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## ON RESIDUALLY FINITE GENERALIZED FREE PRODUCTS

R. J. GREGORAC

Let  $S = A \overset{*}{U} B$  be a nontrivial generalized free product of the groups  $A$  and  $B$  with amalgamated subgroup  $U$  and suppose  $A$  and  $B$  are residually finite groups. Baumslag [1] has given conditions sufficient for  $S = A \overset{*}{U} B$  to itself be residually finite and these have been used to investigate the residual finiteness of  $S$  by Baumslag [1] and Dyer [2], when the factors  $A$  and  $B$  of  $S$  are assumed to satisfy certain additional properties. The question then arises as to the necessity of these conditions. It is shown here that Baumslag's conditions are in fact necessary, provided  $A$  and  $B$  satisfy suitable identical relations.

A group  $G$  is *residually finite* if there exists a set  $\{G_i \mid i \in I\}$  of normal subgroups of  $G$  such that  $G/G_i$  is finite for each  $i \in I$  and  $\bigcap_{i \in I} G_i = 1$ ; the set  $\{G_i \mid i \in I\}$  is called a *filter* of  $G$  [1].

Baumslag has shown

**THEOREM 1 [1].** *Suppose  $A$  and  $B$  are residually finite groups with filters  $\{A_i \mid i \in I\}$  and  $\{B_j \mid j \in J\}$ , respectively. The group  $S = A \overset{*}{U} B$  will be residually finite provided*

$$(i) \{A_i \cap U \mid i \in I\} = \{B_j \cap U \mid j \in J\}$$

and

$$(ii) \bigcap_I A_i U = U = \bigcap_J B_j U.$$

Now suppose the nontrivial product  $S = A \overset{*}{U} B$  is residually finite with filter  $\{S_k \mid k \in K\}$ . Set  $A_k = S_k \cap A$  and  $B_k = S_k \cap B$  for each  $k \in K$ . Clearly  $\{A_k \mid k \in K\}$  and  $\{B_k \mid k \in K\}$  are filters of  $A$  and  $B$  satisfying (i) above. When do they also satisfy (ii)? Using the preceding notation we state

**THEOREM 2.** *Let  $w(x_1, \dots, x_n) = 1$  be a nontrivial identical relation on  $B$ . Then  $\bigcap_{k \in K} A_k U = U$  if*

(a) *the index of  $U$  in  $B$  is greater than two, or*

(b)  *$w(x_1, \dots, x_n) = 1$  is not an identical relation of the infinite dihedral group.*

**PROOF.** First note that  $\bigcap S_i = 1$  implies that any identical relation of  $B$  is an identical relation of  $\bigcap_K B S_k = R$ , for the map defined by  $r \mapsto (r S_k)$  is a monomorphism of  $R$  into the cartesian product  $\prod B S_k / S_k$ .

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Now  $R_1 = \langle \bigcap_K UA_k, B \rangle \subseteq R$ , because  $A_k \subseteq S_k$ , so  $R_1$  satisfies the identical relation  $w(x_1, \dots, x_n) = 1$ . Put  $X = \bigcap_K UA_k$ . Then  $R_1 = X \overset{*}{U} B$ . That  $X = U$  now follows from

**LEMMA 3.** *Suppose the generalized free product  $G = H \overset{*}{K} L$ , where  $H \neq K \neq L$ , satisfies a nontrivial identical relation. Then  $[H:K] = [L:K] = 2$ , and the infinite dihedral group satisfies the identical relation.*

**PROOF.** Suppose  $[H:K] > 2$  and let  $h_1$  and  $h_2$  be elements in two different cosets of  $K$  in  $H$  but not in  $K$ . Choose  $y \in L \setminus K$ . Then  $h_1 y h_1 y$  and  $h_2 y h_2 y$  freely generate a free subgroup  $F$  of  $G$ , so in this case  $G$  satisfies no nontrivial identical relation.

Thus if  $G$  does satisfy a nontrivial identical relation, then  $[H:K] = [L:K] = 2$ , so  $K$  is normal in both  $H$  and  $L$ . Thus  $G/K$ , which is the infinite dihedral group, must satisfy the identical relation as required.

As examples we note

**THEOREM 4.** *Let  $A$  and  $B$  be finitely generated infinite nilpotent groups. Then  $S = A \overset{*}{U} B$  is residually finite if and only if*

- (i)  $A$  and  $B$  have normal series  $A = A_0 \supseteq A_1 \supseteq \dots$ ,  $B = B_0 \supseteq B_1 \supseteq \dots$ , such that  $1 < [A_i: A_{i+1}]$ ,  $[B_i: B_{i+1}] < \infty$  for all  $i$ ,
- (ii)  $\bigcap_i A_i = 1 = \bigcap_i B_i$ ,
- (iii)  $\{U \cap A_i\} = \{U \cap B_i\}$  and
- (iv)  $\bigcap_i UA_i = U = \bigcap_i UB_i$ .

**THEOREM 5.** *Let  $A$  and  $B$  be finitely generated infinite nilpotent groups and let  $p$  be a prime. Then  $S = A \overset{*}{U} B$  is residually a finite  $p$ -group if and only if (i)  $A$  and  $B$  have normal series  $A = A_0 \supseteq A_1 \supseteq \dots$ ,  $B = B_0 \supseteq B_1 \supseteq \dots$ , such that  $[A_i: A_{i+1}] = p = [B_i: B_{i+1}]$  for all  $i$ , and such that (ii), (iii) and (iv) of Theorem 4 hold.*

**COROLLARY 6.** *Let  $S = gp(a, b \mid a^h = b^k)$  and let  $p$  be a prime. Then  $S$  is residually a finite  $p$ -group if and only if*

- (a) both  $h$  and  $k$  are powers of  $p$ , or
- (b)  $h = 1$  or  $k = 1$ .

Theorem 4 follows immediately from Theorem 1 and Theorem 2. Theorem 5 follows from an easy extension of Theorem 1 using the main result of Higman [3]. The (well-known) corollary to Theorem 5 follows because property (iv) fails when  $h \neq 1$  and  $k \neq 1$  are not both powers of  $p$ .

Although special cases of Lemma 3 are well known (see for example [4, p. 217, Problem 10]), the proof of the general case given here is

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