delta$ such that any local $(\lambda_0, \epsilon_0, L)$-quasigeodesic in Cayley$(G)$ is a global $(\lambda, \epsilon)$-quasigeodesic in Cayley$(G)$. These are the $\lambda$ and $\epsilon$ we choose, and we choose $C = \max(L, \frac{\delta}{\epsilon})$.

Let $p$ be a good path representative of $l \in \langle H, K \rangle$ such that all subpaths $\eta_i$ are longer than $C$. In order to show that $p$ is a $(\lambda, \epsilon)$-quasigeodesic in Cayley$(G)$, it is enough to show that $p$ is a local $(1/6, (4\mu \cdot A + \delta + N(2\delta)), L)$-quasigeodesic in Cayley$(G)$.

As $|\eta_i| > C \geq L$, it follows that any subpath $t$ of $p$ with $|t| < L$ has a (unique) decomposition $t_1t_2t_3$, where $t_1$ and $t_3$ are subpaths of some $\eta_i$ and $\eta_{i+1}$, and $t_2$ is a subpath of $\kappa_i$ (some of $t_1$ might be empty). Let $t_4$ be a geodesic in Cayley$(G)$ connecting the endpoints of $t$.

If $|t_2| > \frac{2|t_1|}{3}$, then $|t_1| + |t_3| \leq |t|/3$; hence $|t_4| \geq |t_2| - (|t_1| + |t_3|) \geq \frac{2|t|}{3} - \frac{|t|}{3} = \frac{|t|}{3}$.

If $|t_2| \leq \frac{2|t_1|}{3}$, then without loss of generality assume that $|t_1| \geq |t_3|$; hence $|t_3| > \frac{|t|}{2}$. As $t_1t_2t_3t_4$ is a geodesic $4$-gon in a $\delta$-negatively curved group $G$, there exists a decomposition $t_1 = s_2s_3s_4$ such that $s_2$ belongs to the $\delta$-neighborhood of $t_2$, $s_3$ belongs to the $2\delta$-neighborhood of $t_3$, and $s_4$
belongs to the $\delta$-neighborhood of $t_4$. According to Lemma 2, $|s_3| < N(2\delta)$, and according to Lemma 5 (below), $|s_2| \leq 4\mu \cdot A$. But then $|t_4| + \delta \geq |s_4| = |t_1| - |s_2| - |s_3| \geq |t_1| - 4\mu \cdot A - N(2\delta) \geq \frac{|t_1|}{6} - 4\mu \cdot A - N(2\delta).

Hence $|t_4| \geq \frac{|t_1|}{6} - (N(2\delta) + \delta + 4\mu \cdot A)$, so the path $p$ is a local $(\frac{1}{6}, N(2\delta) + \delta + 4\mu \cdot A, L)$-quasigeodesic in $G$, hence it is a $(\lambda, \epsilon)$-quasigeodesic in $G$.

**Lemma 5.** Using the notation of the proof of Lemma 4, $|s_2| \leq A \cdot 4\mu$.

**Proof.** To simplify notation, we drop the subscript $i$, so $t_1$ is a subpath of $\eta$, $t_2$ is a subpath of $\kappa$, Lab($\eta$) = $h$, and Lab($\kappa$) = $k$. Without loss of generality, assume that $\kappa$ begins at 1 (so it ends at $k$), then $\eta$ begins at $h^{-1}$ and ends at 1. As $K$ and $H$ are $\mu$-quasiconvex in $G$, any vertex $v_i$ on $\eta$ is in the $\mu$-neighborhood of $H$, and any vertex $w_i$ on $\kappa$ is in the $\mu$-neighborhood of $K$. Hence we can find vertices $v_1$ and $v_2$ in $s_2$, $w_1$ and $w_2$ in $t_2$, $h'$ and $h''$ in $H$, and $k'$ and $k''$ in $K$ such that $|v_i, w_i| < \delta$, $|v_2, (h'')^{-1}| < \mu$, $|w_1, k'| < \mu$, and $|w_2, k''| < \mu$. Then $|h'k'| < 2\mu + \delta$ and $|h''k''| < 2\mu + \delta$.

Assume that $|s_2| > A \cdot 4\mu$. Then we can find vertices, as above which, in addition, satisfy $|v_2, v_3| > 4\mu$ and $h'k' = h''k''$. But then $(h'')^{-1}h' = k''(k')^{-1}$, so both products are in $H \cap K$. As $h$ is a shortest element in the double coset $(H \cap K)h(H \cap K)$, it follows that $|h| \leq |h(h'')^{-1}h'|$. Let $r$ be a geodesic joining $h''^{-1}$ to $v_2$, let $s'$ be a subpath of $\eta$ joining $v_2$ to 1, and let $s''$ be a subpath of $\eta$ joining $h^{-1}$ to $v_2$. Then $|s''| = |s'| + |s''|$, and $|h(h'')^{-1}h'| \leq |h(h''h')^{-1}| + |h'| \leq |s'| + |s''| + |r| + |h'|$; hence $|s'| + |s''| \leq |s'| + |s''| + |r| + |h'|$, so $|s'| + |r| \leq 2|r| + |h'|$. As $|h'| \leq |s'| + |r|$ and as $|r| \leq \mu$, it follows that $|h''| \leq 2\mu + |h'|$.

However, as $|v_2, v_3| > 4\mu$, the triangle inequality implies that $|h''| = (h'')^{-1} \geq |s'| - |r| = |s'| + |v_1, v_2| - |r| \geq |s', v_3| + 4\mu - \mu = |s', v_3| + \mu + 2\mu$. Let $a$ be a geodesic joining $(h'')^{-1}$ to $v_3$. As $|a| \leq \mu$, the triangle inequality implies that $|a| = (h'')^{-1} \leq |s', v_3| + |a| < |s', v_3| + \mu$. Hence, $|h''| > |h''| + 2\mu$, a contradiction. Therefore, $|s_2| \leq A \cdot 4\mu$.

**Proof of Theorem 1.** Let $H$, $K$, $G$, $\lambda$, and $\epsilon$ be as in Lemma 4, and let $g$ be an element of $G$ which does not belong to $KH$. As $G$ is $\delta$-negatively curved, there exists a positive constant $\nu$ which depends only on $\delta$, $\lambda$, and $\epsilon$, such that any $(\lambda, \epsilon)$-quasigeodesic in Cayley($G$) belongs to the $\nu$-neighborhood of the geodesic with the same endpoints [C-D-P, p. 24]. Let $N(\delta + \nu + |g|)$ be, as in Lemma 2, for $H$ in $G$ with $\rho = \delta + \nu + |g|$.
As $H$ is LERF, there exists a finite index subgroup $H_g$ of $H$ which contains $H \cap K$ such that if $h \in H_g$, but $h \not\in H \cap K$, then $h$ is longer than $\max[|N(\delta + \nu + |g|)|, C]$.

Let $\{a_i, 1 \leq i \leq m\}$ be a set of shortest right coset representatives of $H_g$ in $H$. We claim that $ga_i^{-1} \not\in \langle H_g, K \rangle$ for all $a_i$. Indeed, assume without loss of generality that $ga_i^{-1} \in \langle H_g, K \rangle$. Then $ga_i^{-1}$ has a good path representative $p = \eta_1 \eta_2 \cdots \eta_k$ with $\text{Lab}(\eta) \in H_g$ which begins at 1 in Cayley$(G)$ and ends at $ga_i^{-1}$. As $g \not\in KH$, it follows that $ga_i^{-1} \not\in KH$; in particular $ga_i^{-1} \not\in H \cap K$, so the definitions of $H_g$ and of $p$ imply that $|\eta| > C$. Hence Lemma 4 implies that $p$ is a $(\lambda, \varepsilon)$-quasigeodesic in Cayley$(G)$. Also, the definition of $H_g$ implies that $\eta$ is longer than $N(\delta + \nu + |g|)$.

Let $\gamma$ be a geodesic in Cayley$(G)$ joining 1 to $g$, let $\gamma_a$ be a geodesic in Cayley$(G)$ joining $g$ to $ga_i^{-1}$, and let $\gamma_p$ be a geodesic in Cayley$(G)$ joining 1 to $ga_i^{-1}$. As $\gamma_a$, $\gamma_p$, and $\gamma$ form a geodesic triangle and $G$ is $\delta$-negatively curved, it follows that $\gamma$ belongs to the $\delta$-neighborhood of $\gamma_a \cup \gamma_p$. As was mentioned above, $p$ belongs to the $\nu$-neighborhood of $\gamma_p$; hence it belongs to the $(\nu + \delta + |g|)$-neighborhood of $\gamma_a$.

If $\eta_1$ is non-trivial, then as $\text{Lab}(\eta_1) = g \not\in H, \text{Lab}(\eta_k) = a_i^{-1} \in H$, and $\eta_1$ is a geodesic which belongs to the $(\delta + \nu + |g|)$-neighborhood of $\gamma_a$, Lemma 2 implies that $|\eta_1| < N(\delta + \nu + |g|)$, a contradiction. But if $\eta_1$ is trivial, then as $g \not\in KH$, it follows that $\eta_1$ and $\eta_2$ are non-trivial. As $g \not\in KH$ and $\text{Lab}(\eta_1) \in K$, it follows that $g^{-1}\text{Lab}(\eta_1) \not\in H$. Then $\eta_2$ is a geodesic which belongs to the $(\delta + \nu + |g|)$-neighborhood of $\gamma_a$, so Lemma 2 implies that $|\eta_2| < N(\delta + \nu + |g|)$, a contradiction.

Therefore $ga_i^{-1} \not\in \langle H_g, K \rangle$ for all $a_i$, and Theorem 1 follows from Remark 5.

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