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On the Profinite Topology on Negatively Curved Groups

Rita Gitik*

MSRI, 1000 Centennial Drive, Berkeley, California 94720
E-mail: ritagtk@math.lsa.umich.edu

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Let H and K be quasiconvex subgroups of a negatively curved locally extended residually finite (LERF) group G. It is shown that if H is malnormal in G, then the double coset KH is closed in the profinite topology of G. In particular, this is true if G is the fundamental group of an atoroidal LERF hyperbolic 3-manifold, and H is the fundamental group of a totally geodesic boundary component of such manifold.

INTRODUCTION

The profinite topology on a group G is defined by proclaiming all finite index subgroups of G to be the base open neighborhoods of the identity in G. We denote it by $PT(G)$. A group G is residually finite (RF) if the trivial subgroup is closed in $PT(G)$ and a group G is locally extended residually finite (LERF) if any finitely generated subgroup of G is closed in $PT(G)$.

E. Rips and the author showed in [G-R] that for any finitely generated subgroups H and K of a free group F the double coset KH is closed in $PT(F)$. G. A. Niblo generalised that result in [Niblo] showing that finitely generated Fuchsian groups have this property.

In this paper we prove the following:

THEOREM 1. *Let H and K be quasiconvex subgroups of a negatively curved LERF group G. If H is malnormal in G then the double coset KH is closed in the profinite topology on G.*

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Recall that a subgroup H is malnormal in G if for any $g \notin H$ the intersection of H and gHg^{-1} is trivial, and a group G is locally quasiconvex if all finitely generated subgroups of G are quasiconvex in G.

Remarks. (1) Let G_1 be a finite index subgroup of a negatively curved group G containing H, and let K be a quasiconvex subgroup of G. Then the intersection K_1 of K and G_1 is quasiconvex in G_1. As KH is a finite union of cosets $k_i K_1 H$, $k_i \in K$, it follows that if $K_1 H$ is closed in $PT(G_1)$ (hence $K_1 H$ is also closed in $PT(G)$), then KH is closed in $PT(G)$.

(2) A theorem of M. Hall in [Hall] implies that any finitely generated subgroup H of a free group F is a free factor of some finite index subgroup F_1 of F. As a free factor is malnormal and as free groups are locally quasiconvex, Theorem 1 and Remark 1 imply the result in [G-R].

(3) A theorem of P. Scott in [Scott] implies that given a finitely generated subgroup H of a surface group S there exists a finite index subgroup S_1 of S and a graph of groups decomposition of S_1 such that H is a vertex group and all edge groups are infinite cyclic groups malnormal in S_1. Then H is malnormal in S_1. As surface groups are locally quasiconvex, negatively curved, and LERF, Theorem 1 and Remark 1 imply the result in [Niblo].

(4) If G is the fundamental group of an atoroidal hyperbolic 3-manifold then G is negatively curved. If H is the fundamental group of a totally geodesic boundary component of such manifold, then H is malnormal in G. Moreover, the result of Thurston [Thu] states that such G is locally quasiconvex; hence Theorem 1 implies that for any finitely generated subgroup K of G, the double coset KH is closed in $PT(G)$.

(5) Denote the minimal subgroup of G containing H and K by $\langle H, K \rangle$.

In order to show that KH is closed in $PT(G)$, it is enough to show that for any $g \in G$ such that $g \notin KH$ there exists a set S closed in $PT(G)$ which contains KH, but does not contain g.

Assume that G is LERF and H and K are finitely generated. Fix $g \notin KH$. We can easily find the required closed set S, if we can find a finite index subgroup H_g of H and a set $\{a_i,\ 1 \le i \le m\}$ of right coset representatives of H_g in H such that $ga_i^{-1} \notin \langle H_g, K \rangle$ for all a_i. Indeed, as G is LERF, there exists a finite index subgroup G_1 of G which contains the (finitely generated) subgroup $\langle H_g, K \rangle$, but does not contain the elements ga_i^{-1} for all a_i. Then the coset $G_1 a_i$ contains $KH_g a_i$, but does not contain g, so the closed set $S = \cup G_1 a_i$ contains the set $KH = \cup KH_g a_i$, but does not contain g.

If $H \cap K$ has finite index in H, we can take H_g to be $H \cap K$. The general case is treated below.

PRELIMINARIES

Let X be a set, let $X^* = \{x, x^{-1} \mid x \in X\}$, and for $x \in X$ define $(x^{-1})^{-1} = x$. Denote the set of all words in X by $W(X)$, and denote the equality of two words by "$\equiv$." Let G be a group generated by the set X^*, and let Cayley(G) be the Cayley graph of G with respect to the generating set X^*. The set of vertices of Cayley(G) is G, the set of edges of Cayley(G) is $G \times X^*$, and the edge (g, x) joins the vertex g to gx.

DEFINITION 1. The label of the path $p = (g, x_1)(gx_1, x_2) \cdots (gx_1x_2 \cdots x_{n-1}, x_n)$ in Cayley(G) is the word $\mathrm{Lab}(p) \equiv x_1 \cdots x_n \in W(X^*)$. As usual, whenever it is convenient, we identify the word Lab(p) with the corresponding element in G. Denote the length of the path p by $|p|$, where $|(g, x_1)(gx_1, x_2) \cdots (gx_1x_2 \cdots x_{n-1}, x_n)| = n$.

A geodesic in the Cayley graph is a shortest path joining two vertices. A group G is δ-negatively curved if any side of any geodesic triangle in Cayley(G) belongs to the δ-neighborhood of the union of two other sides (see [Gr] and [C-D-P]). Let $\lambda \leq 1$, $L > 0$ and let $\epsilon > 0$. A path p in Cayley(G) is a (λ, ϵ)-quasigeodesic if for any subpath p' of p and for any geodesic γ with the same endpoints as p', $|\gamma| > \lambda|p'| - \epsilon$. A path p is a local (λ, ϵ, L)-quasigeodesic in Cayley(G) if any subpath of p which is shorter than L is a (λ, ϵ)-quasigeodesic (cf. [C-D-P, p. 24]).

Theorem 1.4 [p. 25 of C-D-P] (see also Gr, p. 187]) states that for any $\lambda_0 \leq 1$ and for any $\epsilon_0 > 0$ there exist constants (L, λ, ϵ) which depend only on (λ_0, ϵ_0) and on δ, such that any local $(\lambda_0, \epsilon_0, L)$-quasigeodesic in Cayley(G) is a global (λ, ϵ)-quasigeodesic in Cayley(G).

Recall that H is a μ-quasiconvex subgroup of G if any geodesic in Cayley(G) which has its endpoints in H belongs to the μ-neighborhood of H.

We use the following property of malnormal quasiconvex subgroups of finitely generated groups proven in [Gi 1].

LEMMA 2. *Let H be a malnormal μ-quasiconvex subgroup of a finitely generated group G. Let $\gamma_1 t \gamma_2$ be a path in* Cayley(G) *such that γ_1 and γ_2 are geodesics in Cayley(G), Lab(γ_1) $\in H$,* Lab(γ_2) $\in H$, *and* Lab(t) $\notin H$. *Then for any $\rho \geq 0$, there exists a positive constant $N(\rho)$ which depends only on ρ, on G, and on μ such that any subpath of γ_1 which belongs to the ρ-neighborhood of γ_2 is shorter than $N(\rho)$.*

DEFINITION 3. Let H and K be subgroups of G. Consider an element l of $\langle H, K \rangle$ such that $l \notin H \cap K$. There exists a word $w_l \in W(X)$ representing l of the following form: $w_l \equiv h_1 k_1 \cdots k_{m-1} h_m$, where h_i represents an element in H, but not in $H \cap K$, k_i represents an element in K, but not in $H \cap K$, k_i and h_i are geodesics in G, h_1 is a shortest representative of the coset $h_1(H \cap K)$, h_m is a shortest representative of the coset $(H \cap K)h_m$, and for $1 < i < m$, h_i is a shortest representative of the double coset $(H \cap K)h_i(H \cap K)$. (The words h_1 or h_m might be trivial.) We call such w_l a good word representative of l. Let p be a path in Cayley(G) with the decomposition of the following form: $p = \eta_1 \kappa_1 \cdots \kappa_{m-1} \eta_m$, where Lab$(\eta_i) \equiv h_i$ and Lab$(\kappa_i) \equiv k_i$. If Lab$(p) \equiv h_1 k_1 \cdots h_m = l$ is a good word representative of l, we will call such p a good path representative of l.

We need the following result, which was proven, but not explicitly stated in [Gi 2]. We use the notation of Definition 3.

LEMMA 4. *Let H and K be μ-quasiconvex subgroups of a δ-negatively curved group G, and let H be malnormal in G. There exist constants C, λ, and ϵ which depend only on G, δ, and μ such that if $p = \eta_1 \kappa_1 \cdots \kappa_{m-1} \eta_m$ is a good path representative of $l \in \langle H, K \rangle$ and all subpaths η_i are longer than C, then p is a (λ, ϵ)-quasigeodesic in Cayley(G).*

Proof. We choose the constants C, λ, and ϵ, as follows. Let A be the number of words in G which are shorter than $2\mu + \delta$, and let $N(2\delta)$ be the constant defined in Lemma 2 for H in G with $\rho = 2\delta$. Let $\lambda_0 = \frac{1}{6}$, and let $\epsilon_0 = 4\mu \cdot A + N(2\delta)$. As mentioned above, there exist constants (L, λ, ϵ) which depend only on (λ_0, ϵ_0) and on δ such that any local $(\lambda_0, \epsilon_0, L)$-quasigeodesic in Cayley$(G)$ is a global (λ, ϵ)-quasigeodesic in Cayley(G). These are the λ and ϵ we choose, and we choose $C = \max(L, \frac{\epsilon}{\lambda})$.

Let p be a good path representative of $l \in \langle H, K \rangle$ such that all subpaths η_i are longer than C. In order to show that p is a (λ, ϵ)-quasigeodesic in Cayley(G), it is enough to show that p is a local $(1/6, (4\mu \cdot A + \delta + N(2\delta)), L)$-quasigeodesic in Cayley$(G)$.

As $|\eta_i| > C \geq L$, it follows that any subpath t of p with $|t| < L$ has a (unique) decomposition $t_1 t_2 t_3$, where t_1 and t_3 are subpaths of some η_i and η_{i+1}, and t_2 is a subpath of κ_i (some of t_i might be empty). Let t_4 be a geodesic in Cayley(G) connecting the endpoints of t.

If $|t_2| > \frac{2|t|}{3}$, then $|t_1| + |t_3| \leq \frac{|t|}{3}$; hence $|t_4| \geq |t_2| - (|t_1| + |t_3|) \geq \frac{2|t|}{3} - \frac{|t|}{3} = \frac{|t|}{3}$.

If $|t_2| \leq \frac{2|t|}{3}$, then without loss of generality assume that $|t_1| \geq |t_3|$; hence $|t_1| > \frac{|t|}{6}$. As $t_1 t_2 t_3 t_4$ is a geodesic 4-gon in a δ-negatively curved group G, there exists a decomposition $t_1 = s_2 s_3 s_4$ such that s_2 belongs to the δ-neighborhood of t_2, s_3 belongs to the 2δ-neighborhood of t_3, and s_4

belongs to the δ-neighborhood of t_4. According to Lemma 2, $|s_3| < N(2\delta)$, and according to Lemma 5 (below), $|s_2| \leq 4\mu \cdot A$. But then $|t_4| + \delta \geq |s_4| = |t_1| - |s_2| - |s_3| \geq |t_1| - 4\mu \cdot A - N(2\delta) \geq \frac{|t|}{6} - 4\mu \cdot A - N(2\delta)$.

Hence $|t_4| \geq \frac{|t|}{6} - (N(2\delta) + \delta + 4\mu \cdot A)$, so the path p is a local $(\frac{1}{6}, (N(2\delta) + \delta + 4\mu \cdot A), L)$-quasigeodesic in G, hence it is a (λ, ϵ)-quasigeodesic in G.

LEMMA 5. *Using the notation of the proof of Lemma* 4, $|s_2| \leq A \cdot 4\mu$.

Proof. To simplify notation, we drop the subscript i, so t_1 is a subpath of η, t_2 is a subpath of κ, $\mathrm{Lab}(\eta) = h$, and $\mathrm{Lab}(\kappa) = k$. Without loss of generality, assume that κ begins at 1 (so it ends at k), then η begins at h^{-1} and ends at 1. As K and H are μ-quasiconvex in G, any vertex v_i on η is in the μ-neighborhood of H, and any vertex w_i on κ is in the μ-neighborhood of K. Hence we can find vertices v_1 and v_2 in s_2, w_1 and w_2 in t_2, h' and h'' in H, and k' and k'' in K such that $|v_i, w_i| < \delta$, $|v_1, (h')^{(-1)}| < \mu$, $|v_2, (h'')^{(-1)}| < \mu$, $|w_1, k'| < \mu$, and $|w_2, k''| < \mu$. Then $|h'k'| < 2\mu + \delta$ and $|h''k''| < 2\mu + \delta$.

Assume that $|s_2| > A \cdot 4\mu$. Then we can find vertices, as above which, in addition, satisfy $|v_2, v_1| > 4\mu$ and $h'k' = h''k''$. But then $(h'')^{(-1)}h' = k''(k')^{(-1)}$, so both products are in $H \cap K$. As h is a shortest element in the double coset $(H \cap K)h(H \cap K)$, it follows that $|h| \leq |h(h'')^{(-1)}h'|$. Let r be a geodesic joining $(h'')^{(-1)}$ to v_2, let s' be a subpath of η joining v_2 to 1, and let s'' be a subpath of η joining h^{-1} to v_2. Then $|h| = |\eta| = |s'| + |s''|$, and $|h(h'')^{(-1)}h'| \leq |h(h'')^{(-1)}| + |h'| \leq |s''| + |r| + |h'|$; hence $|s'| + |s''| \leq |s''| + |r| + |h'|$, so $|s'| + |r| \leq 2|r| + |h'|$. As $|h''| \leq |s'| + |r|$ and as $|r| \leq \mu$, it follows that $|h''| \leq 2\mu + |h'|$.

However, as $|v_2, v_1| > 4\mu$, the triangle inequality implies that $|h''| = (h'')^{-1}| \geq |s'| - |r| = |1, v_1| + |v_1, v_2| - |r| \geq |1, v_1| + 4\mu - \mu = |1, v_1| + \mu + 2\mu$. Let a be a geodesic joining $(h')^{-1}$ to v_1. As $|a| < \mu$, the triangle inequality implies that $|h'| = |(h')^{-1}| \leq |1, v_1| + |a| < |1, v_1| + \mu$. Hence, $|h''| > |h'| + 2\mu$, a contradiction. Therefore, $|s_2| \leq A \cdot 4\mu$.

Proof of Theorem 1. Let H, K, G, C, λ, and ϵ be as in Lemma 4, and let g be an element of G which does not belong to KH. As G is δ-negatively curved, there exists a positive constant ν which depends only on δ, λ, and ϵ, such that any (λ, ϵ)-quasigeodesic in Cayley(G) belongs to the ν-neighborhood of the geodesic with the same endpoints [C-D-P, p. 24]. Let $N(\delta + \nu + |g|)$ be, as in Lemma 2, for H in G with $\rho = \delta + \nu + |g|$.

As H is LERF, there exists a finite index subgroup H_g of H which contains $H \cap K$ such that if $h \in H_g$, but $h \notin H \cap K$, then h is longer than $\max([N(\delta + \nu + |g|)], C)$.

Let $\{a_i, 1 \leq i \leq m\}$ be a set of shortest right coset representatives of H_g in H. We claim that $ga_i^{-1} \notin \langle H_g, K \rangle$ for all a_i. Indeed, assume without loss of generality that $ga_1^{-1} \in \langle H_g, K \rangle$. Then ga_1^{-1} has a good path representative $p = \eta_1 \kappa_1 \cdots \eta_n$ with $\mathrm{Lab}(\eta_i) \in H_g$ which begins at 1 in Cayley(G) and ends at ga_1^{-1}. As $g \notin KH$, it follows that $ga_1^{-1} \notin KH$; in particular $ga_1^{-1} \notin H \cap K$, so the definitions of H_g and of p imply that $|\eta_i| > C$. Hence Lemma 4 implies that p is a (λ, ϵ)-quasigeodesic in Cayley(G). Also, the definition of H_g implies that η_i is longer than $N(\delta + \nu + |g|)$.

Let γ_g be a geodesic in Cayley(G) joining 1 to g, let γ_a be a geodesic in Cayley(G) joining g to ga_1^{-1}, and let γ_p be a geodesic in Cayley(G) joining 1 to ga_1^{-1}. As γ_p, γ_a, and γ_g form a geodesic triangle and G is δ-negatively curved, it follows that γ_p belongs to the δ-neighborhood of $\gamma_a \cup \gamma_g$. As was mentioned above, p belongs to the ν-neighborhood of γ_p; hence it belongs to the $(\nu + \delta + |g|)$-neighborhood of γ_a.

If η_{h1} is non-trivial, then as $\mathrm{Lab}(\gamma_g) = g \notin H$, $\mathrm{Lab}(\gamma_a) = a_1^{-1} \in H$, and η_1 is a geodesic which belongs to the $(\delta + \nu + |g|)$-neighborhood of γ_a, Lemma 2 implies that $|\eta_1| < N(\delta + \nu + |g|)$, a contradiction. But if η_1 is trivial, then as $g \notin KH$, it follows that κ_1 and η_2 are non-trivial. As $g \notin KH$ and $\mathrm{Lab}(\kappa_1) \in K$, it follows that $g^{-1}\mathrm{Lab}(\kappa_1) \notin H$. Then η_2 is a geodesic which belongs to the $(\delta + \nu + |g|)$-neighborhood of γ_a, so Lemma 2 implies that $|\eta_2| < N(\delta + \nu + |g|)$, a contradiction.

Therefore $ga_i^{-1} \notin \langle H_g, K \rangle$ for all a_i, and Theorem 1 follows from Remark 5.

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