ON RESIDUALLY FINITE GENERALIZED FREE PRODUCTS

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Let $S = A \overset{*}{_U} B$ be a nontrivial generalized free product of the groups A and B with amalgamated subgroup U and suppose A and B are residually finite groups. Baumslag [1] has given conditions sufficient for $S = A \overset{*}{_U} B$ to itself be residually finite and these have been used to investigate the residual finiteness of S by Baumslag [1] and Dyer [2], when the factors A and B of S are assumed to satisfy certain additional properties. The question then arises as to the necessity of these conditions. It is shown here that Baumslag's conditions are in fact necessary, provided A and B satisfy suitable identical relations.

A group G is *residually finite* if there exists a set $\{G_i \mid i \in I\}$ of normal subgroups of G such that G/G_i is finite for each $i \in I$ and $\bigcap_{i\in I} G_i = 1$; the set $\{G_i \mid i \in I\}$ is called a *filter* of G [1].

Baumslag has shown

THEOREM 1 [1]. *Suppose A and B are residually finite groups with filters $\{A_i \mid i \in I\}$ and $\{B_j \mid j \in J\}$, respectively. The group $S = A \overset{*}{_U} B$ will be residually finite provided*

(i) $\{A_i \cap U \mid i \in I\} = \{B_j \cap U \mid j \in J\}$

and

(ii) $\bigcap_I A_i U = U = \bigcap_J B_j U$.

Now suppose the nontrivial product $S = A \overset{*}{_U} B$ is residually finite with filter $\{S_k \mid k \in K\}$. Set $A_k = S_k \cap A$ and $B_k = S_k \cap B$ for each $k \in K$. Clearly $\{A_k \mid k \in K\}$ and $\{B_k \mid k \in K\}$ are filters of A and B satisfying (i) above. When do they also satisfy (ii)? Using the preceding notation we state

THEOREM 2. *Let $w(x_1, \cdots, x_n) = 1$ be a nontrivial identical relation on B. Then $\bigcap_{k\in K} A_k U = U$ if*

(a) *the index of U in B is greater than two, or*

(b) *$w(x_1, \cdots, x_n) = 1$ is not an identical relation of the infinite dihedral group.*

PROOF. First note that $\bigcap S_i = 1$ implies that any identical relation of B is an identical relation of $\bigcap_K BS_k = R$, for the map defined by $r \mapsto (rS_k)$ is a monomorphism of R into the cartesian product $\prod BS_k/S_k$.

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Now $R_1 = \langle \bigcap_K UA_k, B \rangle \subseteq R$, because $A_k \subseteq S_k$, so R_1 satisfies the identical relation $w(x_1, \cdots, x_n) = 1$. Put $X = \bigcap_K UA_k$. Then $R_1 = X *_U B$. That $X = U$ now follows from

LEMMA 3. *Suppose the generalized free product $G = H *_K L$, where $H \neq K \neq L$, satisfies a nontrivial identical relation. Then $[H:K] = [L:K] = 2$, and the infinite dihedral group satisfies the identical relation.*

PROOF. Suppose $[H:K] > 2$ and let h_1 and h_2 be elements in two different cosets of K in H but not in K. Choose $y \in L \backslash K$. Then $h_1 y h_1 y$ and $h_2 y h_2 y$ freely generate a free subgroup F of G, so in this case G satisfies no nontrivial identical relation.

Thus if G does satisfy a nontrivial identical relation, then $[H:K] = [L:K] = 2$, so K is normal in both H and L. Thus G/K, which is the infinite dihedral group, must satisfy the identical relation as required.

As examples we note

THEOREM 4. *Let A and B be finitely generated infinite nilpotent groups. Then $S = A *_U B$ is residually finite if and only if*

(i) *A and B have normal series $A = A_0 \supseteq A_1 \supseteq \cdots$, $B = B_0 \supseteq B_1 \supseteq \cdots$, such that $1 < [A_i : A_{i+1}]$, $[B_i : B_{i+1}] < \infty$ for all i,*

(ii) $\bigcap_i A_i = 1 = \bigcap_i B_i$,

(iii) $\{U \cap A_i\} = \{U \cap B_i\}$ *and*

(iv) $\bigcap_i UA_i = U = \bigcap_i UB_i$.

THEOREM 5. *Let A and B be finitely generated infinite nilpotent groups and let p be a prime. Then $S = A *_U B$ is residually a finite p-group if and only if* (i) *A and B have normal series $A = A_0 \supseteq A_1 \supseteq \cdots$, $B = B_0 \supseteq B_1 \supseteq \cdots$, such that $[A_i : A_{i+1}] = p = [B_i : B_{i+1}]$ for all i, and such that* (ii), (iii) *and* (iv) *of Theorem 4 hold.*

COROLLARY 6. *Let $S = gp(a, b \mid a^h = b^k)$ and let p be a prime. Then S is residually a finite p-group if and only if*

(a) *both h and k are powers of p, or*

(b) $h = 1$ *or* $k = 1$.

Theorem 4 follows immediately from Theorem 1 and Theorem 2. Theorem 5 follows from an easy extension of Theorem 1 using the main result of Higman [3]. The (well-known) corollary to Theorem 5 follows because property (iv) fails when $h \neq 1$ and $k \neq 1$ are not both powers of p.

Although special cases of Lemma 3 are well known (see for example [4, p. 217, Problem 10]), the proof of the general case given here is due to Professor B. H. Neumann. I thank Professor Neumann for his help with this.

References

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