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Polycyclic groups with isomorphic finite quotients

By F. J. GRUNEWALD, P. F. PICKEL and D. SEGAL

1. Introduction

Let $\mathcal{F}(G)$ denote the set of isomorphism classes of finite quotients of a group G. For polycyclic-by-finite groups G and H, $\mathcal{F}(G) = \mathcal{F}(H)$ if and only if the profinite completions $\hat{G}$ and $\hat{H}$ of G and H are isomorphic (see 2.3 below). When this holds we say that *G and H belong to the same $\hat{}$-class.* The purpose of this paper is to prove the following

THEOREM. *Every $\hat{}$-class of polycyclic-by-finite groups is the union of finitely many isomorphism classes.*

This result represents the culmination of work that the authors have been engaged in independently for some time; preliminary results on which the proof is based are summarized below. The authors are grateful to the Universities of Warwick and Bielefeld for bringing them together and making the present outcome possible.

Preliminary results

$\mathfrak{P}$ denotes the class of polycyclic groups and $\mathfrak{P}\mathfrak{F}$ the class of polycyclic-by-finite groups.

THEOREM A (Segal [S]). *Let G be a $\mathfrak{P}\mathfrak{F}$ group and d a positive integer. Then up to isomorphism there exist only finitely many groups containing G as a normal subgroup of index d.*

THEOREM B (Pickel [P1]). *Every $\hat{}$-class of finitely generated nilpotent groups is the union of finitely many isomorphism classes.*

THEOREM C (Pickel [P3]). *Let G be a $\mathfrak{P}\mathfrak{F}$ group with Fitting subgroup N. Then $\hat{N}$ is the Fitting subgroup of $\hat{G}$.*

To state the next result we need a definition. Let $\mathcal{G}$ be an algebraic matrix group of degree n, say, defined over $\mathbf{Q}$. To $\mathcal{G}$ one associates the group

$$\mathcal{G}^{\infty} = \prod_p \mathcal{G}(\mathbf{Z}_p) \ ,$$

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the Cartesian product over all primes p of the groups $\mathcal{G}(\mathbf{Z}_p)$; here $\mathbf{Z}_p$ denotes the ring of p-adic integers and $\mathcal{G}(\mathbf{Z}_p)$ the zero-set in $\mathrm{GL}_n(\mathbf{Z}_p)$ of the rational polynomials which define the algebraic group $\mathcal{G}$. The group $\mathcal{G}(\mathbf{Z})$ appears as a subgroup of $\mathcal{G}^\infty$ if one identifies $g \in \mathcal{G}(\mathbf{Z})$ with the element of $\mathcal{G}^\infty$ each of whose coordinates is equal to g. Now for each positive integer m there is a canonical ring epimorphism

$$\pi_m\colon \prod_p \mathbf{Z}_p \longrightarrow \mathbf{Z}/m\mathbf{Z}$$

(by the Chinese Remainder Theorem), and it induces a homomorphism—to which we give the same symbol—

$$\pi_m\colon \mathcal{G}^\infty \longrightarrow \mathrm{GL}_n(\mathbf{Z}/m\mathbf{Z}) .$$

(Note that this homomorphism is *surjective* in the special case where $\mathcal{G} = \mathrm{GL}_n$; this is easy to prove.) We define an equivalence relation $\sim_{\mathcal{G}}$ on the set of all subgroups of $\mathcal{G}(\mathbf{Z})$ as follows:

$X \sim_{\mathcal{G}} Y$ if and only if $X\pi_m$ is conjugate to $Y\pi_m$ in $\mathcal{G}^\infty\pi_m$ for every positive integer m.

We can now state

THEOREM D (Grunewald, Segal [GS3]). *Every $\sim_{\mathcal{G}}$-class of soluble subgroups of $\mathcal{G}(\mathbf{Z})$ is the union of finitely many conjugacy classes in $\mathcal{G}(\mathbf{Z})$.*

A quick alternative proof of Theorem B, based on Theorem D, is indicated in Section 3. Further special cases of the Theorem were established in [P2] and [GS1]; these will not be needed here.

Outline of the paper

To prove the Theorem, we consider a set $\mathcal{C}$ of $\mathfrak{PF}$ groups contained in a single $\hat{\ }$-class. In view of Theorems C and B, we may assume that the groups in $\mathcal{C}$ all have the same Fitting subgroup N, say. We apply Theorem D a first time to reduce to the situation where the groups in $\mathcal{C}$ all act in the same way on N; in particular, if Z denotes the centre of N, we may assume that the groups G/Z are all isomorphic. This information is used in order to bound various parameters that play an important technical role in the following argument.

So far the proof is straightforward. To go further we have to introduce an auxiliary construction, which has some interest in itself. For $G \in \mathcal{C}$ we produce a finitely generated abelian subgroup $T_G \leqq \mathrm{Aut}\, G$, such that the split extension $G]T_G$ is equal to $M_G]T_G$, where M_G is the Fitting subgroup of $G]T_G$. This is a so-called *semisimple splitting* of G; the idea has been used by several authors, notably H.C. Wang [Wa], Tolimieri [T] and L. Auslander [A], Chapter IV. However their methods are too unwieldy for our purpose, and we present a more streamlined approach which we feel sheds some new light on the matter. Specifically, we show in Section 5 that (under suitable conditions) there exists a *nilpotent supplement* C for N in G. Then C induces by conjugation a group Δ of automorphisms of N; as Δ is nilpotent, the Jordan semisimple components of the elements of Δ form a group Δ_s, say. In Section 7 we show how to define a group $T_G \leqq \operatorname{Aut} G$ which acts like Δ_s on N and centralizes C, and establish that T_G has the required properties. Incidentally, the reader interested only in the existence of semisimple splittings can extract a fairly simple proof of this by ignoring many of the finer details in Sections 5 and 7; in the present context we also need to know that the splittings can be constructed in a canonical manner, and that they then enjoy a certain uniqueness property. Some finiteness lemmas connected with this are established in Sections 4 and 6.

There is a well known method for embedding a group of the form $M]T$ into a suitable $\mathrm{GL}_n(\mathbf{Z})$, when M is a finitely generated nilpotent group and $T \leqq \operatorname{Aut} M$: one makes M act by right multiplication and T act by conjugation on a suitable factor ring of the group ring $\mathbf{Z}M$. The details are recalled in Section 3. To finish the proof of the theorem, we use this technique to embed the groups $M_G]T_G$ into $\mathrm{GL}_n(\mathbf{Z})$. The degree n will be the same for all $G \in \mathcal{C}$, and we are able to show that the images in $\mathrm{GL}_n(\mathbf{Z})$ of the groups $G \in \mathcal{C}$ lie in finitely many $\sim_{\mathrm{GL}_n}$-classes of $\mathrm{GL}_n(\mathbf{Z})$. Another application of Theorem D then finishes the proof.

The main body of the proof occupies Sections 8–15. The proof is divided into six steps in Section 8 and these are proved in the succeeding sections. Section 2 begins with the definitions and elementary properties of some technical constructions which are used throughout.

Notation

$\mathfrak{P}$, $\mathfrak{P}\mathfrak{F}$, $\mathfrak{T}$ denote respectively the classes of all polycyclic, polycyclic-by-finite, and finitely generated torsion-free nilpotent groups.

$\operatorname{Fitt}(G)$ = Fitting subgroup of G; if $G \in \mathfrak{P}\mathfrak{F}$, $\operatorname{Fitt}(G)$ is the unique maximal nilpotent normal subgroup of G.

G' = derived group of G, $\gamma_i(G) = i^{\text{th}}$ term of the lower central series of G.

$\zeta_1(G)$ = centre of G, $\zeta_i(G) = i^{\text{th}}$ term of the upper central series of G,
$\zeta(G)$ = hypercentre of G.

$G^k = \langle g^k \mid g \in G \rangle$ for a positive integer k.

$g^h = h^{-1}gh$; $[g, h] = g^{-1}g^h$; $[G, H] = \langle [g, h] \mid g \in G, h \in H \rangle$; $[G,_m H] = [[G,_{m-1} H], H]$ for $m > 1$.

$\hat{G}$ = profinite completion of G (see § 2).

If N is a $\mathfrak{T}$ group,

$N^{\mathbf{Q}}$ = Mal'cev completion of N (see § 2),

$N^{1/k} = \langle g^{1/k} \mid g \in N \rangle \leqq N^{\mathbf{Q}}$ for a positive integer k.

$i_N(H)$ = isolator in N of H for $H \leqq N$ (see § 2).

$K]H$ = split extension of (normal subgroup) K by H.

$H \leqq_f G$ means H is a subgroup of finite index in G,

$H \lhd_f G$ means $H \lhd G$ and $H \leqq_f G$.

$\mathbf{Z}_p$ = p-adic integers, $\mathbf{Q}_p$ = p-adic numbers.

For a linear algebraic group $\mathcal{G}$ defined over $\mathbf{Q}$,

$\mathcal{G}^\infty = \prod_p \mathcal{G}(\mathbf{Z}_p)$, the Cartesian product being over all (finite) primes of $\mathbf{Q}$.

References within a Section n take the form "Lemma k" for Lemma k of Section n, "Lemma m.k" for Lemma k of Section m when m $\neq$ n.

2. Notes on completions

In this section we recall definitions and properties of some constructions that will be used in the paper.

Profinite completions. The *profinite topology* on a group Γ is the uniform topology on Γ with a neighbourhood basis of the identity given by the normal subgroups of finite index in Γ. The *profinite completion* $\hat{\Gamma}$ of Γ is the completion of Γ in this topology. If Γ is residually finite, the topology is Hausdorff and Γ embeds naturally in $\hat{\Gamma}$.

Now consider a group $G \in \mathfrak{PF}$; G is residually finite. The normal subgroups of finite index in G form a directed set, and the subgroups G^k for all positive integers k form a cofinal subset, as do the subgroups $G^{n!}$ for all positive integers n. Thus

$$\begin{aligned} \hat{G} &= \varprojlim \{G/N \mid N \lhd_f G\} \\ &= \varprojlim \{G/G^k \mid 0 < k \in \mathbf{Z}\} \qquad (1) \\ &= \varprojlim \{G/G^{n!} \mid 0 < n \in \mathbf{Z}\} . \end{aligned}$$

This last representation gives a representation of $\hat{G}$ in terms of Cauchy sequences (a_n) with $a_n a_{n+1}^{-1} \in G^{n!}$ for each n. A null sequence is a sequence (a_n) with $a_n \in G^{n!}$ for each n, and

$$\hat{G} = \{\textit{Cauchy sequences}\}/\{\textit{null sequences}\} .$$

A Cauchy sequence represents an element of G if and only if it is a constant multiple of some null sequence.

It is a well known theorem of Mal'cev that every subgroup of G is the intersection of the subgroups of finite index which contain it ([Ma]; see also 2.5 below); it follows that if $H \leqq G$ then H is closed in the profinite topology of G and this topology induces the profinite topology on H. Thus $\hat{H}$ may be identified with a subgroup of $\hat{G}$; in fact $\hat{H}$ is the closure in $\hat{G}$ of H (in the natural topology of $\hat{G}$ induced from the profinite topology on G). If $H \lhd G$ then $\hat{H} \lhd \hat{G}$ and by considering Cauchy sequences one sees that $\hat{G}/\hat{H} \cong (G/H)^\wedge$.

In particular, taking $H = G^k$ we obtain

$$G/G^k \cong (G/G^k)^\wedge \cong \hat{G}/(G^k)^\wedge \; ;$$

the first isomorphism comes from the fact that a finite group is equal to its profinite completion. Now it is easy to see that

$$G^k \leqq \hat{G}^k \leqq (G^k)^\wedge \; , \tag{2}$$

so that $(G^k)^\wedge$ is the closure of $\hat{G}^k$. Unfortunately it is not in general an easy matter to identify the closure of a subgroup in a profinite group; thus, for example, given groups Γ and Δ and an isomorphism $\hat{\Gamma} \cong \hat{\Delta}$, we may at once deduce that the isomorphism must send $\hat{\Gamma}^k$ onto $\hat{\Delta}^k$, but without further information one cannot then deduce that the closure of $\hat{\Gamma}^k$ gets mapped to the closure of $\hat{\Delta}^k$ (there is thus a gap in the proof of Proposition 1 of [P2]). When the groups are polycyclic-by-finite, the extra information required is provided by

PROPOSITION 2.1. i) *If G is in $\mathfrak{PF}$ then the natural topology on $\hat{G}$ coincides with the profinite topology.*

ii) *If G is in $\mathfrak{PF}$ then*

$$\hat{G}^k = (G^k)^\wedge$$

for every positive integer k.

Part (i) will often be used implicitly; its most important consequence is that every isomorphism $\hat{G} \cong \hat{H}$ for $\mathfrak{PF}$ groups G and H is a topological isomorphism.

We shall frequently need the following observations, which are easily verified by considering Cauchy sequences:

LEMMA 2.2. *Let G be in $\mathfrak{PF}$.* i) *If $H \leqq_f G$ and $N \leqq G$ then*

$$\hat{H} \cap \hat{N} = (H \cap N)^\wedge \; .$$

ii) *If $H \leqq_f G$ and $N \lhd G$ then*

$$(HN)^\wedge = \hat{H}N \; .$$

iii) *If $H \leqq G$ and $N \lhd G$ then*

$$(HN)\hat{} = \hat{H}\hat{N} .$$

Proof of 2.1. (ii) In view of (2) above, it remains to show that $(G^k)\hat{} \leqq \hat{G}^k$. If G is either finite or free abelian this is easy to see. Suppose G is infinite and let $N \neq 1$ be a free abelian normal subgroup of G; then the result holds for N, and by induction on the Hirsch number we may assume that it holds for G/N also.

Now there exists $l > 0$ such that $G^l \cap N \leqq N^k$, and without loss of generality we take l so that $k \mid l$. By inductive hypothesis,

$$\big((G/N)\hat{}\big)^l = \big((G/N)^l\big)\hat{} ,$$

from which it follows, by Lemma 2.2 (ii), that

$$\hat{G}^l\hat{N} = (G^lN)\hat{} = (G^l)\hat{}N .$$

Therefore

$$(G^l)\hat{} = \hat{G}^l\big(\hat{N} \cap (G^l)\hat{}\big) = \hat{G}^l(N \cap G^l)\hat{}$$

by Lemma 2.2 (i). But $N \cap G^l \leqq N^k$ and $(N^k)\hat{} = \hat{N}^k$ by the choice of N, so

$$(G^l)\hat{} \leqq \hat{G}^l\hat{N}^k \leqq \hat{G}^k .$$

Now taking $N = G$ and $H = G^l$ in Lemma 2.2 (ii), we obtain

$$\hat{G} = (G^l)\hat{}G; \text{ thus } \hat{G} = \hat{G}^kG \text{ and so}$$

$$|\hat{G}: \hat{G}^k| = |G: G \cap \hat{G}^k| \leqq |G: G^k| = |\hat{G}: (G^k)\hat{}| .$$

Since $\hat{G}^k \leqq (G^k)\hat{}$ it follows that $\hat{G}^k = (G^k)\hat{}$.

Proof of 2.1. (i) The subgroups $(G^k)\hat{}$ form a base for the neighbourhoods of 1 in the natural topology on $\hat{G}$. Now by part (ii), each subgroup $\hat{G}^k$ has finite index in $\hat{G}$, so the subgroups $\hat{G}^k$ form a base for the neighbourhoods of 1 in the profinite topology on $\hat{G}$. Hence by part (ii) again, the two topologies coincide.

COROLLARY 2.3. *If G and H are in $\mathfrak{PF}$, then $\mathcal{F}(G) = \mathcal{F}(H)$ if and only if $\hat{G} \cong \hat{H}$.*

Proof. "If" follows from Proposition 2.1 (ii) and the remarks preceding it. "Only if" follows from the representation (1) above; cf. [P2], Proposition 1.

The congruence topology. If G is a subgroup of $\mathrm{GL}_n(\mathbf{Z})$, one defines the *congruence topology* on G by taking as a base for the neighbourhoods of 1 the *congruence subgroups*

$$G_k = \{g \in G \mid g \equiv 1 \bmod k\}$$

for positive integers k. It is clear that this topology is contained in the profinite topology, for any G; in fact we have

THEOREM 2.4 (Formanek [F], Theorem 2; Wehrfritz [We 2], Theorem 2). *If $G \in \mathfrak{PF}$ then G is closed in the congruence topology on $\mathrm{GL}_n(\mathbf{Z})$ and the congruence topology on G coincides with the profinite topology.*

Remark 2.5. This shows that every subgroup H of G is closed in the profinite topology on G and that this topology induces the profinite topology on H; as every $\mathfrak{PF}$ group G can be embedded in some $\mathrm{GL}_n(\mathbf{Z})$ (see Chapter IV of [A]) one gets an alternative proof of the theorem of Mal'cev mentioned above.

The completion of $\mathrm{GL}_n(\mathbf{Z})$ in the congruence topology is contained in

$$\mathrm{GL}_n^\infty = \prod_p \mathrm{GL}_n(\mathbf{Z}_p)$$

where the Cartesian product is taken over all primes p. As a consequence of Theorem 2.4 we have

COROLLARY 2.6. *Suppose $G \leqq \mathrm{GL}_n(\mathbf{Z})$ and $G \in \mathfrak{PF}$. Then $\hat{G}$ can be identified with the closure of G in the congruence topology on GL_n^∞.*

Completions of $\mathfrak{T}$ groups. Next we consider groups in $\mathfrak{T}$, that is, finitely generated torsion-free nilpotent groups. In any $\mathfrak{T}$ group N we can choose a generating set $\{x_0 = 1, x_1, \cdots, x_n\}$ such that for each $i \geqq 1$, x_i is central in N modulo $\langle x_0, \cdots, x_{i-1}\rangle$ and $N/\langle x_0, \cdots, x_{i-1}\rangle$ is again in $\mathfrak{T}$. Such a system $(x_1, \cdots, x_n)$ is called a *Mal'cev basis* for N; each element of N can then be uniquely expressed in the form

$$w(\boldsymbol{r}) = w(r_1, \cdots, r_n) = x_1^{r_1} \cdots x_n^{r_n} \tag{3}$$

with $r_1, \cdots, r_n \in \mathbf{Z}$. The group operations in N are given by

$$\begin{aligned} w(\boldsymbol{r}) \cdot w(\boldsymbol{s}) &= w\big(p_1(\boldsymbol{r}, \boldsymbol{s}), \cdots, p_n(\boldsymbol{r}, \boldsymbol{s})\big) , \\ w(\boldsymbol{r})^m &= w\big(q_1(m, \boldsymbol{r}), \cdots, q_n(m, \boldsymbol{r})\big) , \end{aligned} \tag{4}$$

where the p_i and q_i are certain polynomials in $2n$, respectively $n + 1$, variables. If we allow $r_1, \cdots, r_n$ to range over $\mathbf{Q}$, the set of expressions (3) with the operations (4) forms a group $N^{\mathbf{Q}}$, the *Mal'cev completion of* N: this is the unique (up to isomorphism) minimal radicable torsion-free nilpotent group containing N. Every element of $N^{\mathbf{Q}}$ has some positive power in N, and every automorphism of N extends uniquely to an automorphism of $N^{\mathbf{Q}}$. See Chapter 6 of [H].

For a positive integer k, we define $N^{1/k}$ to be the subgroup of $N^{\mathbf{Q}}$ generated by the k^{th} roots of all elements of N. Then automorphisms of N extend uniquely to automorphisms of $N^{1/k}$.

LEMMA 2.7. *N has finite index in $N^{1/k}$, and $N^{1/k} \in \mathfrak{T}$.*

Proof. The second claim follows from the first. Now it is easy to see that the factors of the upper central series of $N^{\mathbf{Q}}$ are finite-dimensional $\mathbf{Q}$-vector spaces; from this it follows that if $H \leqq K \leqq N^{\mathbf{Q}}$ and $K^m \leqq H$ for some $m > 0$, then $|K: H|$ is finite. Thus for the first claim it will suffice to show that there exists $m > 0$ such that $(N^{1/k})^m \leqq N$. In fact we shall prove that this holds with

$$m = k^{c(c+1)/2}$$

where c is the nilpotency class of N. Put $M = N^{1/k}$. Then every element of M is of the form $y_1 y_2 \cdots y_s$ for some s with $y_i^k \in N$ for each i. Since $\gamma_{c+1} M = 1$, the following claim implies our result:

Claim. If $Y = \langle y_1, \cdots, y_s \rangle \leqq M$ and $1 \leqq j \leqq c + 1$,

$$(y_1 y_2 \cdots y_s)^{k^{j(j+1)/2}} \in \langle y_1^k, \cdots, y_s^k \rangle \cdot \gamma_{j+1} Y . \tag{5}$$

Proof of claim. Clearly (5) holds with $j = 1$. Suppose that (5) holds for some $j \geqq 1$. Then there exist $w \in \langle y_1^k, \cdots, y_s^k \rangle$ and certain $(j + 1)$-fold commutators $r_1, \cdots, r_t$, say in the elements $y_1, \cdots, y_s$, such that the left-hand member of (5) is equal to

$$w r_1 r_2 \cdots r_t .$$

Raise this to the power k^{j+1} to obtain

$$(y_1 y_2 \cdots y_s)^{k^{(j+1)(j+2)/2}} = (w r_1 r_2 \cdots r_t)^{k^{j+1}} . \tag{6}$$

Now modulo $\gamma_{j+2} Y$, each $(j + 1)$-fold commutator is central and is homomorphic as a function of each argument; this shows that the right-hand member of (6) is an element of

$$\langle y_1^k, \cdots, y_s^k \rangle \cdot \gamma_{j+2} Y .$$

We deduce that (5) holds with $j + 1$ in place of j, and the claim follows by induction.

We turn now to profinite completions of $\mathfrak{T}$ groups. For a prime p, define the *pro-p topology* on the $\mathfrak{T}$ group N by taking the normal subgroups of p-power index in N to be a base for the neighbourhoods of 1. The completion of N in this topology is denoted $\hat{N}_p$. This group can be realized as the set of expressions (3) with $r_1, \cdots, r_n$ ranging over $\mathbf{Z}_p$, and group operations given by (4), see [P1], Lemma 1.3. Since every finite nilpotent group is the direct product of its Sylow subgroups, we have

$$\begin{aligned}\hat{N} &= \varprojlim \{N/N^k \mid 0 < k \in \mathbf{Z}\} \\ &= \textstyle\prod_p \varprojlim \{N/N^{p^i} \mid 0 < i \in \mathbf{Z}\} \\ &= \textstyle\prod_p \hat{N}_p \,.\end{aligned}$$

It is evident that $\hat{N}$ is nilpotent, of the same class as N.

$\mathfrak{T}$ *groups and Lie algebras.* For details of what follows, see [Ba], Chapter 4 and [P1], Section 3. Let N be a $\mathfrak{T}$ group. To the group $N^{\mathbf{Q}}$ is associated a finite-dimensional Lie algebra Λ over $\mathbf{Q}$ and a bijective map $\log\colon N^{\mathbf{Q}} \to \Lambda$ with inverse $\exp\colon \Lambda \to N^{\mathbf{Q}}$. (These may be visualized as applying the usual power series to a unipotent representation of $N^{\mathbf{Q}}$ by matrices over $\mathbf{Q}$, as in Section 3 below; as the matrices are unipotent the series will turn out to be polynomials.) We have

$$\log(xy) = \log(x) * \log(y)$$

where $*$ is given by the Baker-Campbell-Hausdorff formula, and for $q \in \mathbf{Q}$ we have

$$\log(x^q) = q\log(x)\,. \tag{7}$$

Group automorphisms of $N^{\mathbf{Q}}$ correspond bijectively to Lie algebra automorphisms of Λ via

$$\log(x^\phi) = (\log(x))^\phi\,.$$

We shall usually identify Λ with $N^{\mathbf{Q}}$ via log, and hence identify $\mathrm{Aut}(N^{\mathbf{Q}})$ with $\mathrm{Aut}(\Lambda)$. Now there is an algebraic $\mathbf{Q}$-group $\mathcal{G}$ such that

$$\mathrm{Aut}(\Lambda) = \mathcal{G}(\mathbf{Q})\,;$$

($\mathcal{G}(\mathbf{Q})$ is the subgroup of $\mathrm{GL}(\Lambda)$ consisting of those matrices which satisfy certain polynomial equations, namely the equations which say that the Lie multiplication is preserved).

The group N in $\mathfrak{T}$ is called *lattice nilpotent* if the subset $\log(N)$ of $\Lambda = \log(N^{\mathbf{Q}})$ is an additive subgroup of Λ. While not every N in $\mathfrak{T}$ is lattice nilpotent, each such N is contained as a subgroup of finite index in a lattice nilpotent group M with $N \leqq M \leqq N^{\mathbf{Q}}$ ([Mo], Theorem 2). The intersection $\bar{N}$ of all lattice nilpotent subgroups of $N^{\mathbf{Q}}$ containing N is called the *lattice hull* of N; clearly $\bar{N}$ is lattice nilpotent, $|\bar{N}\colon N|$ is finite, and every automorphism of N extends uniquely to an automorphism of $\bar{N}$.

Now assume N is lattice nilpotent. Then $\log(N)$ really is a lattice in Λ, i.e., a free $\mathbf{Z}$-submodule which spans Λ over $\mathbf{Q}$; this follows from (7) and the fact that $\log(N)$ is then additively finitely generated. We need the following result of Blackburn:

LEMMA 2.8 [Bl]. *For each prime p and each positive integer c there is an integer $d \geqq 0$ such that for every $\mathfrak{T}$ group N of class c and every $i \geqq d$, every product of p^{i}'th powers in N is a $p^{(i-d)}$'th power.*

Applying this to our lattice nilpotent N we deduce that for each $i \geqq d$,

$$p^{i}\log(N) \leqq \log(N^{p^{i}}) \leqq p^{i-d}\log(N) .$$

This shows that log is a homeomorphism of N with its pro-p topology onto $\log(N)$ with its pro-p topology (as additive group), i.e., its p-adic topology. We may thus extend log to a homeomorphism between the respective completions

$$\log\colon \hat{N}_{p} \longrightarrow \mathbf{Z}_{p}\otimes_{\mathbf{Z}}\log(N) ;$$

as before we shall use this to identify $\hat{N}_{p}$ with $\mathbf{Z}_{p}\otimes_{\mathbf{Z}}\log(N)$. Under this identification, the various automorphism groups are related as follows (we choose a $\mathbf{Z}$-basis of $\log(N)$ to be a $\mathbf{Q}$-basis of Λ, thereby specifying a concrete representation of $\mathcal{G} = \mathrm{Aut}(\Lambda)$ by matrices):

LEMMA 2.9 ([P1], Proposition 2.4). *Suppose N is lattice nilpotent. Then there exist an algebraic $\mathbf{Q}$-group $\mathcal{G}$ and compatible isomorphisms*

(a) $\mathrm{Aut}(N^{\mathbf{Q}}) \cong \mathcal{G}(\mathbf{Q})$;

(b) $\mathrm{Aut}(N) \cong \mathcal{G}(\mathbf{Z})$;

(c) $\mathrm{Aut}(\hat{N}_{p}) \cong \mathcal{G}(\mathbf{Z}_{p})$ *for each prime p*;

(d) $\mathrm{Aut}(\hat{N}) = \prod_{p}\mathrm{Aut}(\hat{N}_{p}) \cong \prod_{p}\mathcal{G}(\mathbf{Z}_{p}) = \mathcal{G}^{\infty}$.

Extensions of $\mathfrak{PF}$ groups. Consider next a $\mathfrak{PF}$ group G with a normal $\mathfrak{T}$ subgroup N. Let $\{g_{\alpha} \mid \alpha \in A\}$ be a transversal to the cosets of N in G. Then every element of G can be written uniquely in the form $x \cdot g_{\alpha}$ with $x \in N$ and $\alpha \in A$. We have

$$(xg_{\alpha})(yg_{\beta}) = xy^{\phi_{\alpha}}f(\alpha, \beta)g_{\gamma} \tag{8}$$

where ϕ_{α} is the automorphism of N given by conjugation with g_{α}^{-1}, g_{γ} is the coset representative of $g_{\alpha}g_{\beta}$, and $f\colon A \times A \to N$ is the 2-cocycle defined by $g_{\alpha}g_{\beta} = f(\alpha, \beta)g_{\gamma}$. Using this data we define the group $N^{\mathbf{Q}}G$ to consist of expression xg_{α} with $x \in N^{\mathbf{Q}}$, $\alpha \in A$, and with multiplication defined by (8): this is possible because $f(\alpha, \beta) \in N \leqq N^{\mathbf{Q}}$ and ϕ_{α} extends uniquely to an automorphism of $N^{\mathbf{Q}}$. If $N \leqq \bar{N} \leqq N^{\mathbf{Q}}$ and $\bar{N}$ is "normalized by G", i.e., $\bar{N}$ is invariant under the automorphisms ϕ_{α}, we define $\bar{N}G \leqq N^{\mathbf{Q}}G$ similarly. Any automorphism of G stabilizing N has an obvious extension to $N^{\mathbf{Q}}G$.

LEMMA 2.10. *Let $\bar{N}$ and G be as above, with $\bar{N}$ normalized by G. Then*

(a) *$\bar{N} \lhd \bar{N}G$ and $\bar{N}G/\bar{N} \cong G/N$;*

(b) *If $|\bar{N}\colon N| < \infty$, then $|\bar{N}G\colon G| = |\bar{N}\colon N| = |(\bar{N}G)^{\wedge}\colon \hat{G})|$;*

(c) *Suppose* $M \lhd H$ *with* $M \in \mathfrak{T}$ *and* $H \in \mathfrak{PF}$, *and* $\theta\colon \hat{H} \xrightarrow{\sim} \hat{G}$ *sends* $\hat{M}$ *to* $\hat{N}$. *Then* θ *extends to an isomorphism* $(M^{1/k})^\wedge \hat{H} \to (N^{1/k})^\wedge \hat{G}$.

Proof. (a) is clear from (8). For (b), let $\{t_i \mid i \in I\}$ be a transversal to the left cosets of N in $\bar{N}$. The same set is then a transversal to the left cosets of G in $\bar{N}G$. Since $\bar{N}$ is residually finite, there exists $k > 0$ such that the elements t_i lie in distinct cosets of N modulo $\bar{N}^k$. Thus if $(t_{i(j)}x_j)$ with $x_j \in N$ is a Cauchy sequence in $\bar{N}$, the sequence $(t_{i(j)})$ must be eventually constant. Hence the t_i are representatives for the left cosets of $\hat{N}$ in $(\bar{N})^\wedge$ and thus also for the left cosets of $\hat{G}$ in $(\bar{N}G)^\wedge$. A similar argument using also Lemma 2.2 (ii), (iii) gives (c).

Recall that the *isolator* $i(H) = i_N(H)$ of a subgroup H in a nilpotent group N is the set of elements $x \in N$ such that x has some positive power in H; then $i(H)$ is a subgroup of N, $ii(H) = i(H)$, and if N is finitely generated then $|i(H)\colon H| < \infty$. We say H is *isolated* if $i(H) = H$. For all this, see Chapter 4 of [H]. Note that if $H \leqq N \in \mathfrak{T}$ then

$$i_N(H) = H^{\mathbf{Q}} \cap N\,.$$

LEMMA 2.11. *Suppose* $G \in \mathfrak{PF}$, $N \lhd G$ *is in* $\mathfrak{T}$ *and* Q *is an isolated subgroup of* N *with* $Q \lhd G$. *Let* k *be a positive integer and put*

$$Q^* = i_{N^{1/k}}(Q) = Q^{\mathbf{Q}} \cap N^{1/k}\,.$$

Then $N^{1/k}/Q^*$ *can be identified with* $(N/Q)^{1/k}$ *and* $(N^{1/k}G)/Q^*$ *can be identified with* $(N/Q)^{1/k}\,(G/Q)$.

Proof. Consider the map $\theta\colon N \to N^{1/k}/Q^*$ induced by the inclusion $N \to N^{1/k}$. Then $\operatorname{Ker}\theta = N \cap Q^* = Q$ since Q is isolated in N. Since each element of $N\theta$ has a k^{th} root, and $N^{1/k}/Q^*$ is generated by k^{th} roots of elements of $N\theta$, θ induces an isomorphism of $(N/Q)^{1/k}$ onto $N^{1/k}/Q^*$.

For the second part, we use the first part and the observation that a transversal to the right cosets of N in G maps onto a transversal to the right cosets of N/Q in G/Q; thus we may identify the right cosets of $N^{1/k}/Q^*$ in $(N^{1/k}G)/Q^*$ with corresponding cosets of $(N/Q)^{1/k}$ in $(N/Q)^{1/k}\,(G/Q)$.

LEMMA 2.12. *Let* $G \in \mathfrak{PF}$ *and suppose* $N = \operatorname{Fitt}(G) \in \mathfrak{T}$. *Let* $Q = \zeta(G)$ *be the hypercentre of* G; *then* $Q \leqq N$. *Let* $N \leqq_f \bar{N} \leqq N^{\mathbf{Q}}$ *with* $\bar{N}$ *normalized by* G. *Then*

(a) $\bar{N} = \operatorname{Fitt}(\bar{N}G)$;

(b) $Q^* = \bar{N} \cap Q^{\mathbf{Q}}$ *is the hypercentre of* $\bar{N}G$;

(c) $\hat{N} = \operatorname{Fitt}(\hat{G})$;

(d) $\hat{Q}$ *is the hypercentre of* $\hat{G}$.

Proof. (a) and (b) are straightforward and (c) is Theorem C. We prove (d). There exists m such that $[Q, {}_mG] = 1$; considering Cauchy sequences, one deduces that $[\hat{Q}, {}_m\hat{G}] = 1$, so certainly $\hat{Q} \leqq \zeta(\hat{G})$. In view of (c), it will therefore suffice to show that $\hat{G}$ acting by conjugation has no fixed points in $\hat{N}/\hat{Q}$. Now it is not hard to verify that N/Q is torsion-free, so we may choose a Mal'cev basis $x_1, \cdots, x_n$ for N such that $Q = \langle x_1, \cdots, x_r\rangle$ for some r, with $r < n$ (if $r = n$ the result follows from (c)); we may also suppose that $x_{r+1}, \cdots, x_{r+s}$, say, generate the centre of N modulo Q, which we shall call Z, i.e.,

$$Z/Q = \zeta_1(N/Q) = Q\langle x_{r+1}, \cdots, x_{r+s}\rangle/Q .$$

Now it follows from the structure of $\hat{N}$ that $\hat{Z}/\hat{Q}$ is the centre of $\hat{N}/\hat{Q}$; hence if $\hat{G}$ has nontrivial fixed points in $\hat{N}/\hat{Q}$, it also has some in $\hat{Z}/\hat{Q}$. Hence there exist, for each prime p, p-adic integers $\lambda_1, \cdots, \lambda_s$, not all zero, such that $\prod x_{r+i}^{\lambda_i}$ is centralized by G modulo $\hat{Q}_p$. This is equivalent to the existence of a nontrivial p-adic solution to a certain finite set of homogeneous linear equations over $\mathbf{Z}$, corresponding to conjugation by a finite set of generators for G. But then there must exist a nontrivial solution in $\mathbf{Z}$ for the equations, which means that G has a nontrivial fixed point in Z/Q. This is impossible by the definition of Q. (One uses a similar argument to show that the centre of $\hat{N}/\hat{Q}$ is $\hat{Z}/\hat{Q}$.)

3. Representations of polycyclic groups

Let N be a $\mathfrak{T}$ group of class c, with integral augmentation ideal $\mathfrak{n}$. Let $T_N/\mathfrak{n}^{c+1}$ be the $\mathbf{Z}$-torsion submodule of $\mathbf{Z}N/\mathfrak{n}^{c+1}$, and define

$$V(N) = \mathbf{Z}N/T_N .$$

LEMMA 1. *There is an injective homomorphism $\beta = \beta_N\colon N]\mathrm{Aut}\,N \to \mathrm{Aut}\,V(N)$, such that*

$$(r + T_N)^{(\mu x)\beta} = (r\mu)^x + T_N$$

for all $r \in \mathbf{Z}N$, $\mu \in N$, $x \in \mathrm{Aut}\,N$.

We omit the proof as this is well known. It depends on the theorem of Jennings [J] that

$$(1 + \mathfrak{n}^{c+1}) \cap N = 1 .$$

LEMMA 2. *Suppose $0 \neq q \in \mathbf{Z}$ satisfies $qT_N \leqq \mathfrak{n}^{c+1}$. Let $0 \neq m \in \mathbf{Z}$. Then there exists $k > 0$ such that*

$$N^k \leqq (1 + \mathfrak{n}^{c+1} + mq\mathbf{Z}N) \cap N .$$

Put $\bar{N} = N/N^k$ and write $\bar{\mathfrak{n}}$ for the integral augmentation ideal of $\bar{N}$. Then there is a canonical isomorphism

$$V(N)/mV(N) \cong (q\mathbf{Z}\bar{N} + \bar{\mathfrak{n}}^{c+1})/(mq\mathbf{Z}\bar{N} + \bar{\mathfrak{n}}^{c+1}) .$$

Proof. Multiplication by q gives an isomorphism

$$V(N)/mV(N) \longrightarrow (q\mathbf{Z}N + \mathfrak{n}^{c+1})/(mq\mathbf{Z}N + \mathfrak{n}^{c+1}) .$$

Compose this map with that induced by the canonical isomorphism

$$\mathbf{Z}N/(N^k - 1)\mathbf{Z}N \longrightarrow \mathbf{Z}\bar{N} .$$

LEMMA 3. *Let M and N be $\mathfrak{X}$ groups and suppose $\hat{M} \cong \hat{N}$. Then there exists an isomorphism σ: $V(M) \to V(N)$, and if σ^*:* Aut $V(M) \to$ Aut $V(N)$ *denotes the induced isomorphism, then*

$$M\beta_M\sigma^* \sim N\beta_N \textit{ in } \operatorname{Aut} V(N) .$$

Here we are identifying $V(N)$ with $\mathbf{Z}^n$ by choosing a $\mathbf{Z}$-basis, so that Aut $V(N) = \mathrm{GL}_n(\mathbf{Z})$; and $\sim$ denotes $\sim_{\mathrm{GL}_n}$.

Proof. M and N have the same class, c, say. Choose $q > 0$ so that $qT_M \leqq \mathfrak{m}^{c+1}$ and $qT_N \leqq \mathfrak{n}^{c+1}$, and choose $k > 0$ so that

$$M^k \leqq (1 + \mathfrak{m}^{c+1} + 2q\mathbf{Z}M) \cap M \quad \text{and} \quad N^k \leqq (1 + \mathfrak{n}^{c+1} + 2q\mathbf{Z}N) \cap N .$$

As $\hat{M} \cong \hat{N}$ we have $M/M^k \cong N/N^k$; hence by Lemma 2 it follows that $V(M)/2V(M) \cong V(N)/2V(N)$. As $V(M)$ and $V(N)$ are free $\mathbf{Z}$-modules, they are isomorphic. Pick any isomorphism σ: $V(M) \to V(N)$.

Now let $0 \neq m \in \mathbf{Z}$. We must show that

$$M\beta_M\sigma^*\pi_m \quad \textit{and} \quad N\beta_N\pi_m$$

are conjugate in Aut $V(N)/mV(N)$, where π_m denotes the canonical map Aut $V(N) \to$ Aut $V(N)/mV(N)$.

Arguing as above with m in place of 2, we obtain an isomorphism

$$\varphi_m\colon V(M)/mV(M) \longrightarrow V(N)/mV(N) .$$

Denote by σ_m: $V(M)/mV(M) \to V(N)/mV(N)$ the map induced by σ, and put $\gamma = \sigma_m^{-1}\varphi_m \in$ Aut $V(N)/mV(M)$. Now recall that φ_m was defined via isomorphisms

$$\mathbf{Z}M/(M^k - 1)\mathbf{Z}M \longrightarrow \mathbf{Z}(M/M^k) , \quad \mathbf{Z}N/(N^k - 1)\mathbf{Z}M \longrightarrow \mathbf{Z}(N/N^k)$$

and

$$\varphi^{(k)}\colon M/M^k \longrightarrow N/N^k , \quad \text{say,}$$

for a suitable $k > 0$. By checking the definitions we find that if $a \in M$ and $b \in N$ satisfy

$$(aM^k)\varphi^{(k)} = bN^k$$

then

$$(a\beta_M\sigma^*\pi_m) \cdot \gamma = \gamma \cdot (b\beta_N\pi_m) .$$

It follows that

$$(M\beta_M\sigma^*\pi_M)^\gamma = N\beta_N\pi_m$$

and the lemma is proved.

We now digress briefly to prove Theorem B:

THEOREM (Pickel [P2]). *Every $\wedge$-class of finitely generated nilpotent-by-finite groups is the union of finitely many isomorphism classes.*

Proof. By Theorem A, it will suffice to prove this for a $\wedge$-class $\mathcal{C}$, say, of $\mathfrak{T}$ groups. Fix $N \in \mathcal{C}$. By Lemma 3, there exists for each $M \in \mathcal{C}$ an injective homomorphism $\beta_M\sigma_M^*: M \to \operatorname{Aut} V(N) = \mathrm{GL}_n(\mathbf{Z})$ such that the images $M\beta_M\sigma_M^*$ all lie in the $\sim$-class of $N\beta_N$. Theorem D then shows that these images lie in finitely many conjugacy classes of $\operatorname{Aut} V(N)$, and a fortiori in finitely many isomorphism classes.

Remark. This proof of Theorem B is more economical than it may appear, because it only needs the special case of Theorem D concerning unipotent subgroups of $\mathrm{GL}_n(\mathbf{Z})$. This is very much simpler that the general case and in fact follows easily from Theorem 2 of [GS2].

LEMMA 4. *Let M, N be $\mathfrak{T}$ groups and S, T be polycyclic subgroups of* $\operatorname{Aut} M$, $\operatorname{Aut} N$ *respectively. Let $H \leqq M]S$ and $G \leqq N]T$, and suppose there exists an isomorphism $\varphi: (M]S)^\wedge \to (N]T)^\wedge$ such that $\hat{M}\varphi = \hat{N}$, $\hat{S}\varphi = \hat{T}$ and $\hat{H}\varphi = \hat{G}$. Let $\sigma^*: \operatorname{Aut} V(M) \to \operatorname{Aut} V(N)$ be as given in Lemma* 3. *Then*

$$H\beta_M\sigma^* \sim G\beta_N \quad in \quad \operatorname{Aut} V(N)\,.$$

Proof. Let $0 \neq m \in \mathbf{Z}$. We must show that

$$H\beta_M\sigma^*\pi_m \ \textit{is conjugate to}\ G\beta_N\pi_m \ \textit{in}\ \operatorname{Aut} V(N)/mV(N)\,.$$

To this end, define $\gamma \in \operatorname{Aut} V(N)/mV(N)$ as in Lemma 3; only make sure that the positive integer k, chosen before so that $M^k \subseteqq 1 + \mathfrak{m}^{c+1} + mq\mathbf{Z}M$ etc., is so large that we also have

$$(M]S)^k\beta_M\pi_m = 1 = (N]T)^k\beta_N\pi_m\,.$$

Now φ induces an isomorphism $\varphi^{(k)}: H/H^k \to G/G^k$, and it will suffice to show that if $h \in H$, $g \in G$ satisfy

$$(hH^k)\varphi^{(k)} = gG^k$$

then

$$(h\beta_M\sigma^*\pi_m)\cdot\gamma = \gamma(g\beta_N\pi_m)\,.$$

Now choose $l \geqq k$ so that

$(M]S)^l \cap H \leqq H^k$, $(M]S)^l \cap M \leqq M^k$, $(M]S)^l \cap S \leqq S^k$ and $(N]T)^l \cap G \leqq G^k$ etc.

Then $\varphi^{(k)}$ is induced by an isomorphism $\varphi^{(l)}: (M]S)/(M]S)^l \to (N]T)/(N]T)^l$.

Without loss of generality we may assume that

$$(h \cdot (M]S)^l)\varphi^{(l)} = g \cdot (N]T)^l .$$

We may then write $h = as$, $g = bt$ where $a \in M$, $s \in S$, $b \in N$, $t \in T$ satisfy

$$(aM^k)\varphi^{(k)} = bN^k , \quad (sS^k)\varphi^{(k)} = tT^k$$

($\varphi^{(k)}$ here denoting the various maps induced by φ on the relevant quotient groups). It will now suffice to check that

$$(a\beta_M\sigma^*\pi_m)\cdot\gamma = \gamma(b\beta_N\pi_m) \quad \text{and} \quad (s\beta_M\sigma^*\pi_m)\cdot\gamma = \gamma(t\beta_N\pi_m) ,$$

a routine verification which we shall omit.

4. A cohomological lemma

Here we give some finiteness results, mainly due to D. J. S. Robinson.

LEMMA 1. *Let M be a finitely generated free $\mathbf{Z}$-module and X a nilpotent group acting on M with $C_M(X) = 0$, i.e., $H^0(X, M) = 0$. Then $H^1(X, M)$ and $H^1(\hat{X}, \hat{M})$ are finite. If X is also finitely generated, then $H^2(X, M)$ is finite.*

LEMMA 2. *Let M be a finite-dimensional $\mathbf{Q}$-module and X a nilpotent group acting on M. If $M(X-1) = M$ then*

$$H^0(X, M) = H^2(X, M) = 0 .$$

Lemma 2 is a special case of [R], Corollary AB. Now let M and X be as in Lemma 1. Theorem F of [R] shows that $H^1(X, M)$ is finite, and Theorem D of that paper shows that $H^2(X, M)$ has finite exponent. The final claim of Lemma 1 will therefore follow from

LEMMA 3. *If A is a finitely generated $\mathbf{Z}$-module and G is a finitely presented group acting on A then $H^2(G, A)$ is finitely generated.*

Note that every finitely generated nilpotent group is finitely presented.

Proof. Suppose G is presented as F/R where F is a free group of finite rank. Then the relation module R/R' is a finitely generated G-module, and so $\mathrm{Hom}_G(R/R', A)$ is finitely generated as a $\mathbf{Z}$-module. But $H^2(G, A)$ is a homomorphic image of $\mathrm{Hom}_G(R/R', A)$, by [G], Chapter 3, Proposition 6.

It remains to discuss the second claim of Lemma 1, concerning the completions. Here $\hat{M}$ is given the structure of an $\hat{X}$-module via the identification

$$(M]X)\hat{} = \hat{M}]\hat{X} .$$

Proof that $H^1(\hat{X}, \hat{M})$ is finite. Assume that $M \neq 0$ and put $Y = C_X(M)$. Then $Y \lhd X$ so there exists $x \in X \setminus Y$ with x central modulo Y. The map

$\mu \mapsto \mu(x-1)$ is then an X-module endomorphism of M; let N be its kernel. Then $N \neq M$ and $M/N \cong M(x-1)$. So $H^0(X, M/N) = H^0(X, N) = 0$. We distinguish two cases.

Case 1. $N \neq 0$. By induction on the $\mathbf{Z}$-rank of M, we may suppose that $H^1(\hat{X}, \hat{N})$ and $H^1(\hat{X}, (M/N)\hat{})$ are both finite. As $(M/N)\hat{} \cong \hat{M}/\hat{N}$ it follows that $H^1(\hat{X}, \hat{M})$ is also finite.

Case 2. $N = 0$. Then $M(x-1) \cong M$, so there exists $m \neq 0$ in $\mathbf{Z}$ such that $M(x-1) \geqq mM$.

Suppose to start with that X is abelian. Let $\delta: \hat{X} \to \hat{M}$ be a derivation. Since $mM \leqq M(x-1)$, it is also the case that $m\hat{M} \leqq \hat{M}(x-1)$, hence there exists $\mu \in \hat{M}$ with

$$m \cdot x\delta = \mu(x-1) \ .$$

If $y \in \hat{X}$ then $xy = yx$ and we deduce that

$$0 = (xy)\delta - (yx)\delta = (x\delta)(y-1) - y\delta(x-1) \ ,$$

whence

$$\bigl(m \cdot y\delta - \mu(y-1)\bigr)(x-1) = 0 \ .$$

But $x - 1$ kills nothing in $\hat{M}$ (as $N = 0$, $x - 1$ is nonsingular as an endomorphism of $\mathbf{Q} \otimes M$, hence also as an endomorphism of $\mathbf{Z}_p \otimes M$ for each prime p); so

$$m \cdot y\delta = \mu(y-1) \ .$$

Thus $m\delta$ is an inner derivation. We have now shown that

$$mH^1(\hat{X}, \hat{M}) = 0 \ .$$

It follows that $m \operatorname{Der}(\hat{X}, \hat{M}) \subseteqq \hat{M}\psi$ where ψ is the map sending each element of $\hat{M}$ to the corresponding inner derivation. Since $\hat{M}/m\hat{M}$ is finite, so is $\hat{M}\psi/m\hat{M}\psi$. Since $\hat{M}$ is $\mathbf{Z}$-torsion free, we get

$$H^1(\hat{X}, \hat{M}) \cong \operatorname{Der}(\hat{X}, \hat{M})/\hat{M}\psi \cong m \operatorname{Der}(\hat{X}, \hat{M})/m\hat{M}\psi \subseteqq \hat{M}\psi/m\hat{M}\psi \ .$$

Thus $H^1(\hat{X}, \hat{M})$ is finite.

In general, suppose X is nilpotent of class $c > 1$ and argue by induction on c. Put

$$K = \zeta_{c-1}(X) \cdot \langle x \rangle \ .$$

Then $H^0(K, M) = 0$ since $x \in K$, and K is nilpotent of class at most $c - 1$, so by inductive hypothesis $H^1(\hat{K}, \hat{M})$ is finite. Now there is an exact sequence

$$H^1(\hat{X}/\hat{K}, \hat{M}^{\hat{K}}) \longrightarrow H^1(\hat{X}, \hat{M}) \longrightarrow H^1(\hat{K}, \hat{M})$$

(where A^B is shorthand for $H^0(B, A)$); the maps are the obvious ones and exactness is easy to verify. As $\hat{M}^{\hat{K}} \subseteqq C_{\hat{M}}(x) = 0$, the left-hand term in the sequence is zero. As the right-hand term is finite, $H^1(\hat{X}, \hat{M})$ is also finite and the proof is finished.

5. Nilpotent supplements

In this section we lay the groundwork for the construction of semisimple splittings.

PROPOSITION 1. *Let N be a $\mathfrak{T}$ group and G a $\mathfrak{P}$ group with $G' \leqq N \leqq G$. Then there exists $k > 0$ such that for every subgroup $\bar{N}$ of $N^{\mathbf{Q}}$ containing $N^{1/k}$ and normalized by G, there is a nonempty set $\mathfrak{X} = \mathfrak{X}(\bar{N})$ of subgroups of $\bar{N}G$ with the following properties*:

1) *for each $X \in \mathfrak{X}$, X is nilpotent and $\bar{N}X = \bar{N}G$;*

2) *assume $|\bar{N}: N| < \infty$. Then given $X \in \mathfrak{X}$, the groups $\hat{X}\lambda$ as λ runs through automorphisms of $(\bar{N}G)\hat{}$ fixing $(\bar{N})\hat{}$ lie in finitely many conjugacy classes in $(\bar{N}G)\hat{}$;*

3) *suppose $\bar{N} \leqq \tilde{N}$ are two subgroups of $N^{\mathbf{Q}}$ normalized by G and containing $N^{1/k}$. Then for each $X \in \mathfrak{X}(\bar{N})$ there exists $Y \in \mathfrak{X}(\tilde{N})$ such that*

$$X = Y \cap \bar{N}G \quad \textit{and} \quad Y = (Y \cap \tilde{N})X .$$

Remark. We have only stated the properties of $\mathfrak{X}$ which we are going to need. In fact $\mathfrak{X}$ is in many ways like the set of nilpotent projectors, in the sense of formation theory. For example, $\mathfrak{X}$ is in fact the set of maximal nilpotent supplements for $\bar{N}$ in $\bar{N}G$, it consists of finitely many conjugacy classes of subgroups in $\bar{N}G$, and each $X \in \mathfrak{X}$ is self-normalizing. After writing the first version of this paper we discovered that part (1) of the proposition, or rather an essentially equivalent result asserting the existence of a nilpotent supplement to N in some subgroup of finite index in G, has already been proved by D.I. Zaicev in [Z]; it also follows from work of Newell, [N]. In fact part (1) can be proved very quickly using Lemma 4.2 and a simple inductive argument; the rather complicated machinery we are about to introduce is needed only for part (2).

Let M be a torsion-free nilpotent group and G a group operating by automorphisms on M. Define

$$\Gamma(M, G) = i_M[M, G] .$$

Thus $\Gamma(M, G)$ *is the smallest isolated G-invariant normal subgroup K of M such that G acts trivially on the quotient M/K.* For $i > 1$ define inductively

$$\Gamma^i(M, G) = \Gamma(\Gamma^{i-1}(M, G), G) .$$

Then

$$M \geqq \Gamma(M, G) \geqq \cdots \geqq \Gamma^i(M, G) \geqq \cdots$$

is a descending series of isolated normal subgroups of M. If M has finite rank this series becomes stationary after finitely many, say n, steps, and we define the subgroup $\mathrm{B}(M, G)$ by

$$\Gamma^n(M, G) = \Gamma^{n+1}(M, G) = \mathrm{B}(M, G) .$$

Then $\mathrm{B}(M, G)$ *is the smallest isolated G-invariant normal subgroup B of M such that G acts nilpotently on M/B.* An argument similar to the proof of Lemma 2.11 enables one to verify the following:

LEMMA 1. *Suppose $N \in \mathfrak{T}$ and G acts on N. Then*
a) $\Gamma(N, G) \;\; = N \cap \Gamma(N^{\mathbf{Q}}, G)$;
b) $\Gamma(N^{\mathbf{Q}}, G) = i_{N^{\mathbf{Q}}}\Gamma(N, G)$;
c) $\mathrm{B}(N, G) \;\; = N \cap \mathrm{B}(N^{\mathbf{Q}}, G)$;
d) $\mathrm{B}(N^{\mathbf{Q}}, G) = i_{N^{\mathbf{Q}}}\mathrm{B}(N, G)$.

LEMMA 2. *Suppose $N \in \mathfrak{T}$ and B is an isolated normal subgroup of $N^{\mathbf{Q}}$. Put $K = \Gamma(B, N^{\mathbf{Q}})$. Then*

$$K \cap N = \Gamma(B \cap N, N) .$$

Proof. The containment $\geqq$ follows from the definition. For a subgroup C of $N^{\mathbf{Q}}$, write $\bar{C} = i_{N^{\mathbf{Q}}}(C)$. The corollary to Theorem 4.6 of [H] shows that

$$[(B \cap N)^-, \bar{N}] \leqq [B \cap N, N]^- .$$

Now $(B \cap N)^- = B$ and $\bar{N} = N^{\mathbf{Q}}$, so we get $K \leqq [B \cap N, N]^-$ and therefore

$$K \cap N \leqq N \cap [B \cap N, N]^- = \Gamma(B \cap N, N) .$$

LEMMA 3. *Suppose $G \in \mathfrak{P}\mathfrak{F}$ and $N \lhd G$ is in $\mathfrak{T}$. Then $\Gamma(N, G)^\wedge$ is the closure in $\hat{G}$ of $\Gamma(\hat{N}, \hat{G})$, and $\mathrm{B}(N, G)^\wedge$ is the unique smallest isolated, closed, $\hat{G}$-invariant normal subgroup B of $\hat{N}$ such that $\hat{G}$ acts nilpotently on $\hat{N}/B$.*

Proof. $[N, G] \leqq [\hat{N}, \hat{G}] \leqq \Gamma(\hat{N}, \hat{G})$ and $\Gamma(\hat{N}, \hat{G}) \cap N$ is isolated in N, so $\Gamma(N, G) \leqq \Gamma(\hat{N}, \hat{G})$. It is easy to see that $[\hat{N}, \hat{G}] \leqq [N, G]^\wedge \leqq \Gamma(N, G)^\wedge$, and as $\Gamma(N, G)^\wedge$ is isolated in $\hat{N}$ it follows that $\Gamma(\hat{N}, \hat{G}) \leqq \Gamma(N, G)^\wedge$. Thus

$$\Gamma(N, G) \leqq \Gamma(\hat{N}, \hat{G}) \leqq \Gamma(N, G)^\wedge ,$$

which establishes the first claim. For the second claim, it is easy to see that $\mathrm{B}(N, G)^\wedge$ is an isolated, closed, $\hat{G}$-invariant normal subgroup of $\hat{N}$ and that $\hat{G}$ acts nilpotently on the quotient $\hat{N}/\mathrm{B}(N, G)^\wedge$. Suppose $C \leqq \mathrm{B}(N, G)^\wedge$ is another such subgroup of $\hat{N}$; we must show that $C = \mathrm{B}(N, G)^\wedge$. Now G acts nilpotently on $N/(C \cap N)$ and $C \cap N$ is isolated in N, so $C \cap N \geqq \mathrm{B}(N, G)$. Therefore since C is closed we have $C \geqq (C \cap N)^\wedge \geqq \mathrm{B}(N, G)^\wedge$, as required.

For the proof of Proposition 1, we start by making a construction in the Mal'cev completion

$$V = N^{\mathbf{Q}} ,$$

where $N \geqq G'$ is a $\mathfrak{T}$ subgroup of the $\mathfrak{P}$ group G. Define

$$\begin{aligned} K_0 &= V , \\ B_i &= \mathrm{B}(K_{i-1}, G) \qquad \text{for} \quad i \geqq 1 ; \\ K_i &= \Gamma(B_i, V) \qquad \text{for} \quad i \geqq 1 . \end{aligned}$$

From the definition, it follows that K_i has smaller rank than B_i if $B_i \neq 1$, so there exists n such that

$$V = K_0 \geqq B_1 > K_1 \geqq \cdots > K_n \geqq B_{n+1} = 1 .$$

LEMMA 4. *In the group VG there is a series of subgroups*

$$VG = C_0 > C_1 > \cdots > C_n = C \tag{1}$$

such that

$$\begin{cases} B_i \cap C_i = K_i,\ B_i C_i = C_{i-1} \text{ for } 1 \leqq i \leqq n, \\ \text{and } C_{i-1}/B_i \text{ is nilpotent for } 1 \leqq i \leqq n+1. \end{cases} \tag{2}$$

The group C is nilpotent and $VC = VG$.

Proof. The groups C_i are constructed inductively. It is clear that $C_0/B_1 = VG/B_1$ is nilpotent. So suppose $C_0, \cdots, C_{i-1}$ have been found and have the stated properties. Write $H = C_{i-1}$ and put $M = B_i/K_i$. Note that since $B_jC_j = C_{j-1}$ for $j = 1, \cdots, i-1$, we have $VH = VG$. Now it follows from the definitions that M is a $\mathbf{Q}(VG)$-module on which V acts trivially, and that $M(VG - 1) = M$. Hence $M(H-1) = M$, and as H/B_i is nilpotent it follows by Lemma 4.2 that

$$H^2(H/B_i, M) = 0 .$$

Therefore the extension

$$1 \longrightarrow B_i/K_i \longrightarrow H/K_i \longrightarrow H/B_i \longrightarrow 1$$

splits, so there exists a group $C_i \leqq H$ such that

$$B_iC_i = H = C_{i-1} , \quad B_i \cap C_i = K_i .$$

Also $C_i/K_i \cong H/B_i$ which is nilpotent, and C_i acts nilpotently on K_i/B_{i+1} by definition of B_{i+1}. Hence C_i/B_{i+1} is nilpotent.

Thus C_i has all the required properties. The final statement of the lemma is clear.

Now define $\mathfrak{X}(V)$ to be the set of all subgroups C of VG which occur at the end of a series like (1) satisfying (2) in the above lemma. Thus the content of Lemma 4 is just that $\mathfrak{X}(V)$ is nonempty. With this at our disposal we are ready for the

Proof of Proposition 1. Choose $C \in \mathfrak{X}(V)$. Then $VG = VC$. As G is finitely generated, there exists $k > 0$ such that $G \leqq N^{1/k} C$. Then if $N^{1/k} \leqq \bar{N} \leqq V$ and $\bar{N}$ is normalized by G, we get

$$\bar{N}G = \bar{N}(C \cap \bar{N}G) .$$

Thus putting

$$X_C(\bar{N}) = C \cap \bar{N}G$$

we have

$$\bar{N}G = \bar{N}X_C(\bar{N}) \quad \text{and} \quad X_C(\bar{N}) \text{ is nilpotent.}$$

Now define

$$\mathfrak{X}(\bar{N}) = \{X_C(\bar{N}) \mid C \in \mathfrak{X}(V) \quad \text{and} \quad \bar{N}C \geqq G\} .$$

The choice of k above ensures that $\mathfrak{X}(\bar{N})$ has at least one member provided $N^{1/k} \leqq \bar{N} \leqq V$ and G normalizes $\bar{N}$, and part (1) of the proposition is established.

Next we prove part (3). Suppose $N^{1/k} \leqq \bar{N} \leqq \tilde{N}$ and $X = X_C(\bar{N}) \in \mathfrak{X}(\bar{N})$. Put $Y = X_C(\tilde{N})$; it is clear that $Y \in \mathfrak{X}(\tilde{N})$. Moreover

$$Y \cap \bar{N}G = (C \cap \tilde{N}G) \cap \bar{N}G = C \cap \bar{N}G = X;$$

and

$$\begin{aligned} Y &= Y \cap \tilde{N}G = Y \cap \tilde{N}(\bar{N}G) \\ &= Y \cap \tilde{N}(\bar{N}X) = Y \cap \tilde{N}X = (Y \cap \tilde{N})X . \end{aligned}$$

To prove part (2) we must refer to the details of Lemma 4. To simplify the notation let us assume that $\bar{N} = N$ (that is, we replace G by $\bar{N}G$; this will not upset the definition of the B_i and K_i). Then $X = C \cap G$ where C is the last term of a series (1) satisfying the conditions (2) of Lemma 4. Write

$$\bar{B}_i = B_i \cap N , \quad \bar{K}_i = K_i \cap N , \quad X_i = \bar{K}_i X .$$

Then $X_i \leqq G \cap C_i$ for each i, and

$$G = NX = X_0 > X_1 > \cdots > X_n = X .$$

First claim.

$$\bar{B}_i \cap X_i = \bar{K}_i , \quad \bar{B}_i X_i = \bar{B}_i X .$$

This follows at once from the definitions and (2).

Second claim. $\bar{B}_i X_i$ is a subgroup of finite index in X_{i-1}. To prove this it will suffice to show that every element of $\bar{K}_{i-1}$ has some power in $\bar{B}_i X_i$, since $\bar{K}_{i-1} X = X_{i-1}$ and $X_{i-1}/\bar{B}_i$ is nilpotent. Now

$$\bar{K}_{i-1} = N \cap B_{i-1} \cap C_{i-1} = N \cap B_i(B_{i-1} \cap C) ,$$

so let us take $w = bc \in \bar{K}_{i-1}$ with $b \in B_i$ and $c \in B_{i-1} \cap C$. There exists $r > 0$

such that $c^r \in B_{i-1} \cap C \cap N = \bar{B}_{i-1} \cap X$, and then

$$w^r = b'c^r \quad \text{where} \quad b' = bb^{c^{-1}} \cdots b^{c^{1-r}} \in B_i \cap N = \bar{B}_i \ .$$

Thus $w^r \in \bar{B}_i(\bar{B}_{i-1} \cap X) \leqq \bar{B}_i X$ and the claim is established.

Third claim. $\bar{B}_i/\bar{K}_i$ *is a finitely generated free* **Z**-*module,* $X_i/\bar{K}_i$ *is nilpotent, and* $X_i/\bar{K}_i$ *acts fixed-point freely on* $\bar{B}_i/\bar{K}_i$.

The first two statements are immediate from the definitions. For the third, recall that $M(VG - 1) = M$ where $M = B_i/K_i$. As $VG = VX$ and V acts trivially on M it follows that $M(X - 1) = M$, and hence by Lemma 4.2 that $H^0(X, M) = 0$. In other words X acts fixed-point freely on M; as $X_i \geqq X$ and $\bar{B}_i/\bar{K}_i$ is isomorphic to a submodule of M, the claim follows.

Now we consider what happens in the profinite completions. Let λ be an automorphism of $\hat{G}$ fixing $\hat{N}$.

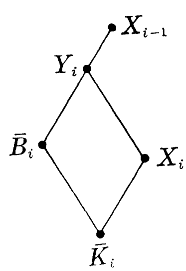
Fourth claim.

$$(\bar{B}_i)^\wedge\lambda = (\bar{B}_i)^\wedge \quad \textit{and} \quad (\bar{K}_i)^\wedge\lambda = (\bar{K}_i)^\wedge \quad \textit{for each } i.$$

This is because $(\bar{B}_i)^\wedge$ is the smallest closed isolated subgroup of $(\bar{K}_{i-1})^\wedge$ which is normal in $\hat{G}$ and such that $\hat{G}$ acts nilpotently on $(\bar{K}_{i-1})^\wedge/(\bar{B}_i)^\wedge$, and $(\bar{K}_i)^\wedge$ is the closure in $(\bar{B}_i)^\wedge$ of $\Gamma\big((\bar{B}_i)^\wedge, \hat{N}\big)$, by Lemmas 1(c), 2 and 3. As $(\bar{K}_0)^\wedge\lambda = \hat{N}\lambda = \hat{N} = (\bar{K}_0)^\wedge$, the claim follows by induction on i.

We can now prove part (2) of the proposition: λ fixes $\hat{X}_0 = \hat{G}$. Suppose inductively that $i > 0$ and that there are only finitely many possibilities up to conjugacy for $\hat{X}_{i-1}\lambda$. Then it will suffice to show that there are only finitely many possibilities for $\hat{X}_i\lambda$ up to conjugacy under the assumption that λ fixes $\hat{X}_{i-1}$ as well as $\hat{N}$; the result will follow by induction since $\hat{X}_n = \hat{X}$.

Put $Y_i = \bar{B}_i X_i$. Claim 2 shows that $|X_{i-1}: Y_i|$ is finite. Therefore $|\hat{X}_{i-1}: \hat{Y}_i|$ is finite, and as $\hat{X}_{i-1}$ has only finitely many subgroups of any given finite index and $\hat{X}_{i-1}\lambda = \hat{X}_{i-1}$, there are only finitely many possibilities for $\hat{Y}_i\lambda$. So we may assume that λ fixes $\hat{Y}_i$ also.



Thus $\hat{X}_i\lambda/(\bar{K}_i)\hat{}$ is a complement for $(\bar{B}_i)\hat{}/(\bar{K}_i)\hat{}$ in $\hat{Y}_i/(\bar{K}_i)\hat{}$. The set of conjugacy classes of such complements in $\hat{Y}_i/(\bar{K}_i)\hat{}$ is in $1-1$ correspondence with

$$H^1\big((\hat{X}_i)/(\bar{K}_i)\hat{}, (\bar{B}_i)\hat{}/(\bar{K}_i)\hat{}\big) \cong H^1\big((X_i/\bar{K}_i)\hat{}, (\bar{B}_i/\bar{K}_i)\hat{}\big) .$$

By claim 3 and Lemma 4.1 this cohomology group is finite. Hence up to conjugacy in $\hat{Y}_i$ there are only finitely many possibilities for $\hat{X}_i\lambda$, and we are done.

6. On Jordan decomposition

Let k be a perfect field and $x \in \mathrm{GL}_n(k)$. Then there exists a unique pair (x_u, x_s) of matrices in $\mathrm{GL}_n(k)$ such that x_u is unipotent, x_s is diagonalizable (over some extension field of k), and

$$x_u x_s = x_s x_u = x ;$$

this is the multiplicative Jordan decomposition of x. See Wehrfritz [We1], Chapter 7 for a discussion (he writes x_d where we write x_s); the fact that the decomposition takes place in k follows by elementary Galois theory from the uniqueness. If X is a subgroup of $\mathrm{GL}_n(k)$ we write

$$X_u = \{x_u \mid x \in X\} ; \quad X_s = \{x_s \mid x \in X\}$$

(this differs from Wehrfritz's notation). If now X is nilpotent, then it follows from [We1], 7.11 and 7.13, that X_u and X_s are subgroups of $\mathrm{GL}_n(k)$ and

$$X \leqq \langle X_u, X_s \rangle = X_u \times X_s .$$

If $\mathcal{G} \leqq \mathrm{GL}_n$ is an algebraic group defined over k and $X \leqq \mathcal{G}(k)$, then we also have, by [We1], 7.3,

$$\langle X_u, X_s \rangle \leqq \mathcal{G}(k) .$$

In this section we establish two main results. The second, Proposition 2, says that $(X_s)\hat{} = (\hat{X})_s$; this is explained below. First we deal with

PROPOSITION 1. *Let Γ be a soluble subgroup of $\mathrm{GL}_n(\mathbf{Z})$. Then there exists a positive integer q with the following property: whenever X is a nilpotent subgroup of Γ such that $UX = \Gamma$, where U is the maximal unipotent normal subgroup of Γ,*

$$\mathbf{Z}^n\langle X, X_s \rangle \subseteqq q^{-1}\mathbf{Z}^n .$$

Of course we know that $\mathbf{Z}^n\langle X, X_s \rangle$ is an additive subgroup of $\mathbf{Q}^n$; the point is that one has some control over how far away it can get from $\mathbf{Z}^n$. Another way of putting the result is that there is a bound for the denominators in the entries of matrices in X_s.

We start with some preparations.

LEMMA 1. *Suppose X is a nilpotent subgroup of* $\mathrm{GL}_n(k)$ *such that X' is unipotent. Then X_s is abelian.*

Proof. The map $x \mapsto x_s$ is a homomorphism of X onto X_s, so $(X_s)' = (X')_s = 1$.

DEFINITION. *Suppose k is an algebraic number field with ring of integers $\mathfrak{o}$, and $x \in \mathrm{GL}_n(\mathfrak{o})$. Let $\lambda_1, \cdots, \lambda_r$ be the distinct eigenvalues of x, and assume that they lie in $\mathfrak{o}$. Define the positive integer $m(x)$ by*

$$m(x)\mathbf{Z} = \mathbf{Z} \cap \prod_{i \neq j}(\lambda_i - \lambda_j)^n \mathfrak{o} .$$

LEMMA 2. *If $\mathfrak{o}$ and x are as above, then x_s stabilizes an $\mathfrak{o}$-module L with*

$$\mathfrak{o}^n \subseteqq L \subseteqq m(x)^{-1}\mathfrak{o}^n .$$

Proof. Put $V = k^n$ and $E = \mathfrak{o}^n$. Then V is the direct sum of generalized eigenspaces for x:

$$V = \bigoplus_{i=1}^r V_i , \qquad V_i(x - \lambda_i)^{l_i} = 0 \quad \text{for} \quad 1 \leqq i \leqq r$$

with $1 \leqq l_i \leqq n$ for each i. Put $L_i = E \cap V_i$ and

$$L = m(x)^{-1} \bigoplus_{i=1}^r L_i .$$

Then $L \subseteqq m(x)^{-1}\mathfrak{o}^n$; since x_s acts as the scalar λ_i on V_i, and λ_i is a unit of $\mathfrak{o}$, x_s stabilizes each L_i; so x_s stabilizes L. It remains to show that $L \supseteqq \mathfrak{o}^n = E$.

Suppose

$$a = v_1 + \cdots + v_r \in E , \qquad v_i \in V_i \quad \text{for each } i .$$

For a fixed i,

$$a \prod_{j \neq i}(x - \lambda_j)^{l_j} = v_i \prod_{j \neq i}(\lambda_i - \lambda_j)^{l_j} .$$

By the definition of $m(x)$, there exists $\rho_i \in \mathfrak{o}$ with $\rho_i \prod_{j \neq i}(\lambda_i - \lambda_j)^{l_j} = m(x)$, and then

$$m(x)v_i = \rho_i a \prod_{j \neq i}(x - \lambda_j)^{l_j} \in E .$$

Thus $m(x)v_i \in E \cap V_i = L_i$ for each i, so $m(x)a \in \bigoplus L_i$ and $a \in L$.

LEMMA 3. *Let X be a nilpotent subgroup of $\mathrm{GL}_n(\mathfrak{o})$ such that X' is unipotent, and choose $x_1, \cdots, x_t \in X$ such that $(x_1)_s, \cdots, (x_t)_s$ generate X_s. Assume that the eigenvalues of $x_1, \cdots, x_t$ lie in $\mathfrak{o}$, and put*

$$m = \prod_{i=1}^t m(x_i) .$$

Then

$$\langle X_u, X_s \rangle \subseteqq m^{-1} M_n(\mathfrak{o}) .$$

Proof. Put $E = \mathfrak{o}^n$. Lemma 2 shows that

$$E\langle (x_i)_s \rangle \subset m(x_i)^{-1}E \quad \text{for each } i .$$

As $X_s = \langle (x_1)_s, \cdots, (x_t)_s \rangle$ is abelian by Lemma 1, it follows that

$$EX_s \subseteqq m^{-1}E .$$

Now $\langle X_u, X_s \rangle = XX_s$ and X_u centralizes X_s, so

$$E\langle X_u, X_s \rangle = EXX_s = EX_s \subseteqq m^{-1}E .$$

The result follows.

Proof of Proposition 1. Γ is a soluble subgroup of $\mathrm{GL}_n(\mathbf{Z})$, so by Mal'cev's Theorem ([Wel], 3.6) there exist an algebraic number field k, a matrix $\alpha \in \mathrm{GL}_n(k)$, and a positive integer e such that

$$(\Gamma^e)^\alpha \leqq \mathrm{Tr}\,(n, k) .$$

Take $0 \neq c \in \mathbf{Z}$ such that $c\alpha$ and $c\alpha^{-1}$ are in $M_n(\mathfrak{o})$; there exists $f \neq 0$ such that

$$\Gamma^f \leqq \Gamma^e \cap \big(1 + c^2 M_n(\mathbf{Z})\big) ,$$

and then it is easy to see that

$$(\Gamma^f)^\alpha \leqq \mathrm{Tr}\,(n, \mathfrak{o}) .$$

Put $\bar{U} = U \cap \Gamma^f$. Then $UX = \Gamma$ implies $\bar{U}X^f = \Gamma^f$, so

$$\bar{U}^\alpha \cdot (X^f)^\alpha = (\Gamma^f)^\alpha . \tag{1}$$

Let $d\colon \mathrm{Tr}\,(n, k) \to D_n(k)$ be the map sending each matrix to its diagonal. Since $\bar{U}^\alpha \leqq \mathrm{Tr}_1(n, k)$, any X satisfying (1) also satisfies

$$d\big((X^f)^\alpha\big) = d\big((\Gamma^f)^\alpha\big) = D, \text{ say.}$$

Now $x \mapsto d(x)$ maps $(X^f)^\alpha_s$ isomorphically onto D, since for an upper-triangular matrix x we have $d(x_s) = d(x)$; and the map preserves eigenvalues. Hence if we choose a generating set $y_1, \cdots, y_t$ for D and put $m = \prod m(y_i)$, we may apply Lemma 3 to the nilpotent group $(X^f)^\alpha$ and obtain

$$\langle (X^f)^\alpha_u, (X^f)^\alpha_s \rangle \subseteqq m^{-1} M_n(\mathfrak{o}) .$$

Therefore

$$\langle X^f_u, X^f_s \rangle \subseteqq c^{-2} m^{-1} M_n(\mathfrak{o}) \cap \mathrm{GL}_n(\mathbf{Q}) = c^{-2} m^{-1} M_n(\mathbf{Z}) .$$

Let $x \in X$. Then $(x_u)^f = (x^f)_u \in c^{-2} m^{-1} M_n(\mathbf{Z})$, so

$$(n-1)!\, f \log x_u = (n-1)! \log (x_u)^f \in (c^{-2} m^{-1})^{n-1} M_n(\mathbf{Z})$$

and so

$$x_u = \exp \log x_u \in q^{-1} M_n(\mathbf{Z})$$

where

$$q = (n-1)!\,\big((n-1)!\, f c^{2(n-1)} m^{(n-1)}\big)^{n-1} .$$

Thus

$$X_u \subseteqq q^{-1}M_n(\mathbf{Z}) \quad \text{and}$$
$$\langle X_u, X_s \rangle = X_u X \subseteqq q^{-1}M_n(\mathbf{Z})\mathrm{GL}_n(\mathbf{Z}) = q^{-1}M_n(\mathbf{Z}) .$$

The result follows.

We turn now to the second topic of this section. Let $\mathfrak{G} \leqq \mathrm{GL}_n$ be a **Q**-group, and recall (§ 2) that we write

$$\mathfrak{G}^\infty = \prod_p \mathfrak{G}(\mathbf{Z}_p) .$$

For $x = (x_p) \in \mathfrak{G}^\infty$, we define x_s to be the element of $\mathfrak{G}^\infty$ with p-coordinate $(x_p)_s$ for each prime p, provided that $(x_p)_s \in \mathfrak{G}(\mathbf{Z}_p)$ for each p; otherwise x_s is undefined. x_u is defined likewise, so that we have

$$x_s x_u = x_u x_s = x .$$

As usual, we then define

$$X_s = \{x_s \mid x \in X\} , \quad X_u = \{x_u \mid x \in X\}$$

for any subgroup $X \leqq \mathfrak{G}^\infty$ such that x_s is defined for all $x \in X$.

If $X \leqq \mathfrak{G}(\mathbf{Z})$, denote by $\bar{X}$ the closure of X in the congruence topology on $\mathfrak{G}^\infty$. (Thus if $X \in \mathfrak{P}\mathfrak{F}$, 2.6 shows that $\bar{X} = \hat{X}$.)

PROPOSITION 2. *Let X be a nilpotent subgroup of $\mathfrak{G}(\mathbf{Z})$ such that X' is unipotent and $X_s \leqq \mathfrak{G}(\mathbf{Z})$. Then*

$$(\bar{X})_s = (X_s)^-$$

(which means in particular that $(\bar{X})_s$ is defined).

The proof is based on two lemmas. From now on X is supposed to be a group satisfying the hypotheses of Proposition 2.

LEMMA 4. *If $(x(i))$ is a Cauchy sequence in X relative to the congruence topology, then the sequence $(x(i)_s)$ is also Cauchy and* $\lim (x(i)_s) = (\lim (x(i)))_s$.

Proof. By Lemma 1, X_s is abelian. So there exist a number field k and a matrix $\alpha \in \mathrm{GL}_n(k)$ such that X_s^α is diagonal. As X_u^α centralizes X_s^α and is unipotent, it is easy to see that there exists $\beta \in \mathrm{GL}_n(k)$ centralizing X_s^α such that $X_u^{\alpha\beta}$ is upper unitriangular. Then for $x \in X$ we have

$$(x^{\alpha\beta})_s = d(x^{\alpha\beta})$$

where $y \mapsto d(y)$ is the map sending a matrix to its diagonal. Now if $(x(i))$ is a Cauchy sequence relative to the congruence topology in $M_n(\mathbf{Z})$, then $(x(i)^{\alpha\beta})$ is a Cauchy sequence relative to the congruence topology in $M_n(\mathfrak{o})$, where $\mathfrak{o}$ denotes the ring of integers of k; hence so is the sequence $(d(x(i)^{\alpha\beta}))$. It follows that

$$(x(i)_s) = ((d(x(i)^{\alpha\beta}))^{\beta^{-1}\alpha^{-1}})$$

is a Cauchy sequence relative to the congruence topology in $M_n(\mathfrak{o})$. As each $x(i)_s \in M_n(\mathbf{Z})$, the first claim follows.

Now put $\hat{x} = \lim(x(i))$, $\hat{y} = \lim(x(i)_s) \in \mathcal{G}^\infty$. Then

$$\hat{x}\hat{y}^{-1} = \lim(x(i)_u) \text{ is unipotent,}$$

and $\hat{x}$ clearly commutes with $\hat{y}$. So to show that $\hat{y} = \hat{x}_s$, it only remains to check that for each prime p, the p-component $\hat{y}_p$ of $\hat{y}$ is diagonalizable (over some extension field of $\mathbf{Q}_p$). But this is clear since $\hat{y}_p^\alpha$ is easily seen to be diagonal.

LEMMA 5. *Let $(x(i))$ be a sequence in X such that $(x(i)_s)$ is a Cauchy sequence relative to the congruence topology. Then there exists a Cauchy sequence $(y(i))$ in X such that* $\lim(y(i)_s) = \lim(x(i)_s)$.

Proof. We may assume that $x(i+1)_s \equiv x(i)_s \bmod i!$ for each i. For each i there exists $a_i > 0$ such that $X_s^{a_i} \equiv 1 \bmod i!$. Conversely, the congruence topology induces the profinite topology on X_s (see 2.4) so for each i there exists $b_i > 0$ such that

$$z \in X,\ z_s \equiv 1 \bmod b_i! \implies z_s \in X_s^{a_i i! n!}\ .$$

We can choose the numbers b_i so that

$$b_{i+1} > b_i \geqq i \quad \text{for all } i.$$

Now for each j put

$$z(j) = x(b_j)x(b_{j+1})^{-1}\ .$$

Then $z(j)_s \equiv 1 \bmod j!$, and for $j = i$ we have

$$z(i)_s \in X_s^{a_i i! n!}\ .$$

Now the map $x \mapsto x_s$ is a homomorphism of X onto X_s, with kernel K, say. Hence

$$z(i) = c(i)g(i)^{a_i i! n!}$$

with $c(i) \in K$ and $g(i) \in X$. Writing $g(i)_u = 1 + \nu$ where $\nu^n = 0$ and expanding $(1+\nu)^{a_i i! n!}$ by the binomial theorem, we obtain

$$g(i)_u^{a_i i! n!} \equiv 1 \bmod i!\ .$$

Also $\qquad g(i)_s^{a_i i! n!} \equiv 1 \bmod i!$ by the choice of a_i;

thus $\qquad g(i)^{a_i i! n!} \equiv 1 \bmod i!$ and so

$$z(i) \equiv c(i) \bmod i!\ .$$

Now define

$$y(i) = c(1)c(2)\cdots c(i-1)x(b_i)\ .$$

As the $c(j)$ are in K, we get $y(i)_s = x(b_i)_s$ for each i, so certainly

$$\lim\left(y(i)_s\right) = \lim\left(x(i)_s\right) .$$

Now

$$c(i) \equiv z(i) = x(b_i)x(b_{i+1})^{-1} \bmod i! ,$$

therefore

$$\begin{aligned} y(i+1) &= c(1) \cdots c(i-1)c(i)x(b_{i+1}) \\ &\equiv c(1) \cdots c(i-1)x(b_i) \bmod i! \\ &= y(i) . \end{aligned}$$

Thus $\left(y(i)\right)$ is a Cauchy sequence in X and the lemma is proved.

Proof of Proposition 2. If $\hat{x} \in \bar{X}$ then $\hat{x}$ is the limit of a Cauchy sequence $\left(x(i)\right)$ in X. By Lemma 4,

$$\hat{x}_s = \lim\left(x(i)_s\right) \in (X_s)^- .$$

Conversely, suppose $\hat{g} \in (X_s)^-$. Then $\hat{g}$ is the limit of a Cauchy sequence $\left(x(i)_s\right)$ in X_s. By Lemma 5 there exists a Cauchy sequence $\left(y(i)\right)$ in X such that $\hat{g} = \lim\left(y(i)_s\right)$. Putting $\hat{y} = \lim\left(y(i)\right) \in \bar{X}$, we have by Lemma 4 that $\hat{y}_s = \hat{g}$. Thus $\hat{g} \in (\bar{X})_s$. The proof is complete.

7. Semisimple splitting

Suppose G is a polycyclic group with a normal $\mathfrak{T}$ subgroup N such that G/N is free abelian. Then G can be embedded in a split extension of a $\mathfrak{T}$ group by a finitely generated abelian group. This is the main point of semisimple splitting, and forms the basis for example (together with Lemma 3.1) of L. Auslander's proof (see [A], Chapter IV) that G can be embedded in some $\mathrm{GL}_n(\mathbf{Z})$. Thus the bare existence of the splitting is of some interest; to prove it one needs only part (1) of Proposition 5.1, which in turn can be proved more simply than we did in Section 5.

In what follows we make some notational conventions. If N is a normal $\mathfrak{T}$ subgroup of a group G, we denote by

$$\operatorname{Inn} g|_N , \quad \operatorname{Inn} g|_{N^{\mathbf{Q}}}$$

the automorphisms of N and of $N^{\mathbf{Q}}$ respectively induced by conjugation by g, for each $g \in G$. Also we identify $N^{\mathbf{Q}}$ with its Lie algebra, so that

$$(\operatorname{Inn} g|_{N^{\mathbf{Q}}})_s$$

has a meaning, and is considered as an automorphism of $N^{\mathbf{Q}}$. Analogous conventions apply to subgroups of G.

PROPOSITION 1. *Let $G \in \mathfrak{P}$ and let N be a $\mathfrak{T}$ subgroup of G containing G'. Let C be a nilpotent supplement for N in G and put*

$$X = \operatorname{Inn} C|_{N^{\mathbf{Q}}} \leqq \operatorname{Aut} N^{\mathbf{Q}} .$$

Then the following hold:

1) *$X_s \leqq \operatorname{Aut} N^{\mathbf{Q}}$ is abelian.*

2) *There exists $e > 0$, depending only on N and on $\operatorname{Inn} G|_{N^{\mathbf{Q}}}$, and a lattice nilpotent group $\bar{N}$, with $N \leqq \bar{N} \leqq N^{1/e}$, such that G normalizes $\bar{N}$ and X_s stabilizes $\bar{N}$. Now write $\bar{G} = \bar{N}G$.*

3) *There is an injective homomorphism $\tau\colon X_s \to \operatorname{Aut} \bar{G}$ given by*

$$(\mu c)^{x\tau} = \mu^x \cdot c,\ \mu \in \bar{N},\ c \in C,\ x \in X_s .$$

Now put $T = X_s\tau \leqq \operatorname{Aut}\bar{G}$.

4) *Let $M = \operatorname{Fitt}(\bar{G}]T)$. Then $\bar{G}]T = M]T$. Also T acts faithfully on M, and if G/N is torsion-free then $M \in \mathfrak{T}$.*

5) *If D is any nilpotent subgroup of $\bar{G}$ with $D \cap G = C$ then T centralizes D and*

$$(\operatorname{Inn} D|_{N^{\mathbf{Q}}})_s = X_s .$$

Proof. Write

$$g\psi = \operatorname{Inn} g|_{N^{\mathbf{Q}}} ,$$

and put $\Gamma = G\psi$. Then $X = C\psi \leqq \Gamma$ and $\Gamma' \leqq N\psi$ which is unipotent. Therefore X_s is abelian by Lemma 6.1. Also $X \cdot N\psi = \Gamma$, so by Proposition 6.1 there exists a positive integer q, depending only on N and Γ, such that

$$L\langle X, X_s\rangle \subseteqq q^{-1}L$$

where L is the lattice in (the Lie algebra) $N^{\mathbf{Q}}$ generated by N. Then

$$N \leqq \langle L\rangle \leqq \langle L\rangle^{\langle X, X_s\rangle} \leqq \langle q^{-1}L\rangle \leqq N^{1/f}$$

for some f depending only on N and q (where $\langle L\rangle$ denotes the subgroup of $N^{\mathbf{Q}}$ generated by L, etc.). Let $\bar{N}$ be the lattice hull of $\langle L\rangle^{\langle X, X_s\rangle}$. Then $\bar{N}$ is stabilized by X_s and X, hence $\bar{N}$ is normalized by $G = NC$. Moreover,

$$N \leqq \bar{N} \leqq N^{1/e}$$

where e, depending only on N and f, is chosen so that $N^{1/e}$ contains the lattice hull of $N^{1/f}$.

We have now proved parts (1) and (2). For part (3), we will show that τ as described is a well defined map; the other properties of τ can then be trivially verified. Thus we have to show only that X_s acts trivially on $\bar{N} \cap C$. As C is nilpotent, inner automorphisms of C act unipotently on $\bar{N} \cap C$. But X_s is a diagonalizable subgroup of $\operatorname{Aut} \bar{N}$, hence it acts trivially on $\bar{N} \cap C$.

Now we prove (4). For $c \in C$, define

$$u(c) = c \cdot ((c\psi)_s \tau)^{-1} \in \bar{G}]T .$$

Since (by definition) T centralizes C, this map u is an isomorphism of C onto a subgroup $u(C)$ of $\bar{G}]T$. Now put

$$M = \bar{N} \cdot u(C) .$$

From the definition of $u(C)$ we have

$$MT = \bar{N}u(C)T = \bar{N}CT = \bar{N}GT = \bar{G}T .$$

For $c \in C$, the automorphism of $\bar{N}$ induced by conjugating with $u(c)$ is the same as $(c\psi)_u$, so the group $u(C)$ acts unipotently on $\bar{N}$. Thus M is nilpotent. By definition, T acts faithfully as a diagonalizable group of automorphisms of $\bar{N}$; as M acts unipotently on $\bar{N}$ it follows that $M \cap T = 1$. As T centralizes C and is abelian, we have

$$(\bar{G}T)' = (\bar{N}CT)' \leqq \bar{N}C' = \bar{N} \leqq M .$$

So M is normal in $\bar{G}T$, and it follows that $M = \mathrm{Fitt}(GT)$ since no element of T acts unipotently on $\bar{N}$.

It remains to show that $M \in \mathfrak{X}$ if G/N is torsion-free. Now a trivial check shows that

$$M/\bar{N} \cong C/(C \cap \bar{N}) \cong \bar{N}C/\bar{N} = \bar{N}G/\bar{N} \cong G/N .$$

As $\bar{N} \in \mathfrak{X}$ the result follows.

For (5), suppose $D \leqq \bar{G}$ is nilpotent and $D \cap G = C$. Then $D = (D \cap \bar{N})C$. As D is nilpotent the map $y \mapsto y_s$ is a homomorphism of $D\psi$ onto $(D\psi)_s$; so

$$(D\psi)_s = ((D \cap \bar{N})\psi)_s (C\psi)_s = (C\psi)_s = X_s .$$

To see that T centralizes D, note that T acts by conjugation as $(D\psi)_s$ does on $D \cap \bar{N}$, hence trivially, since $D\psi$ acts unipotently on $D \cap \bar{N}$. As T also centralizes C the claim follows.

8. Strategy of the proof

Having made all the preparations we may now embark on the proof of the Theorem. From now on, we let $\mathcal{C}$ denote a set of $\mathfrak{PF}$ groups contained in a single $\hat{\ }$-class; the aim is to prove that $\mathcal{C}$ is contained in the union of finitely many isomorphism classes. This will be done in six steps which we describe below. Several of these involve a reduction of one of the following types:

Reduction of type 1. *Divide $\mathcal{C}$ into finitely many subsets and restrict attention to one of them, which is renamed $\mathcal{C}$.*

Reduction of type 2. *Replace each group* $G \in \mathcal{C}$ *by* G^k *for some fixed positive integer* k, *and rename the resulting set* $\mathcal{C}$.

Reduction of type 3. *Assuming that the groups* $N_G = \mathrm{Fitt}(G)$ *are all isomorphic and in* $\mathfrak{X}$, *replace each* $G \in \mathcal{C}$ *by* $\bar{N}_G G$, *where* $\bar{N}_G$ *is a subgroup of* $N_G^{\mathbf{Q}}$ *normalized by* G *and satisfying* $N_G \leqq \bar{N}_G \leqq N_G^{1/k}$ *for some fixed positive integer* k; *rename the resulting set* $\mathcal{C}$.

These reductions are justified in Section 9. The steps of the proof are as follows:

Step A. *We may assume that for each* $G \in \mathcal{C}$,

$$N_G =_{df} \mathrm{Fitt}(G) \in \mathfrak{X}$$

and G/N_G *is free abelian.*

Step B. *We may assume that the groups* N_G *for* $G \in \mathcal{C}$ *are lattice nilpotent and are all isomorphic.*

Step C. *Write* $Z_G = \zeta_1(N_G)$. *We may assume that for each* G *and* $H \in \mathcal{C}$ *there exist isomorphisms*

$$\theta_{GH}\colon N_G \longrightarrow N_H\,, \quad \psi_{GH}\colon G/Z_G \longrightarrow H/Z_H$$

such that the diagram

$$\begin{array}{ccc} G/Z_G & \xrightarrow{\psi_{GH}} & H/Z_H \\ \downarrow & & \downarrow \\ \mathrm{Aut}\,N_G & \xrightarrow[\theta^*_{GH}]{} & \mathrm{Aut}\,N_H \end{array}$$

commutes, where θ^*_{GH} *is the isomorphism induced by* θ_{GH} *and the vertical maps are the canonical ones coming from inner automorphisms; and for* $G, H, K \in \mathcal{C}$ *we also have*

$$\theta_{GK} = \theta_{GH}\theta_{HK}\,, \quad \psi_{GK} = \psi_{GH}\psi_{HK}\,.$$

Step D. *Write* $Q_G = \zeta(G)$. *We may assume that for each* $G, H \in \mathcal{C}$ *there exists an isomorphism*

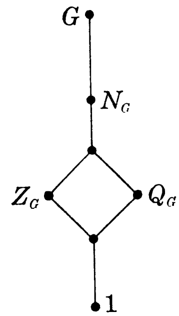
$$\sigma_{GH}\colon G/Q_G \longrightarrow H/Q_H$$

such that

$$\sigma_{GH}|_{N_G/Q_G} = \textit{the isomorphism induced by } \theta_{GH} \textit{ from } N_G/Q_G \to N_H/Q_H.$$

(Note that it follows from Step C that $Q_G\theta_{GH} = Q_H$.) *For* $G, H, K \in \mathcal{C}$ *we also have* $\sigma_{GK} = \sigma_{GH}\sigma_{HK}$.

Step D concludes what might be called the first leg of the proof. Indeed, if the original set $\mathcal{C}$ consisted of groups G such that every subgroup of finite index in G has trivial centre, the proof would be finished at this point. In the general case, we have at least reduced to the situation where the groups N_G, the groups G/Z_G, and the groups G/Q_G can be identified, in a compatible way, for all $G \in \mathcal{C}$. With this information we can embark on the second leg of the proof.



Step E. *We may assume that for each $G \in \mathcal{C}$ there exists a finitely generated abelian subgroup $T_G \leqq \operatorname{Aut} G$ such that the following hold:*

1) $G\,]T_G = M_G\,]T_G$ *where* $M_G = \operatorname{Fitt}(G\,]T_G)$; $M_G \in \mathfrak{X}$ *and* T_G *acts faithfully on* M_G.

2) *For each* G, $H \in \mathcal{C}$ *there exists an isomorphism*

$$\mu_{GH}\colon \hat{G} \longrightarrow \hat{H}$$

such that

$$\hat{T}_G \mu^*_{GH} = \hat{T}_H$$

where $\mu^*_{GH}\colon \operatorname{Aut} \hat{G} \to \operatorname{Aut} \hat{H}$ *is induced by* μ_{GH}.

This is the trickiest step, and uses all of Sections 4–7. However, the idea is fairly simple: after adjusting G/Q_G suitably, we choose a canonical nilpotent supplement C_G/Q_G for N_G/Q_G in G/Q_G, using Proposition 5.1; the isomorphism $\sigma_{GH}\colon G/Q_G \to H/Q_H$ given by Step D enables us to choose the groups C_G/Q_G uniformly for all $G \in \mathcal{C}$, i.e., so that $(C_G/Q_G)\sigma_{GH} = C_H/Q_H$ for all G, $H \in \mathcal{C}$. Then C_G is a nilpotent supplement for N_G in G and we use it to define $T_G \leqq \operatorname{Aut} G$ as in Proposition 7.1. Property (2) of the groups $\hat{T}_G$ can then be obtained with the help of part (2) of Proposition 5.1.

Step F. *For each $G \in \mathcal{C}$ define*

$$\beta_{M_G}\colon G\,]T_G = M_G\,]T_G \longrightarrow \mathrm{GL}_n(\mathbf{Z})$$

as in Lemma 3.1, by identifying $V(M_G)$ with $\mathbf{Z}^n$ for some n. Then n is independent of G as G runs through $\mathcal{C}$, and the images $G\beta_{M_G}$ lie in a single $\sim_{\mathrm{GL}_n}$-class of $\mathrm{GL}_n(\mathbf{Z})$.

Conclusion. By Theorem D the groups $G\beta_{M_G}$ lie in finitely many conjugacy classes in $\mathrm{GL}_n(\mathbf{Z})$. As β_{M_G} is injective, the groups $G \in \mathcal{C}$ lie in finitely many isomorphism classes.

9. Reduction steps

Here we justify the reduction steps described in Section 8. Reductions of type 1 need no explanation. A reduction of type 2 is justified by Proposition 2.1 and Theorem A: if $\hat{G} \cong \hat{H}$ then $(G^k)\hat{} = \hat{G}^k \cong \hat{H}^k = (H^k)\hat{}$ so the groups G^k for $G \in \mathcal{C}$ lie in a single $\hat{}$-class; on the other hand, we also have $H/H^k \cong \hat{H}/\hat{H}^k \cong \hat{G}/\hat{G}^k \cong G/G^k$, so the index $|H: H^k|$ is constant and thus if the groups H^k lie in finitely many isomorphism classes, then so do the groups H.

Type 3 reduction. We will only apply this in the situation where the groups $N_G = \mathrm{Fitt}(G)$ are all isomorphic and in $\mathfrak{T}$. We have a fixed positive integer k and for each $G \in \mathcal{C}$ a group $\bar{N}_G$, normalized by G, with $N_G \leqq \bar{N}_G \leqq N_G^{1/k}$. Write $G^* = \bar{N}_G G$ and put

$$\mathcal{C}^* = \{G^* \mid G \in \mathcal{C}\} .$$

We must establish

Claim 1. *$\mathcal{C}^*$ is contained in the union of finitely many $\hat{}$-classes;*

and

Claim 2. *If $\mathcal{C}^*$ is contained in the union of finitely many isomorphism classes, then so is $\mathcal{C}$.*

Proof of Claim 2. Suppose $G, H \in \mathcal{C}$ *and* $G^* \cong H^*$. Since

$$|H^*: H| = |\bar{N}_H: N_H| \leqq |N_H^{1/k}: N_H| ,$$

H is isomorphic to one of the finitely many subgroups of index at most $|N_H^{1/k}: N_H|$ in G^*. Since we are assuming that the groups N_H for $H \in \mathcal{C}$ are all isomorphic, this index is independent of H and the claim follows.

Proof of Claim 1. Let G, H be in $\mathcal{C}$ and suppose $\theta: \hat{H} \to \hat{G}$ is an isomorphism. Then $\hat{N}_H\theta = \hat{N}_G$ by Theorem C, so θ extends to an isomorphism

$$\theta^*: (N_H^{1/k})\hat{}\,\hat{H} \longrightarrow (N_G^{1/k})\hat{}\,\hat{G}$$

(Lemma 2.10(c)). Now

$$\hat{H} \leqq (H^*)\hat{} \leqq (N_H^{1/k})\hat{}\,\hat{H}$$

so

$$\hat{G} = \hat{H}\theta^* \leqq (H^*)\hat{}\,\theta^* \leqq (N_G^{1/k})\hat{}\,\hat{G} .$$

But

$$|(N_G^{1/k})\hat{}\,\hat{G}: \hat{G}| = |(N_G^{1/k})\hat{}: \hat{N}_G| = |N_G^{1/k}: N_G| ,$$

so there are only finitely many subgroups of $(N_G^{1/k})^\wedge \hat{G}$ containing $\hat{G}$, hence only finitely many possibilities for $(H^*)^\wedge \theta^*$. Thus there are only finitely many possibilities for $(H^*)^\wedge$ up to isomorphism.

10. Steps A and B

Step A. *We may assume that for every $G \in \mathcal{C}$, $N_G \in \mathfrak{T}$ and G/N_G is free abelian.*

Proof. Choose $G \in \mathcal{C}$. There exists $k_1 > 0$ such that G^{k_1} is torsion-free. Choose a multiple k_2 of k_1 such that $\mathrm{Fitt}(G^{k_2})$ has Hirsch number as large as possible. There exists a multiple k of k_2 such that $G^k\,\mathrm{Fitt}(G^{k_2})/\mathrm{Fitt}(G^{k_2})$ is free abelian. The choice of k_2 ensures that

$$\mathrm{Fitt}(G^k) = G^k \cap \mathrm{Fitt}(G^{k_2}) ,$$

so we have $\mathrm{Fitt}(G^k) \in \mathfrak{T}$ and $G^k/\mathrm{Fitt}(G^k)$ is free abelian. Now if $H \in \mathcal{C}$ then Theorem C shows that

$$(\mathrm{Fitt}(H^k))^\wedge \cong (\mathrm{Fitt}(G^k))^\wedge$$

and

$$(H^k/\mathrm{Fitt}(H^k))^\wedge \cong (G^k/\mathrm{Fitt}(G^k))^\wedge .$$

It follows that $\mathrm{Fitt}(H^k)$ is torsion-free and $H^k/\mathrm{Fitt}(H^k)$ is torsion-free abelian. As $H^k \in \mathfrak{P}\mathfrak{F}$ we have $\mathrm{Fitt}(H^k) \in \mathfrak{T}$ and $H^k/\mathrm{Fitt}(H^k)$ free abelian. Finally a type 2 reduction allows us to replace each $H \in \mathcal{C}$ by H^k to finish Step A.

Step B. *We may assume that the groups N_G for $G \in \mathcal{C}$ are all isomorphic, and are lattice nilpotent.*

Proof. Theorem C, Theorem B and a type 1 reduction allow us to assume that the groups N_G are all isomorphic. Write $\bar{N}_G$ for the lattice hull of N_G. Then there exists $k > 0$ such that $N_G \leqq \bar{N}_G \leqq N_G^{1/k}$ for all $G \in \mathcal{C}$, also G normalizes $\bar{N}_G$, $\bar{N}_G = \mathrm{Fitt}(\bar{N}_G G)$, and $\bar{N}_G G/\bar{N}_G \cong G/N_G$ is free abelian for each G. We may thus replace each $G \in \mathcal{C}$ by the group $\bar{N}_G G$, using a reduction of type 3, to finish Step B.

11. Step C

Here we show that we may assume the following: *there exist compatible families of isomorphisms*

$$\theta_{GH}: N_G \longrightarrow N_H \qquad (G,\ H \in \mathcal{C}),$$

and

$$\psi_{GH}: G/Z_G \longrightarrow H/Z_H \qquad (G,\ H \in \mathcal{C})$$

such that the canonical diagrams

$$\begin{array}{ccc} G/Z_G & \xrightarrow{\psi_{GH}} & H/Z_H \\ \downarrow & & \downarrow \\ \operatorname{Aut} N_G & \xrightarrow[\theta^*_{GH}]{} & \operatorname{Aut} N_H \end{array}$$

commute.

(Here and henceforth, to save space we call a family of maps such as $(\theta_{GH})_{G,H\in\mathcal{C}}$ *compatible* if for all G, H, $K \in \mathcal{C}$, we have $\theta_{GK} = \theta_{GH}\theta_{HK}$.)

Proof. Fix $G \in \mathcal{C}$ and put $N = N_G$. By Step B, for each $H \in \mathcal{C}$ there is an isomorphism $\varphi_H: N_H \to N$, inducing say $\varphi_H^*: \operatorname{Aut} N_H \to \operatorname{Aut} N$. Denote by $\rho_H: H \to \operatorname{Aut} N_H$ the map given by taking inner automorphisms. As N is assumed lattice nilpotent, we may identify $\operatorname{Aut} N$ with $\mathcal{G}(\mathbf{Z})$ where $\mathcal{G}$ is the algebraic $\mathbf{Q}$-group $\operatorname{Aut}(\log N^{\mathbf{Q}})$; see Lemma 2.9. Now we make the following

Claim: for each $H \in \mathcal{C}$,

$$H\rho_H\varphi_H^* \sim_{\mathcal{G}} G\rho_G \quad in \quad \mathcal{G}(\mathbf{Z}) .$$

Accepting the claim for now, we may deduce by Theorem D that the groups $H\rho_H\varphi_H^*$ lie in finitely many conjugacy classes in $\operatorname{Aut} N$ as H runs through $\mathcal{C}$. By a reduction of type 1, we may thus assume that the groups $H\rho_H\varphi_H^*$ are all conjugate (as we have subdivided $\mathcal{C}$, the chosen group G may no longer be in $\mathcal{C}$ now). Fix $K \in \mathcal{C}$; then for each $H \in \mathcal{C}$ there exists $\alpha_H \in \operatorname{Aut} N$ such that

$$(H\rho_H\varphi_H^*)^{\alpha_H} = K\rho_K\varphi_K^* .$$

Define θ_{HK} to be the composite isomorphism

$$\theta_{HK} = \varphi_H\alpha_H\varphi_K^{-1}: N_H \longrightarrow N_K ,$$

and let $\theta_{HK}^*: \operatorname{Aut} N_H \to \operatorname{Aut} N_K$ be the induced isomorphism. Then $H\rho_H\theta_{HK}^* = K\rho_K$; as $Z_H = \operatorname{Ker}\rho_H$ and $Z_K = \operatorname{Ker}\rho_K$ we obtain an induced isomorphism

$$\psi_{HK}: H/Z_H \longrightarrow K/Z_K$$

such that the following diagram commutes:

$$\begin{array}{ccc} H/Z_H & \xrightarrow{\psi_{HK}} & K/Z_K \\ \downarrow & & \downarrow \\ \operatorname{Aut} N_H & \xrightarrow[\theta^*_{HK}]{} & \operatorname{Aut} N_K . \end{array}$$

Finally, for any H_1, $H_2 \in \mathcal{C}$ put

$$\theta_{H_1H_2} = \theta_{H_1K}\theta_{H_2K}^{-1} , \quad \psi_{H_1H_2} = \psi_{H_1K}\psi_{H_2K}^{-1}$$

to complete Step C.

Proof of the claim. We are to show that

$$H\rho_H\varphi_H^* \sim_{\mathcal{G}} G\rho_G \quad \text{in} \quad \mathcal{G}(\mathbf{Z}) .$$

Let $\lambda\colon \hat{H} \to \hat{G}$ be an isomorphism. Then $\hat{N}_H\lambda = \hat{N}_G$ by Theorem C; extending φ_H to an isomorphism $\hat{\varphi}_H\colon \hat{N}_H \to \hat{N}_G$, we obtain

$$\gamma = \hat{\varphi}_H^{-1}\lambda \in \operatorname{Aut}\hat{N} = \mathcal{G}^\infty .$$

To establish our claim it will suffice to show that

$$(H\rho_H\varphi_H^*)^\gamma\pi_m = (G\rho_G)\pi_m \qquad \text{for all } m \neq 0$$

where $\pi_m\colon \mathcal{G}^\infty \to \mathrm{GL}_n(\mathbf{Z}/m\mathbf{Z})$ denotes the canonical map. Fix $m \neq 0$. There exist $k > 0$ such that $\log(N^k) \leqq m\log N$ (see Lemma 2.8) and $l > 0$ such that $G^l \cap N \leqq N^k$ and $H^l \cap N_H \leqq N_H^k$. Then λ induces an isomorphism $\bar{\lambda}\colon H/H^l \to G/G^l$ sending the image of N_H/N_H^k onto that of N/N^k; and φ_H induces an isomorphism $\bar{\varphi}_H\colon N_H/N_H^k \to N/N^k$, which induces an isomorphism $\bar{\varphi}_H^*\colon \operatorname{Aut}(N_H/N_H^k) \to \operatorname{Aut}(N/N^k)$. By the choice of l, the composite map

$$H \xrightarrow{\rho_H} \operatorname{Aut}(N_H) \xrightarrow{\text{canon.}} \operatorname{Aut}(N_H/N_H^k)$$

factors through H/H^l, and an analogous statement holds for G. The situation is described by a commutative diagram of solid arrows:

$$\begin{tikzcd}
H \arrow[rrr, two heads] \arrow[dr, "\rho_H"] & & & H/H^l \arrow[dl, "\bar{\rho}_H"'] \arrow[ddd, dashed, "\bar{\lambda}", "\wr"'] \\
& \operatorname{Aut} N_H \arrow[r] \arrow[d, "\varphi_H^*"', "\wr"] & \operatorname{Aut}(N_H/N_H^k) \arrow[d, "\bar{\varphi}_H^*"', "\wr"] \arrow[d, dashed, bend left, "\bar{\lambda}^*"] & \\
& \operatorname{Aut} N \arrow[r, "\tau"] & \operatorname{Aut}(N/N^k) & \\
G \arrow[rrr, two heads] \arrow[ur, "\rho_G"] & & & G/G^l \arrow[ul]
\end{tikzcd}$$

If $\bar{\lambda}^*\colon \operatorname{Aut}(N_H/N_H^k) \to \operatorname{Aut}(N/N^k)$ is the map induced by the isomorphism $N_H/N_H^k \to N/N^k$ which $\bar{\lambda}$ induces, then the dotted square also commutes.

Hence the image in $\operatorname{Aut}(N/N^k)$ of $G\rho_G$ is equal to

$$(H/H^l)\bar{\rho}_H\bar{\lambda}^* = (H\rho_H)\varphi_H^*\tau\bar{\varphi}_H^{*-1}\bar{\lambda}^*$$

(where $\tau\colon \operatorname{Aut} N \to \operatorname{Aut}(N/N^k)$ denotes the canonical map), and this is equal to the image in $\operatorname{Aut}(N/N^k)$ of $(H\rho_H\varphi_H^*)^\gamma$, because for $\alpha \in \operatorname{Aut}(N/N^k)$,

$$\alpha\bar{\varphi}_H^{*-1}\bar{\lambda}^* = \bar{\gamma}^{-1}\alpha\bar{\gamma}$$

where $\bar{\gamma} \in \operatorname{Aut}(N/N^k)$ is induced by γ. Thus $G\rho_G$ and $(H\rho_H\varphi_H^*)^\gamma$ induce the same group of automorphisms on N/N^k. By the choice of k, it follows that they have the same image under π_m, which is what we had to show.

12. Step D

Now $Q_G = \zeta(G) \leqq N_G$ for each $G \in \mathcal{C}$. In this section we show that *we may assume that there exists a compatible family of isomorphisms σ_{GH}: $G/Q_G \to H/Q_H$ for G, $H \in \mathcal{C}$, such that σ_{GH} restricts to the map induced by θ_{GH} from $N_G/Q_G \to N_H/Q_H$.*

Fix $G \in \mathcal{C}$, put $N = N_G$ and $Q = Q_G$. Let

$$\omega \colon G/N \longrightarrow \operatorname{Out}(N/Q) = \operatorname{Aut}(N/Q)/\operatorname{Inn}(N/Q)$$

be the map induced by taking inner automorphisms of G. Now for each $H \in \mathcal{C}$ there is an exact sequence

$$\begin{tikzcd}[column sep=small]
\mathcal{E}(H)\colon 1 \arrow[r] & N/Q \arrow[rr] \arrow[dr, dotted, "\theta^*"'] & & H/Q_H \arrow[rr] \arrow[dr, dotted] & & G/N \arrow[r] & 1 \\
& & N_H/Q_H \arrow[ur, dotted] & & H/N_H \arrow[ur, dotted, "\psi^*"'] & &
\end{tikzcd}$$

defined by specifying that the dotted triangles commute, where θ^* is induced by θ_{GH}, ψ^* is induced by ψ_{GH}^{-1}, and the other dotted arrows are the canonical inclusion and projection maps. The relationship between θ_{GH} and ψ_{GH} established in Step C shows that the map from G/N into $\operatorname{Out}(N/Q)$ induced by $\mathcal{E}(H)$ is equal to ω. Thus in the terminology of Gruenberg [G], Chapter 5, the extensions $\mathcal{E}(H)$ for $H \in \mathcal{C}$ all belong to the class

$$\left(\frac{G/N}{N/Q,\ \omega}\right).$$

With A for the centre of N/Q, Section 5.4 of [G] shows that the equivalence classes of extensions in this class are in bijective correspondence with $H^2(G/N, A)$. Now from the definition of Q we have $H^0(G/N, A) = 0$, hence by Robinson's result, Lemma 4.1, the group $H^2(G/N, A)$ is finite. Thus the extensions $\mathcal{E}(H)$ lie in finitely many equivalence classes.

After a type 1 reduction, we may assume that the extensions $\mathcal{E}(H)$ for $H \in \mathcal{C}$ are all equivalent (as usual, though, the group G may no longer be in $\mathcal{C}$). Fix $K \in \mathcal{C}$. Then for each $H \in \mathcal{C}$ there exists an isomorphism σ_{HK} making the solid diagram below commute:

$$\begin{tikzcd}
& N_H/Q_H \arrow[dr, dashed] & \\
N/Q \arrow[ur, dashed] \arrow[rr] \arrow[d, equal] \arrow[dr, phantom] & & H/Q_H \arrow[d, "\sigma_{HK}"] \\
N/Q \arrow[rr] \arrow[dr, dashed] & & K/Q_K \\
& N_K/Q_K \arrow[ur, dashed] &
\end{tikzcd}$$

Filling in the dotted arrows shows that σ_{HK} does indeed restrict to the map induced by $\theta_{HK} = \theta_{HG}\theta_{KG}^{-1}$ from N_H/Q_H to N_K/Q_K. Finally, for $H_1, H_2 \in \mathcal{C}$ define

$$\sigma_{H_1H_2} = \sigma_{H_1K}\sigma_{H_2K}^{-1}$$

to complete Step D.

13. Step E

Fix $G \in \mathcal{C}$. We apply Proposition 5.1 to the group G/Q and its nilpotent normal subgroup N/Q (where $N = N_G$ and $Q = Q_G$). Thus there exists $k > 0$ such that the set $\mathfrak{X}((N/Q)^{1/k})$ of canonical nilpotent supplements for $(N/Q)^{1/k}$ in $(N/Q)^{1/k}(G/Q)$ is nonempty. Putting $Q^* = N^{1/k} \cap Q^{\mathbf{Q}}$ we may identify $(N/Q)^{1/k}$ with $N^{1/k}/Q^*$ and $(N/Q)^{1/k}(G/Q)$ with $(N^{1/k}G)/Q^*$. Having done so, choose $C/Q^* \in \mathfrak{X}(N^{1/k}/Q^*)$. Then C is nilpotent, as Q^* is the hypercentre of $N^{1/k}G$, and $N^{1/k}C = N^{1/k}G$. Thus we may now apply Proposition 7.1 to the group $N^{1/k}G$ with its nilpotent normal subgroup $N^{1/k}$ and nilpotent supplement C. This gives the following: there exists $e > 0$, depending only on $N^{1/k}$ and on $\operatorname{Inn}(N^{1/k}G)|_{N^{\mathbf{Q}}}$, and a lattice nilpotent group $\bar{N}$, normalized by $N^{1/k}G$, such that

$$N^{1/k} \leqq \bar{N} \leqq (N^{1/k})^{1/e} ;$$

and there exists an abelian subgroup $T_G \leqq \operatorname{Aut}(\bar{N}G)$ such that T_G centralizes C and acts like

$$(\operatorname{Inn} C|_{N^{\mathbf{Q}}})_s$$

on $\bar{N}$; also $(\bar{N}G)\,]\,T_G = M\,]\,T_G$ where $M = \operatorname{Fitt}((\bar{N}G)\,]\,T_G) \in \mathfrak{T}$.

Now put $\bar{Q} = \bar{N} \cap Q^{\mathbf{Q}}$, and identify $\bar{N}/\bar{Q}$ with a subgroup of $(N/Q)^{\mathbf{Q}}$ containing $(N/Q)^{1/k}$. Part (3) of Proposition 5.1 then shows that there exists $D/\bar{Q} \in \mathfrak{X}(\bar{N}/\bar{Q})$ such that $D = (\bar{N} \cap D)C$ and $C = D \cap N^{1/k}G$. Then by part (5) of Proposition 7.1, T_G centralizes D and acts like $(\operatorname{Inn} D|_{N^{\mathbf{Q}}})_s$ on $\bar{N}$.

Let $H \in \mathcal{C}$. The isomorphism $\theta_{GH}\colon N_G \to N_H$ extends to an isomorphism $N^{\mathbf{Q}} \to N_H^{\mathbf{Q}}$ which sends Q^* to $N_H^{1/k} \cap Q_H^{\mathbf{Q}} = Q_H^*$, say. The isomorphism σ_{GH}: $G/Q \to H/Q_H$ extends to an isomorphism $\sigma_{GH}\colon N^{1/k}G/Q^* \to N_H^{1/k}H/Q_H^*$. Define $C_H \leqq N_H^{1/k}H$ by

$$C_H/Q_H^* = (C/Q^*)\sigma_{GH} .$$

Just as we did above for G, we can now find $\bar{N}_H$ with $N_H^{1/k} \leqq \bar{N}_H \leqq (N_H^{1/k})^{1/e}$ such that H normalizes $\bar{N}_H$ and $(\operatorname{Inn} C_H|_{N_H^{\mathbf{Q}}})_s$ stabilizes $\bar{N}_H$; the number e is the same as before because the pairs $(N^{1/k}, \operatorname{Inn}(N^{1/k}G)|_{N^{\mathbf{Q}}})$ and $(N_H^{1/k}, \operatorname{Inn}(N_H^{1/k}H)|_{N_H^{\mathbf{Q}}})$ are isomorphic by the pair of maps (θ_{GH}, ψ_{GH}) (see Step C).

Now we want to ensure that θ_{GH} sends $\bar{N}$ to $\bar{N}_H$. This may not be so in general; however, there are only finitely many subgroups between $N^{1/k}$ and

$(N^{1/k})^{1/e}$, hence only finitely many possibilities for $\bar{N}_H\theta_{GH}^{-1}$ as H runs through $\mathcal{C}$. After a type 1 reduction we may thus assume that $\bar{N}_H\theta_{GH}^{-1}$ is constant as H runs through $\mathcal{C}$; noting that all the families of maps (θ_{GH}), (σ_{GH}), (ψ_{GH}) are compatible, we may safely choose a new "reference group" G and assume now that

$$\bar{N}\theta_{GH} = \bar{N}_H \qquad \text{for all} \quad H \in \mathcal{C} .$$

Put $\bar{Q}_H = \bar{N}_H \cap Q_H^0$. Then $\bar{Q}\theta_{GH} = \bar{Q}_H$ and σ_{GH} extends to an isomorphism $\bar{N}G/\bar{Q} \to \bar{N}_H H/\bar{Q}_H$. Define $D_H \leqq \bar{N}_H H$ by

$$D_H/\bar{Q}_H = (D/\bar{Q})\sigma_{GH} .$$

Then we have $D_H = (\bar{N}_H \cap D_H)C_H$ and $D_H \cap N_H^{1/k}H = C_H$. Define $T_H \leqq \mathrm{Aut}(\bar{N}_H H)$ by applying Proposition 7.1 to $N_H^{1/k}H$ with its subgroups $N_H^{1/k}$ and C_H. Then part (5) of that proposition shows that T_H centralizes D_H and acts like $(\mathrm{Inn}\, D_H|_{N_H^0})_s$ on $\bar{N}_H$.

By a reduction of type 3 we may assume that the groups $\bar{N}_H H$ lie in a single $\hat{}$-class; again we may have to choose a new "reference group" for G, but this is allowed since all the constructions so far are made in terms of compatible families of maps (θ_{GH}), (σ_{GH}). Thus if $H \in \mathcal{C}$ we have an isomorphism

$$\mu = \mu_H \colon (\bar{N}_H H)^\wedge \longrightarrow (\bar{N}G)^\wedge .$$

Then $(\bar{N}_H)^\wedge\mu = (\bar{N})^\wedge$ by Theorem C, and $(\bar{Q}_H)^\wedge\mu = (\bar{Q})^\wedge$ by Lemma 2.12. So there is an induced isomorphism

$$\bar{\mu} \colon (\bar{N}_H H/\bar{Q}_H)^\wedge \longrightarrow (\bar{N}G/\bar{Q})^\wedge .$$

The isomorphism $\sigma_{GH} \colon \bar{N}G/\bar{Q} \to \bar{N}_H H/\bar{Q}_H$ induces an isomorphism

$$\hat{\sigma} = \hat{\sigma}_{GH} \colon (\bar{N}G/\bar{Q})^\wedge \longrightarrow (\bar{N}_H H/\bar{Q}_H)^\wedge .$$

The composite map $\hat{\sigma}\bar{\mu}$ is then an automorphism of $(\bar{N}G/\bar{Q})^\wedge$ fixing $(\bar{N}/\bar{Q})^\wedge$. Now recall that

$$D/\bar{Q} \in \mathfrak{X}(\bar{N}/\bar{Q}) ;$$

hence by part (2) of Proposition 5.1, there are only finitely many possibilities up to conjugacy in $(\bar{N}G)^\wedge$ for

$$(D/\bar{Q})^\wedge \hat{\sigma}_{GH}\bar{\mu}_H$$

as H runs through $\mathcal{C}$. As μ_H was an arbitrary isomorphism $(\bar{N}_H H)^\wedge \to (\bar{N}G)^\wedge$, we may adjust μ_H for each $H \in \mathcal{C}$ by tacking on a suitable inner automorphism of $(\bar{N}G)^\wedge$, so that as H runs through $\mathcal{C}$ only finitely many distinct groups $(D/\bar{Q})^\wedge\hat{\sigma}_{GH}\bar{\mu}_H$ occur. Recalling the definition of D_H, we see that there are only finitely many distinct groups $\hat{D}_H\mu_H$ as H runs through $\mathcal{C}$. Now making reductions of type 1 and 3 we may assume the following:

For each $H \in \mathcal{C}$, N_H *is lattice nilpotent; there exists* $D_H \leqq H$ *and* $T_H \leqq \operatorname{Aut} H$ *such that* $N_H D_H = H$, T_H *centralizes* D_H, *and* T_H *acts like* $(\operatorname{Inn} D_H|_{N_H^0})_s$ *on* N_H. *Also*

$$H]T_H = M_H]T_H \quad \textit{where} \quad M_H = \operatorname{Fitt}(H]T_H) \in \mathfrak{X} ;$$

and for each $H, K \in \mathcal{C}$ *there exists an isomorphism* $\mu_{HK}: \hat{H} \to \hat{K}$ *such that* $\hat{D}_H \mu_{HK} = \hat{D}_K$.

(For μ_{HK} we simply take $\mu_H \mu_K^{-1}$.) To complete Step E we must show that

$$(1) \qquad \hat{T}_H \mu_{HK}^* = \hat{T}_K$$

for all $H, K \in \mathcal{C}$, *where* $\mu_{HK}^*: \operatorname{Aut}\hat{H} \to \operatorname{Aut}\hat{K}$ *is induced by* μ_{HK}. Now $\hat{K} = \hat{N}_K \hat{D}_K$ and it is easy to see that both members of (1) centralize $\hat{D}_K$, so it will suffice to show that the restriction of (1) to $\operatorname{Aut}\hat{N}_K$ is valid. As N_H, N_K are lattice nilpotent we can identify $\operatorname{Aut}\hat{N}_H$ with $\mathcal{G}_H^\infty$ and $\operatorname{Aut}\hat{N}_K$ with $\mathcal{G}_K^\infty$ where $\mathcal{G}_H = \operatorname{Aut}(\log N_H^0)$ and $\mathcal{G}_K = \operatorname{Aut}(\log N_K^0)$, and we have an isomorphism

$$\mu^* = \mu_{HK}^*: \mathcal{G}_H^\infty \longrightarrow \mathcal{G}_K^\infty .$$

As $\hat{D}_H \mu_{HK} = \hat{D}_K$ it follows that

$$\begin{aligned}(\operatorname{Inn} D_H|_{N_H})\hat{}\ \mu^* &= (\operatorname{Inn}\hat{D}_H|_{\hat{N}_H})\mu^* \\ &= \operatorname{Inn}\hat{D}_H \mu_{HK}|_{\hat{N}_K} \\ &= \operatorname{Inn}\hat{D}_K|_{\hat{N}_K} = (\operatorname{Inn} D_K|_{N_K})\hat{}\ .\end{aligned}$$

Now $(\operatorname{Inn} D_H|_{N_H})\hat{}$ is the congruence closure in $\mathcal{G}_H^\infty$ of $\operatorname{Inn} D_H|_{N_H}$, by Corollary 2.6; hence by Proposition 6.2 we have

$$\hat{T}_H|_{\hat{N}_H} = ((\operatorname{Inn} D_H|_{N_H})_s)\hat{}\ = ((\operatorname{Inn} D_H|_{N_H})\hat{}\ )_s .$$

Similarly

$$\hat{T}_K|_{\hat{N}_K} = ((\operatorname{Inn} D_K|_{N_K})\hat{}\ )_s .$$

Therefore

$$(\hat{T}_H\mu^*)|_{\hat{N}_K} = (\hat{T}_H|_{\hat{N}_H})\mu^* = \hat{T}_K|_{\hat{N}_K} ,$$

as it is clear from the definition that μ^* respects the Jordan decomposition.

Thus (1) is established and Step E is complete.

14. Step F

Fix $G \in \mathcal{C}$. For each $H \in \mathcal{C}$ there is an isomorphism $\mu_{HG}: \hat{H} \to \hat{G}$ such that the induced map $\mu_{HG}^*: \operatorname{Aut}\hat{H} \to \operatorname{Aut}\hat{G}$ sends $\hat{T}_H$ to $\hat{T}_G$. Then μ_{HG} extends to an isomorphism

$$\lambda_{HG}: (H]T_H)\hat{}\ \longrightarrow (G]T_G)\hat{}\ .$$

By Theorem C, $\hat{M}_H \lambda_{HG} = \hat{M}_G$, and by construction $\hat{H}\lambda_{HG} = \hat{G}$ and $\hat{T}_H \lambda_{HG} = \hat{T}_G$.

Hence we may apply Lemmas 3.3 and 3.4 to deduce that there exists an isomorphism $\tau_{HG}: V(M_H) \to V(M_G)$ and that

$$H\beta_{M_H}\tau^*_{HG} \sim G\beta_{M_G} \quad \text{in} \quad \operatorname{Aut} V(M_G)$$

where τ^*_{HG}: Aut $V(M_H) \to$ Aut $V(M_G)$ is induced by τ_{HG}. So if we fix a $\mathbf{Z}$-basis in $V(M_G)$ and choose as $\mathbf{Z}$-basis in $V(M_H)$ the image of it under τ^{-1}_{HG}, we then have

$$H\beta_{M_H} \sim G\beta_{M_G} \quad \text{in} \quad \mathrm{GL}_n(\mathbf{Z})\,.$$

This holds for each $H \in \mathcal{C}$, so the groups $H\beta_{M_H}$ lie in a single $\sim$-class of $\mathrm{GL}_n(\mathbf{Z})$.

This concludes the proof.

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