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# Classes of separable two-generator free subgroups of 3-manifold groups

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**Abstract**

In this paper, we consider subgroup separability of free products with amalgamation of Kleinian groups of finite covolume. Specifically, we prove the following. Let $M_1$ and $M_2$ be compact orientable 3-manifolds with non-empty boundary whose interiors admit complete hyperbolic structures of finite volume. Fix boundary components $T_1 \in \partial M_1$ and $T_2 \in \partial M_2$, and let $f : T_1 \to T_2$ be a homeomorphism. Let $M$ be the manifold obtained by identifying $M_1$ and $M_2$ along the fixed boundary components via $f$. If $\alpha$ and $\beta$ are loxodromic elements of $\pi_1(M_1)$ and $\pi_1(M_2)$, respectively, then $\langle \alpha, \beta \rangle$ is a separable subgroup of $\pi_1(M)$.


*MSC:* 20E26; 57M05

*Keywords:* Subgroup separability; Free products with amalgamation; Hyperbolic 3-manifolds

## 1. Introduction

**Definition 1.1.** Let $H$ be a subgroup of a group $G$. $H$ is *separable* in $G$ if, given any element $g \in G \setminus H$, there is a finite index subgroup $K \subset G$, such that $H \subset K$ but $g \notin K$. A group $G$ is *residually finite* if the subgroup consisting of the identity element is separable in $G$. Equivalently $G$ is residually finite if for any non-trivial element $g$ of $G$, there is a homomorphism $\phi$ of $G$ to a finite group with $\phi(g)$ non-trivial. A group $G$ is *subgroup separable*, or LERF (locally extended residually finite), if every finitely generated subgroup of $G$ is separable in $G$.

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Subgroup separability is a powerful property. It enables one to pass from immersions to embeddings in finite sheeted covering spaces. Specifically, if $X$ is a Hausdorff topological space with a regular covering $\widetilde{X}$ and covering group $G$, then $G$ is subgroup separable if and only if given a finitely generated subgroup $S$ of $G$ and a compact subset $C$ of $\widetilde{X}/S$, there is a finite covering $X_1$ of $X$ such that the projection $\widetilde{X}/S \to X$ factors through $X_1$ and $C$ projects homeomorphically into $X_1$ [9].

By a *Haken* 3-manifold we mean a compact, orientable, irreducible 3-manifold which contains a 2-sided incompressible surface. If $M$ is a closed Haken manifold, then by the Jaco–Shalen–Johannson decomposition theorem there exists a finite collection of disjoint, two-sided, incompressible tori $\{T_i\}$ in $M$ which split $M$ into 3-manifolds $\{M_i\}$ which are either Seifert fibered spaces, or have a complete hyperbolic structure of finite volume. Let $\mathcal{T} = \bigcup T_i$. We call each component $M_i$ of $\overline{M \setminus \mathcal{T}}$ a *vertex manifold*. The manifold $M$ is a *graph* manifold if each vertex manifold is a Seifert fibered space.

There are interesting results on separability of graph manifolds. For example, in [8] a criterion is given for an immersed horizontal $\pi_1$-injective surface in a graph manifold to be separable. Moreover, examples are constructed of such surfaces which are not separable. However, very little is known about separability of Haken manifolds where each vertex manifold is hyperbolic. We consider the following open question in this setting. Let $M_1$ and $M_2$ be compact orientable 3-manifolds with non-empty boundary whose interiors admit complete hyperbolic structures of finite volume. Fix boundary components $T_1 \in \partial M_1$ and $T_2 \in \partial M_2$, and let $f : T_1 \to T_2$ be a homeomorphism. Let $M$ be the manifold obtained by identifying $M_1$ and $M_2$ along the fixed boundary components via $f$. If $A$ is a separable subgroup of $\pi_1(M_1)$ and $B$ is a separable subgroup of $\pi_1(M_2)$, is $\langle A, B\rangle$ a separable subgroup of $\pi_1(M)$? In this paper, we give an affirmative answer in the case where $A$ and $B$ are cyclic subgroups generated by loxodromic elements.

**Theorem 1.2.** *Let $M_1$ and $M_2$ be compact orientable* 3*-manifolds with non-empty boundary whose interiors admit complete hyperbolic structures of finite volume. Fix boundary components $T_1 \in \partial M_1$ and $T_2 \in \partial M_2$, and let $f : T_1 \to T_2$ be a homeomorphism. Let $M$ be the manifold obtained by identifying $M_1$ and $M_2$ along the fixed boundary components via $f$. If $\alpha \in \pi_1(M_1)$ and $\beta \in \pi_1(M_2)$ are loxodromic elements, then $\langle \alpha, \beta\rangle$ is a separable subgroup of $\pi_1(M)$.*

We illustrate the idea of the proof of Theorem 1.2 with two examples. First, fix basepoints $x_1 \in T_1 \subset M_1$ and $x_2 = f(x_1) \in M_2$, and let $x \in M$ be the point where $x_1$ and $x_2$ are identified. Suppose that $g \in \pi_1(M_1, x_1) \setminus \langle \alpha\rangle$. To find a subgroup $K$ of finite index in $\pi_1(M, x)$ such that $\langle \alpha, \beta\rangle \subset K$ but $g \notin K$, we find finite sheeted covering spaces $(\widetilde{M}_1, \tilde{x}_1) \to (M_1, x_1)$ and $(\widetilde{M}_2, \tilde{x}_2) \to (M_2, x_2)$ such that: (i) the lift of $\alpha$ in $\widetilde{M}_1$ with initial point $\tilde{x}_1$ is a loop, (ii) the lift of $\beta$ in $\widetilde{M}_2$ with initial point $\tilde{x}_2$ is a loop, and (iii) the lift of $g$ in $\widetilde{M}_1$ with initial point $\tilde{x}_1$ is not a loop. Furthermore, we require that $\widetilde{M}_1$ and $\widetilde{M}_2$ be compatible, as defined in Section 2 of [6]. Then by Theorem 2.2 of [6], there exists a finite sheeted covering space $(\widetilde{M}, \tilde{x}) \to (M, x)$, where $\widetilde{M}$ is the graph product of a graph whose vertex spaces are copies of $\widetilde{M}_1$ and $\widetilde{M}_2$. Let $K$ be the subgroup of $\pi_1(M, x)$ corresponding to $\widetilde{M}$. Then by construction, $\langle \alpha, \beta\rangle \subset K$, but $g \notin K$. For more general words, we require that $\widetilde{M}_1$ and $\widetilde{M}_2$ satisfy additional conditions. For

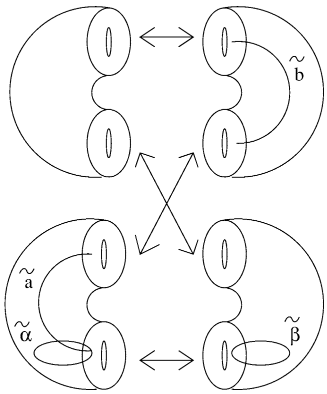


Fig. 1. The covering space $N$.

example, suppose $g = ab$, where $a \in \pi_1(M_1, x_1) \setminus \{\alpha^t\gamma \mid t \in \mathbb{Z},\ \gamma \in \pi_1(T_1, x_1)\}$ and $b \in \pi_1(M_2, x_2) \setminus \{\beta^t\gamma \mid t \in \mathbb{Z},\ \gamma \in \pi_1(T_2, x_2)\}$. Let $T$ denote the torus in $M$ where $T_1$ and $T_2$ are identified. To separate $g$ from $\langle\alpha, \beta\rangle$, we find finite sheeted covering spaces $p_1 : (\widetilde{M}_1, \tilde{x}_1) \to (M_1, x_1)$ and $p_2 : (\widetilde{M}_2, \tilde{x}_2) \to (M_2, x_2)$ that satisfy the conditions above. Furthermore, we require that the terminal point of the lift of $a$ in $\widetilde{M}_1$ with initial point $\tilde{x}_1$ lie on a different component of $p_1^{-1}(T)$ than $\tilde{x}_1$ does. We have a similar requirement for $b$. As in the first example, there exists a finite sheeted covering space $(\widetilde{M}, \tilde{x}) \to (M, x)$, where $\widetilde{M}$ is the graph product of a graph whose vertex spaces are copies of $\widetilde{M}_1$ and $\widetilde{M}_2$. Let $\tilde{a}(1)$ and $\tilde{b}(1)$ denote the terminal points of the lifts of $a$ and $b$, respectively, to $\widetilde{M}$ with initial point $\tilde{x}$. Take two copies of $\widetilde{M}$, cut the first copy along the tori containing $\tilde{x}$ and $\tilde{a}(1)$ and cut the second copy along the tori containing $\tilde{x}$ and $\tilde{b}(1)$. Then identify the eight tori boundary components in pairs appropriately to obtain a covering space $(N, x') \to (M, x)$ such that the lift of $g = ab$ in $N$ with initial point $x'$ is not a loop. (See Fig. 1.) If $K$ is subgroup of $\pi_1(M, x)$ corresponding to $N$, then $\langle\alpha, \beta\rangle \subset K$ but $g \notin K$.

As the examples above indicate, to separate elements from $\langle\alpha, \beta\rangle$, we need to find finite sheeted covering spaces of $M_1$ and $M_2$ that satisfy specified conditions. Since $M_1$ and $M_2$ are orientable hyperbolic 3-manifolds, there exist discrete, faithful representations $\rho_1 : \pi_1(M_1) \to \mathrm{PSL}(2, \mathbb{C})$ and $\rho_2 : \pi_1(M_2) \to \mathrm{PSL}(2, \mathbb{C})$. To produce finite sheeted covering spaces of $M_1$ we work with the representation $\rho_1$. A ring homomorphism from the coefficient ring of $\rho_1(\pi_1(M_1))$ into a finite ring $S$ induces a group homomorphism from $\pi_1(M_1)$ into the finite group $\mathrm{PSL}(2, S)$. The preimage of a subgroup of $\mathrm{PSL}(2, S)$ is then a subgroup of finite index in $\pi_1(M_1)$, which corresponds to a finite sheeted covering space of $M_1$. Similarly for $M_2$. To produce these ring homomorphisms, we apply results from algebraic number theory.

## 2. Algebraic preliminaries

In this section we state algebraic results that will be used in the proof of Theorem 1.2. We assume standard terminology of algebraic number theory. For reference see [7].

**Notation 2.1.** By a *number field* we mean a finite field extension of $\mathbb{Q}$. If $k$ is a number field, let $\mathcal{O}_k$ denote the ring of integers of $k$. If $\mathfrak{p}$ is a non-zero prime ideal of $\mathcal{O}_k$, then we complete $k$ at $\mathfrak{p}$ to obtain the local field $k_\mathfrak{p}$, with ring of integers $\mathcal{O}_{k_\mathfrak{p}}$. The ring $\mathcal{O}_{k_\mathfrak{p}}$ has a unique maximal ideal. The quotient of $\mathcal{O}_{k_\mathfrak{p}}$ by this maximal ideal is called the *residue class field* of $\mathcal{O}_{k_\mathfrak{p}}$. The quotient map is called the *residue class field map* with respect to $\mathfrak{p}$.

The following two Propositions are Corollary 2.4 and Theorem 2.6 of [5].

**Proposition 2.2.** *Let $m \in \mathbb{N}$. Let $k$ be a number field and let $\delta$ be a non-zero element of $k$ that is not a root of unity. Then there exist infinitely many prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$ such that $\delta \in \mathcal{O}_{k_\mathfrak{p}}$ and the multiplicative order of the image of $\delta$ in the residue class field of $\mathcal{O}_{k_\mathfrak{p}}$ is divisible by $m$.*

**Proposition 2.3.** *Let $k$ be a number field. Let $\lambda$ and $\omega$ be non-zero elements of $k$ such that $\lambda$ is not a multiplicative power of $\omega$. Let $P$ be a finite set of prime ideals of $\mathcal{O}_k$. Given a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$, let $\eta_\mathfrak{p}$ denote the residue class field map with respect to $\mathfrak{p}$. Then there exist primes $\mathfrak{p}$ and $\mathfrak{q}$, lying outside of $P$, such that $\lambda, \omega \in \mathcal{O}_{k_\mathfrak{p}} \cap \mathcal{O}_{k_\mathfrak{q}}$ and $(\eta_\mathfrak{p} \times \eta_\mathfrak{q})(\lambda)$ is not a multiplicative power of $(\eta_\mathfrak{p} \times \eta_\mathfrak{q})(\omega)$.*

## 3. Topological preliminaries

We assume standard results and terminology of hyperbolic geometry. For reference see [2] or [10].

We begin with the statement of Lemma 4.1 of [6].

**Lemma 3.1.** *Let $M$ be a compact orientable 3-manifold with non-empty boundary whose interior admits a complete hyperbolic structure of finite volume. Fix a component $T \in \partial M$. Suppose $\delta \in \pi_1(M) \setminus \pi_1(T)$. Then for all but finitely many primes $p$, there exists a normal subgroup $N_p$ of finite index in $\pi_1(M)$ such that*:

(1) *if $G$ is a maximal parabolic subgroup of $\pi_1(M)$, then $G/(G \cap N_p) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, and*
(2) $N_p \cap \{\delta\gamma \mid \gamma \in \pi_1(T)\} = \emptyset$.

To prove Theorem 1.2, we will need the following.

**Lemma 3.2.** *Let $M$ be a compact orientable 3-manifold with non-empty boundary whose interior admits a complete hyperbolic structure of finite volume. Fix a component $T \in \partial M$. Let $\alpha$ be a loxodromic element of $\pi_1(M)$. Suppose $\delta \in \pi_1(M) \setminus \{\alpha^t\gamma \mid t \in \mathbb{Z},\ \gamma \in \pi_1(T)\}$. Then for all but finitely many primes $p$, there exists a subgroup $K_\mathfrak{p}$ of finite index in $\pi_1(M)$ such that*:

(1) $\alpha \in K_\mathfrak{p}$, *and*
(2) *if $G$ is a maximal parabolic subgroup of $\pi_1(M)$, then $G/(G \cap K_\mathfrak{p}) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.*

*Moreover, for infinitely many $p$ and infinitely many $q$*

(3) $(K_{\mathfrak{p}} \cap K_{\mathfrak{q}}) \cap \{\delta\gamma \mid \gamma \in \pi_1(T)\} = \emptyset$.

**Proof.** Express $\alpha = \alpha_o^m$, where $\alpha_o$ is a primitive loxodromic element of $\pi_1(M)$ and $m \in \mathbb{N}$. We begin by introducing a class of group homomorphisms from $\pi_1(M)$ into finite groups.

The group of orientation preserving isometries of $\mathbb{H}^3$ may be identified with $\mathrm{PSL}(2,\mathbb{C})$. Thus there exists a discrete faithful representation

$$\rho_1 : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$$

which is well-defined up to conjugation in $\mathrm{PSL}(2,\mathbb{C})$. For simplicity we use the fact that $\rho_1$ may be lifted to a representation

$$\psi : \pi_1(M) \to \mathrm{SL}(2,\mathbb{C}).$$

(See Proposition 3.1.1 of [4].) By assumption, $\mathrm{int}(M)$ is a hyperbolic manifold of finite volume. It follows from a standard argument (cf. [1, Lemma 2]) that after conjugating $\psi(\pi_1(M))$ in $\mathrm{SL}(2,\mathbb{C})$, if necessary, we may assume that

$$\psi(\alpha_o) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{where } |\omega| \neq 1,$$

and that $\psi(\pi_1(M)) \in \mathrm{SL}(2,k)$, for some number field $k$. Since $M$ is compact, $\pi_1(M)$ is finitely generated. Let $R_1$ be the ring generated by the coefficients of the generators of $\psi(\pi_1(M))$ over $\mathbb{Z}$. Since $R_1$ is finitely generated, $R_1 \subset \mathcal{O}_{k_{\mathfrak{p}}}$ for all but finitely many prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$. For each of these primes $\mathfrak{p}$ the residue class field map $\eta_{\mathfrak{p}} : \mathcal{O}_{k_{\mathfrak{p}}} \to F_{\mathfrak{p}}$ induces a homomorphism

$$\overline{\eta_{\mathfrak{p}}} : \mathrm{SL}(2,\mathcal{O}_{k_{\mathfrak{p}}}) \to \mathrm{SL}(2,F_{\mathfrak{p}}),$$

where $F_{\mathfrak{p}}$ is the residue class field of $\mathcal{O}_{k_{\mathfrak{p}}}$. Let $\psi_{\mathfrak{p}}$ denote the composition

$$\psi_{\mathfrak{p}} : \pi_1(M) \xrightarrow{\psi} \psi\big(\pi_1(M)\big) \overset{\iota}{\hookrightarrow} \mathrm{SL}(2,\mathcal{O}_{k_{\mathfrak{p}}}) \xrightarrow{\overline{\eta_{\mathfrak{p}}}} \mathrm{SL}(2,F_{\mathfrak{p}}).$$

The group of orientation preserving isometries of $\mathbb{H}^3$ may also be identified with $\mathrm{SO}_o(3,1,\mathbb{R})$, the identity component of $\mathrm{SO}(3,1,\mathbb{R})$. Thus there exists a discrete faithful representation

$$\rho_2 : \pi_1(M) \to \mathrm{SO}_o(3,1,\mathbb{R})$$

which is well-defined up to conjugation in $\mathrm{SO}_o(3,1,\mathbb{R})$. Since $\mathrm{int}(M)$ has finite volume, there are a finite number of conjugacy classes of maximal parabolic subgroups of $\pi_1(M)$, each corresponding to a boundary component of $M$. Choose a representative from each conjugacy class $G_1,\ldots,G_m$. For each $1 \leqslant i \leqslant m$, fix generators $\mu_i, \lambda_i$ of $G_i \cong \mathbb{Z} \oplus \mathbb{Z}$. There is a natural isomorphism

$$\mathrm{PSL}(2,\mathbb{C}) \to \mathrm{SO}_o(3,1,\mathbb{R})$$

such that the class of

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad z = x + yi,$$

in $\mathrm{PSL}(2,\mathbb{C})$ corresponds to

$$\begin{pmatrix} \frac{2+|z|^2}{2} & \frac{-|z|^2}{2} & x & y \\ \frac{|z|^2}{2} & \frac{2-|z|^2}{2} & x & y \\ x & -x & 1 & 0 \\ y & -y & 0 & 1 \end{pmatrix} \quad \text{in } \mathrm{SO_o}(3,1,\mathbb{R}).$$

We may conjugate $\rho_1(G_i)$ in $\mathrm{PSL}(2,\mathbb{C})$ such that $\rho_1(\mu_i)$ is represented by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $\rho_1(\lambda_i)$ is represented by

$$\begin{pmatrix} 1 & z_i \\ 0 & 1 \end{pmatrix}, \quad \text{for some } z_i \in \mathbb{C} \setminus \mathbb{R}.$$

Therefore, for each $1 \leqslant i \leqslant m$, there exists an element $Y_i \in \mathrm{SO_o}(3,1,\mathbb{R})$ such that

$$Y_i \rho_2(\mu_i) Y_i^{-1} = \begin{pmatrix} 3/2 & -1/2 & 1 & 0 \\ 1/2 & 1/2 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$Y_i \rho_2(\lambda_i) Y_i^{-1} = \begin{pmatrix} \frac{2+|z_i|^2}{2} & \frac{-|z_i|^2}{2} & x_i & y_i \\ \frac{|z_i|^2}{2} & \frac{2-|z_i|^2}{2} & x_i & y_i \\ x_i & -x_i & 1 & 0 \\ y_i & -y_i & 0 & 1 \end{pmatrix},$$

where $z_i = x_i + y_i i,\ y_i \neq 0$.

Since $\alpha_o$ is loxodromic, there exists an element $X \in \mathrm{SL}(4,\mathbb{C})$ such that $X\rho_2(\alpha_o)X^{-1} = D$, where $D$ is a diagonal matrix of the form

$$D = \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_1^{-1} & 0 & 0 \\ 0 & 0 & \omega_2 & 0 \\ 0 & 0 & 0 & \omega_2^{-1} \end{pmatrix}, \quad |\omega_1| \neq 1,\ |\omega_2| = 1.$$

(See Theorem 3.4.1 of [3].) Define

$$\phi : \pi_1(M) \to \mathrm{SL}(4,\mathbb{C})$$

by $\phi(g) = X\rho_2(g)X^{-1}$, $g \in \pi_1(M)$. Let $R_2$ be the ring generated by the coefficients of the generators of $\rho_2(\pi_1(M))$ along with the coefficients of $X, X^{-1}, Y_i, Y_i^{-1},\ 1 \leqslant i \leqslant m$. By Lemma 4.2 of [6], for all but finitely many primes $p \in \mathbb{Z}$, there is a finite field, $L_p$, of characteristic $p$, and a ring homomorphism

$$\pi_p : R_2 \to L_p$$

such that $\pi_p(y_i) \neq 0$ for all $1 \leqslant i \leqslant m$. This ring homomorphism induces a group homomorphism

$$\overline{\pi_p} : \mathrm{SL}(4, R_2) \to \mathrm{SL}(4, L_p).$$

Let $\phi_p$ denote the composition

$$\phi_p : \pi_1(M) \xrightarrow{\phi} \phi\big(\pi_1(M)\big) \xhookrightarrow{\iota} \mathrm{SL}(4, R_2) \xrightarrow{\overline{\pi_p}} \mathrm{SL}(4, L_p).$$

Consider the map

$$\tau_{\mathfrak{p}} = (\psi_{\mathfrak{p}} \times \phi_p) : \pi_1(M) \to \mathrm{SL}(2, F_{\mathfrak{p}}) \times \mathrm{SL}(4, L_p),$$

where $p$ is the characteristic of $F_{\mathfrak{p}}$. This is defined for all but finitely many prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$.

**Claim 1.** *If $G$ is a maximal parabolic subgroup of $\pi_1(M)$, then $\tau_{\mathfrak{p}}(G) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.*

**Proof.** It suffices to show that for each $1 \leqslant i \leqslant m$, $\tau_{\mathfrak{p}}(G_i) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.

If $g \in G_i$, then $\psi(g)$ is conjugate in $\mathrm{SL}(2, \mathbb{C})$ to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Therefore we write

$$\psi(g) = \begin{pmatrix} 1 - ac & a^2 \\ -c^2 & 1 + ac \end{pmatrix}, \qquad \psi\big(g^n\big) = \begin{pmatrix} 1 - nac & na^2 \\ -nc^2 & 1 + nac \end{pmatrix}, \quad n \in \mathbb{N},$$

for a fixed element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$. Since

$$\psi_{\mathfrak{p}}\big(g^p\big) = \begin{pmatrix} 1 - p\eta_{\mathfrak{p}}(ac) & p\eta_{\mathfrak{p}}(a^2) \\ p\eta_{\mathfrak{p}}(-c^2) & 1 + p\eta_{\mathfrak{p}}(ac) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$g^p$ is contained in the kernel of $\psi_{\mathfrak{p}}$. Thus to prove that $\tau_{\mathfrak{p}}(G_i) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, we need only show $\phi_p(G_i) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. By construction, $\phi(G_i)$ is conjugate in $\mathrm{SL}(4, R_2)$ to the group generated by $Y_i\rho_2(\mu_i)Y_i^{-1}$ and $Y_i\rho_2(\lambda_i)Y_i^{-1}$. Note that

$$\big(Y_i\rho_2(\mu_i)Y_i^{-1}\big)^n = \begin{pmatrix} (2+n^2)/2 & -n^2/2 & n & 0 \\ n^2/2 & (2-n^2)/2 & n & 0 \\ n & -n & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\big(Y_i\rho_2(\lambda_i)Y_i^{-1}\big)^n = \begin{pmatrix} \frac{2+|nz_i|^2}{2} & \frac{-|nz_i|^2}{2} & nx_i & ny_i \\ \frac{|nz_i|^2}{2} & \frac{2-|nz_i|^2}{2} & nx_i & ny_i \\ nx_i & -nx_i & 1 & 0 \\ ny_i & -ny_i & 0 & 1 \end{pmatrix}.$$

Since $\pi_p(y_i) \neq 0$, it follows that $\overline{\pi_p}(\langle Y_i\rho_2(\mu_i)Y_i^{-1}, Y_i\rho_2(\lambda_i)Y_i^{-1}\rangle) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Hence $\phi_p(G_i) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. This proves Claim 1. □

**Claim 2.** *If $g \in \pi_1(M)$ is parabolic and $\tau_{\mathfrak{p}}(g) \in \langle \tau_{\mathfrak{p}}(\alpha_o)\rangle$, then $\tau_{\mathfrak{p}}(g) = \mathrm{id}$.*

**Proof.** Suppose that $\tau_{\mathfrak{p}}(g) \in \langle \tau_{\mathfrak{p}}(\alpha_o)\rangle$. Then $\psi_{\mathfrak{p}}(g) = \psi_{\mathfrak{p}}(\alpha_o^n)$ and $\phi_p(g) = \phi_p(\alpha_o^n)$ for some $n \in \mathbb{Z}$. Recall that

$$\psi(\alpha_o) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \qquad \psi(\alpha_o)^n = \begin{pmatrix} \omega^n & 0 \\ 0 & \omega^{-n} \end{pmatrix}.$$

Since $g$ is parabolic, the trace of $\psi_{\mathfrak{p}}(g)$ is equal to 2. Therefore, $\eta_{\mathfrak{p}}(\omega^n) + \eta_{\mathfrak{p}}(\omega^{-n}) = 2$, implying that $\eta_{\mathfrak{p}}(\omega^n) = 1$. Hence $\psi_{\mathfrak{p}}(g) = \psi_{\mathfrak{p}}(\alpha_o^n) = \mathrm{id}$.

It remains to show that $\phi_p(g) = \mathrm{id}$. By construction, $\phi(\alpha_o) = D$. Since $g$ is parabolic, there exists an element

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \mathrm{SL}(4, R_2)$$

such that $\phi(g) = AUA^{-1}$ where

$$U = \begin{pmatrix} \frac{2+|z|^2}{2} & \frac{-|z|^2}{2} & x & y \\ \frac{|z|^2}{2} & \frac{2-|z|^2}{2} & x & y \\ x & -x & 1 & 0 \\ y & -y & 0 & 1 \end{pmatrix}, \quad \text{for some } z = x + \mathrm{i}y \in \mathbb{C}.$$

Since $\phi_p(g) = \phi_p(\alpha_o^n)$,

$$\overline{\pi_p}(A)\overline{\pi_p}(U) = \overline{\pi_p}\big(D^n\big)\overline{\pi_p}(A). \tag{$*$}$$

Multiplying the matrices in $(*)$ and equating the coefficients of the two right columns,

$$\begin{aligned}
&\pi_p\big((a_{11} + a_{12})x\big) = \pi_p\big(a_{13}\big(\omega_1^n - 1\big)\big), \\
&\pi_p\big((a_{11} + a_{12})y\big) = \pi_p\big(a_{14}\big(\omega_1^n - 1\big)\big), \\
&\pi_p\big((a_{21} + a_{22})x\big) = \pi_p\big(a_{23}\big(\omega_1^{-n} - 1\big)\big), \\
&\pi_p\big((a_{21} + a_{22})y\big) = \pi_p\big(a_{24}\big(\omega_1^{-n} - 1\big)\big), \\
&\pi_p\big((a_{31} + a_{32})x\big) = \pi_p\big(a_{33}\big(\omega_2^n - 1\big)\big), \\
&\pi_p\big((a_{31} + a_{32})y\big) = \pi_p\big(a_{34}\big(\omega_2^n - 1\big)\big), \\
&\pi_p\big((a_{41} + a_{42})x\big) = \pi_p\big(a_{43}\big(\omega_2^{-n} - 1\big)\big), \\
&\pi_p\big((a_{41} + a_{42})y\big) = \pi_p\big(a_{44}\big(\omega_2^{-n} - 1\big)\big).
\end{aligned} \tag{$**$}$$

Since $g$ is parabolic, the trace of $\phi_p(g)$ is equal to 4. Hence $\pi_p(\omega_1^n) + \pi_p(\omega_1^{-n}) + \pi_p(\omega_2^n) + \pi_p(\omega_2^{-n}) = 4$. It follows that $\pi_p(\omega_1^n) = 1$ if and only if $\pi_p(\omega_2^n) = 1$. Our goal is to show that $\pi_p(\omega_1^n) = \pi_p(\omega_2^n) = 1$. Suppose not. Then $\pi_p(\omega_1^n) \neq 1$ and $\pi_p(\omega_2^n) \neq 1$. If $\pi_p(x) = 0$, then by equations $(**)$,

$$\begin{aligned}
&\pi_p\big(a_{13}\big(\omega_1^n - 1\big)\big) = 0, \qquad \pi_p\big(a_{23}\big(\omega_1^{-n} - 1\big)\big) = 0, \\
&\pi_p\big(a_{33}\big(\omega_2^n - 1\big)\big) = 0, \qquad \pi_p\big(a_{43}\big(\omega_2^{-n} - 1\big)\big) = 0.
\end{aligned}$$

Thus $\pi_p(a_{13}) = \pi_p(a_{23}) = \pi_p(a_{33}) = \pi_p(a_{43}) = 0$, implying that the determinant of $\overline{\pi_p}(A)$ is equal to zero, a contradiction. We conclude that $\pi_p(x) \neq 0$. Similarly, $\pi_p(y) \neq 0$. It then follows from equations $(**)$ that

$$\begin{aligned}
&\pi_p(a_{13}) = \pi_p(x/y)\pi_p(a_{14}), \qquad \pi_p(a_{23}) = \pi_p(x/y)\pi_p(a_{24}), \\
&\pi_p(a_{33}) = \pi_p(x/y)\pi_p(a_{34}), \qquad \pi_p(a_{43}) = \pi_p(x/y)\pi_p(a_{44}).
\end{aligned}$$

Hence, the determinant of $\overline{\pi_p}(A)$ is equal to zero, a contradiction. We conclude that $\pi_p(\omega_1^n) = \pi_p(\omega_2^n) = 1$. Thus, $\phi_p(g) = \mathrm{id}$, as required. This proves Claim 2. □

Let $K_\mathfrak{p} = \tau_\mathfrak{p}^{-1}(\langle \tau_\mathfrak{p}(\alpha)\rangle)$. Then $\alpha \in K_\mathfrak{p}$, and by Claims 1 and 2, if $G$ is a maximal parabolic subgroup of $\pi_1(M)$, then $G/(G \cap K_\mathfrak{p}) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Hence $K_\mathfrak{p}$ satisfies conditions 1 and 2. It remains to verify condition 3. Let $\delta \in \pi_1(M) \setminus \{\alpha^t\gamma \mid t \in \mathbb{Z},\ \gamma \in \pi_1(T)\}$ be given.

*Case* 1: $\delta \in \{\alpha_o^t\gamma \mid t \in \mathbb{Z},\ \gamma \in \pi_1(T)\}$.

Recall that

$$\psi(\alpha_o) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad \psi(\alpha) = \psi\big(\alpha_o^m\big) = \begin{pmatrix} \omega^m & 0 \\ 0 & \omega^{-m} \end{pmatrix}.$$

Express $\delta = \alpha_o^s\gamma_1$, where $s \in \mathbb{Z}$ and $\gamma_1 \in \pi_1(T)$. Since $\delta \notin \{\alpha^t\gamma \mid t \in \mathbb{Z},\ \gamma \in \pi_1(T)\}$, $m \nmid s$. By Proposition 2.2, there exist infinitely many prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$ such that the multiplicative order of $\eta_\mathfrak{p}(\omega)$ is divisible by $m$. Therefore, there exist infinitely many primes $\mathfrak{p}$ such that $\tau_\mathfrak{p}$ is defined, and the order of $\tau_\mathfrak{p}(\alpha_o)$ is divisible by $m$. Fix one such prime $\mathfrak{p}$. If $K_\mathfrak{p} \cap \{\delta\gamma \mid \gamma \in \pi_1(T)\} \neq \emptyset$, then $\tau_\mathfrak{p}(\delta\gamma_2) = \tau_\mathfrak{p}(\alpha^t)$, for some $\gamma_2 \in \pi_1(T)$ and $t \in \mathbb{Z}$. This means that $\tau_\mathfrak{p}(\gamma_1\gamma_2) = \tau_\mathfrak{p}(\alpha_o^{-s}\alpha^t) = \tau_\mathfrak{p}(\alpha_o^{mt-s})$. Hence by Claim 2, $\tau_\mathfrak{p}(\alpha_o^{mt-s}) = \mathrm{id}$. Since the order of $\tau_\mathfrak{p}(\alpha_o)$ is divisible by $m$, this implies that $m \mid s$, a contradiction. We conclude that $K_\mathfrak{p} \cap \{\delta\gamma \mid \gamma \in \pi_1(T)\} = \emptyset$. This completes the proof of the theorem in Case 1.

*Case* 2: $\delta \notin \{\alpha_o^t\gamma \mid t \in \mathbb{Z},\ \gamma \in \pi_1(T)\}$.

Express

$$\psi(\delta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

Since $\alpha_o$ is a primitive loxodromic element of $\pi_1(M)$ and $\delta \notin \langle\alpha_o\rangle$, $\delta$ and $\alpha_o$ do not commute. Therefore, $b \neq 0$ and $c \neq 0$. Since $\pi_1(T)$ is a maximal parabolic subgroup of $\pi_1(M)$, $\psi(\pi_1(T))$ is conjugate in $\mathrm{SL}(2, \mathbb{C})$ to a group generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad \text{for some } r \in \mathbb{C}.$$

Hence

$$\psi\big(\pi_1(T)\big) = \left\{ \begin{pmatrix} 1 - sxz & sx^2 \\ -sz^2 & 1 + sxz \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}^{-1} \right\},$$

where $s \in \{n_1 + n_2 r \mid n_1, n_2 \in \mathbb{Z}\}$ and $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ is a fixed element of $\mathrm{SL}(2, \mathbb{C})$. Since $\psi(\pi_1(M))$ is discrete, the fixed point of $\psi(\pi_1(T))$ is distinct from the fixed points of $\psi(\alpha_o)$. Hence, $x \neq 0$ and $z \neq 0$. Note that $\{a, b, c, d, x^2, z^2, xz, rx^2\} \subset R_1 \subset k$. Therefore, $acx^2 + 2bcxz + bdz^2 \in R_1$ and $r \in k$.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_k$ such that $\psi_\mathfrak{p}$ is defined. If there exists an element $\gamma \in \pi_1(T)$ such that $\psi_\mathfrak{p}(\delta\gamma) \in \langle\psi_\mathfrak{p}(\alpha_o)\rangle$, then

$$\begin{pmatrix} \eta_\mathfrak{p}(a) & \eta_\mathfrak{p}(b) \\ \eta_\mathfrak{p}(c) & \eta_\mathfrak{p}(d) \end{pmatrix} \begin{pmatrix} \eta_\mathfrak{p}(1 - sxz) & \eta_\mathfrak{p}(sx^2) \\ \eta_\mathfrak{p}(-sz^2) & \eta_\mathfrak{p}(1 + sxz) \end{pmatrix} = \begin{pmatrix} \eta_\mathfrak{p}(\omega^n) & 0 \\ 0 & \eta_\mathfrak{p}(\omega^{-n}) \end{pmatrix},$$

for some $n \in \mathbb{Z}$ and $s \in \{n_1 + n_2 r \mid n_1, n_2 \in \mathbb{Z}\}$. Equating coefficients,

$$\begin{aligned}
&\eta_{\mathfrak{p}}(a)\eta_{\mathfrak{p}}(1 - sxz) + \eta_{\mathfrak{p}}(b)\eta_{\mathfrak{p}}\big(-sz^2\big) = \eta_{\mathfrak{p}}\big(\omega^n\big),\\
&\eta_{\mathfrak{p}}(a)\eta_{\mathfrak{p}}\big(sx^2\big) + \eta_{\mathfrak{p}}(b)\eta_{\mathfrak{p}}(1 + sxz) = 0,\\
&\eta_{\mathfrak{p}}(c)\eta_{\mathfrak{p}}(1 - sxz) + \eta_{\mathfrak{p}}(d)\eta_{\mathfrak{p}}\big(-sz^2\big) = 0 \quad \text{and}\\
&\eta_{\mathfrak{p}}(c)\eta_{\mathfrak{p}}\big(sx^2\big) + \eta_{\mathfrak{p}}(d)\eta_{\mathfrak{p}}(1 + sxz) = \eta_{\mathfrak{p}}\big(\omega^{-n}\big).
\end{aligned} \tag{$*$}$$

*Subcase* 1: $acx^2 + 2bcxz + bdz^2 \neq 0$.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_k$ such that $\tau_{\mathfrak{p}}$ is defined, $\eta_{\mathfrak{p}}(acx^2 + 2bcxz + bdz^2) \neq 0$, $\eta_{\mathfrak{p}}(b) \neq 0$, $\eta_{\mathfrak{p}}(c) \neq 0$, and $r \in \mathcal{O}_{k_{\mathfrak{p}}}$. Then $\eta_{\mathfrak{p}}(s)$ is defined for each $s \in \{n_1 + n_2 r \mid n_1, n_2 \in \mathbb{Z}\}$. If there exists an element $\gamma \in \pi_1(T)$ such that $\psi_{\mathfrak{p}}(\delta\gamma) \in \langle \psi_{\mathfrak{p}}(\alpha_o)\rangle$, then by the second and third equations of $(*)$,

$$\begin{aligned}
&\eta_{\mathfrak{p}}(-b) = \eta_{\mathfrak{p}}(s)\eta_{\mathfrak{p}}\big(ax^2 + bxz\big) \quad \text{and}\\
&\eta_{\mathfrak{p}}(c) = \eta_{\mathfrak{p}}(s)\eta_{\mathfrak{p}}\big(cxz + dz^2\big).
\end{aligned}$$

Since $\eta_{\mathfrak{p}}(b) \neq 0$ and $\eta_{\mathfrak{p}}(c) \neq 0$, $\eta_{\mathfrak{p}}(ax^2 + bxz)$ and $\eta_{\mathfrak{p}}(cxz + dz^2)$ are units in $F_{\mathfrak{p}}$. Hence

$$\eta_{\mathfrak{p}}(s) = \eta_{\mathfrak{p}}(-b)\eta_{\mathfrak{p}}\big(ax^2 + bxz\big)^{-1} = \eta_{\mathfrak{p}}(c)\eta_{\mathfrak{p}}\big(cxz + dz^2\big)^{-1}.$$

It follows that $\eta_{\mathfrak{p}}(acx^2 + 2bcxz + bdz^2) = 0$, a contradiction. We conclude that, for every element $\gamma \in \pi_1(T)$, $\psi_{\mathfrak{p}}(\delta\gamma) \notin \langle \psi_{\mathfrak{p}}(\alpha_o)\rangle$, implying that $K_{\mathfrak{p}} \cap \{\delta\gamma \mid \gamma \in \pi_1(T)\} = \emptyset$.

*Subcase* 2: $acx^2 + 2bcxz + bdz^2 = 0$.

If $cx + dz = 0$, then $acx^2 + 2bcxz + bdz^2 = -xz \neq 0$. Therefore, $cx + dz \neq 0$ in this case. Setting $t = c/(cxz + dz^2)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 - txz & tx^2 \\ -tz^2 & 1 + txz \end{pmatrix} = \begin{pmatrix} z/(cx + dz) & 0 \\ 0 & (cx + dz)/z \end{pmatrix}$$

in $\mathrm{SL}(2, \mathbb{C})$. Suppose that $z/(cx + dz)$ is a multiplicative power of $\omega$. Then the three matrices above are elements of $\psi(\pi_1(M))$. Furthermore,

$$\begin{pmatrix} 1 - txz & tx^2 \\ -tz^2 & 1 + txz \end{pmatrix}$$

lies in $\psi(\pi_1(T))$ since they have the same fixed point. But this contradicts our assumption that $\delta \notin \{\alpha_o^t\gamma \mid t \in \mathbb{Z},\ \gamma \in \pi_1(T)\}$. We conclude that $z/(cx + dz)$ is not a multiplicative power of $\omega$. Hence, by Proposition 2.3, there exist infinitely many primes $\mathfrak{p}$ and infinitely many primes $\mathfrak{q}$ of $\mathcal{O}_k$ such that $\tau_{\mathfrak{p}}$ and $\tau_{\mathfrak{q}}$ are defined, $\eta_{\mathfrak{p}}(c) \neq 0$, $\eta_{\mathfrak{q}}(c) \neq 0$, and $(\eta_{\mathfrak{p}} \times \eta_{\mathfrak{q}})(z/(cx + dz))$ is not a multiplicative power of $(\eta_{\mathfrak{p}} \times \eta_{\mathfrak{q}})(\omega)$. Suppose there exists an element $\gamma \in \pi_1(T)$ such that $(\psi_{\mathfrak{p}} \times \psi_{\mathfrak{p}})(\delta\gamma) \in \langle(\psi_{\mathfrak{p}} \times \psi_{\mathfrak{q}})(\alpha_o)\rangle$. Then by the third equation of $(*)$,

$$\begin{aligned}
&\eta_{\mathfrak{p}}(c) = \eta_{\mathfrak{p}}(s)\eta_{\mathfrak{p}}\big(cxz + dz^2\big) = 0 \quad \text{and}\\
&\eta_{\mathfrak{q}}(c) = \eta_{\mathfrak{q}}(s)\eta_{\mathfrak{q}}\big(cxz + dz^2\big) = 0.
\end{aligned}$$

Since $\eta_{\mathfrak{p}}(c) \neq 0$ and $\eta_{\mathfrak{q}}(c) \neq 0$, $\eta_{\mathfrak{p}}(cxz + dz^2)$ and $\eta_{\mathfrak{q}}(cxz + dz^2)$ are units in $F_{\mathfrak{p}}$ and $F_{\mathfrak{q}}$, respectively. Therefore,

$$\eta_{\mathfrak{p}}(s) = \eta_{\mathfrak{p}}(c)\eta_{\mathfrak{p}}\big(cxz + dz^2\big)^{-1} \quad \text{and} \quad \eta_{\mathfrak{q}}(s) = \eta_{\mathfrak{q}}(c)\eta_{\mathfrak{q}}\big(cxz + dz^2\big)^{-1}. \tag{$**$}$$

Substituting equations $(**)$ into the first equation of $(*)$,

$$\eta_{\mathfrak{p}}\big(\omega^n\big) = \eta_{\mathfrak{p}}(z)\eta_{\mathfrak{p}}(cx+dz)^{-1} \quad \text{and} \quad \eta_{\mathfrak{q}}\big(\omega^n\big) = \eta_{\mathfrak{q}}(z)\eta_{\mathfrak{q}}(cx+dz)^{-1}.$$

Hence

$$(\eta_{\mathfrak{p}} \times \eta_{\mathfrak{q}})\big(z/(cx+dz)\big) = (\eta_{\mathfrak{p}} \times \eta_{\mathfrak{q}})\big(\omega^n\big),$$

a contradiction. We conclude that for every $\gamma \in \pi_1(T)$, $(\psi_{\mathfrak{p}} \times \psi_{\mathfrak{p}})(\delta\gamma) \notin \langle(\psi_{\mathfrak{p}} \times \psi_{\mathfrak{q}})(\alpha_o)\rangle$. Therefore, $(K_{\mathfrak{p}} \cap K_{\mathfrak{q}}) \cap \{\delta\gamma \mid \gamma \in \pi_1(T)\} = \emptyset$. □

## 4. Proof of Theorem 1.2

**Theorem 4.1.** *Let $M_1$ and $M_2$ be compact orientable* 3*-manifolds with non-empty boundary whose interiors admit complete hyperbolic structures of finite volume. Fix boundary components $T_1 \in \partial M_1$ and $T_2 \in \partial M_2$, and let $f : T_1 \to T_2$ be a homeomorphism. Let $M$ be the manifold obtained by identifying $M_1$ and $M_2$ along the fixed boundary components via $f$. If $\alpha \in \pi_1(M_1)$ and $\beta \in \pi_1(M_2)$ are loxodromic elements, then $\langle\alpha, \beta\rangle$ is a separable subgroup of $\pi_1(M)$.*

**Proof.** Let $k \in \pi_1(M) \setminus \langle\alpha, \beta\rangle$. We must find a subgroup $K$ of finite index in $\pi_1(M)$ such that $\langle\alpha, \beta\rangle \subset K$ but $k \notin K$. By Van Kampen's Theorem, $\pi_1(M)$ can be expressed as the free product with amalgamation,

$$\pi_1(M) \cong \pi_1(M_1) \underset{c}{*} \pi_1(M_2),$$

where $C \cong \pi_1(T_1) \cong \pi_1(T_2)$. Suppose that

$$k = a_1b_1a_2b_2\cdots a_nb_nc \quad \text{where}$$
$$c \in C, \quad a_i \in \pi_1(M_1) \setminus C, \quad \text{and} \quad b_i \in \pi_1(M_2) \setminus C.$$

If $a_1 = \alpha^{s_1}\gamma_1$ for some $s_1 \in \mathbb{Z}$ and $\gamma_1 \in C$, then $k = \alpha^{s_1}(\gamma_1b_1)a_2b_2\cdots a_nb_nc$. If $\gamma_1b_1 = \beta^{t_1}\gamma_2$, for some $t_1 \in \mathbb{Z}$ and $\gamma_2 \in C$, then $k = \alpha^{s_1}\beta^{t_1}(\gamma_2a_2)b_2\cdots a_nb_nc$. Continuing in this way, we see that $k = hg$, where $h \in \langle\alpha, \beta\rangle$ and $g$ satisfies one of the following conditions.

(1) $g \in C \setminus \{\text{id}\}$;
(2) $g = a_1b_1a_2b_2\cdots a_nb_nc$, where $a_i \in \pi_1(M_1) \setminus C$, $b_i \in \pi_1(M_2) \setminus C$, $c \in C$ and $a_1 \notin \{\alpha^m\gamma \mid m \in \mathbb{Z},\ \gamma \in C\}$;
(3) $g = a_1b_1a_2b_2\cdots a_nc$, where $a_i \in \pi_1(M_1) \setminus C$, $b_i \in \pi_1(M_2) \setminus C$, $c \in C$ and $a_1 \notin \{\alpha^m\gamma \mid m \in \mathbb{Z},\ \gamma \in C\}$;
(4) $g = b_1a_2b_2\cdots a_nb_nc$, where $a_i \in \pi_1(M_1) \setminus C$, $b_i \in \pi_1(M_2) \setminus C$, $c \in C$ and $b_1 \notin \{\beta^m\gamma \mid m \in \mathbb{Z},\ \gamma \in C\}$;
(5) $g = b_1a_2b_2\cdots a_nc$, where $a_i \in \pi_1(M_1) \setminus C$, $b_i \in \pi_1(M_2) \setminus C$, $c \in C$ and $b_1 \notin \{\beta^m\gamma \mid m \in \mathbb{Z},\ \gamma \in C\}$.

If there exists a subgroup $K$ of finite index in $\pi_1(M)$ such that $\langle\alpha, \beta\rangle \subset K$ but $g \notin K$, then $k \notin K$, as required. Therefore, it suffices to separate $g$ from $\langle\alpha, \beta\rangle$ in each of the cases above.

*Case* 1: $g \in C \setminus \{\mathrm{id}\}$.

Express $g = u\lambda + v\mu$, where $u, v \in \mathbb{Z}$ and $\lambda$ and $\mu$ are basis elements of $\pi_1(T_1) \cong \mathbb{Z} \oplus \mathbb{Z}$. By Lemma 3.2, for all but finitely many primes $p$, there exist subgroups $H_1$ and $H_2$ of finite index in $\pi_1(M_1)$ and $\pi_1(M_2)$, respectively, such that:

(i) for every maximal parabolic subgroup $G$ of $\pi_1(M_1)$, $G/(G \cap H_1) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$,
(ii) for every maximal parabolic subgroup $G$ of $\pi_1(M_2)$, $G/(G \cap H_2) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, and
(iii) $\alpha \in H_1$, $\beta \in H_2$.

Choose one such prime $p$ such that either $p \nmid u$ or $p \nmid v$. Then by condition (i), $g \notin H_1$.

Let $p_1 : \widetilde{M}_1 \to M_1$ be the covering space corresponding to $H_1$, and $p_2 : \widetilde{M}_2 \to M_2$ be the covering space corresponding to $H_2$. By conditions (i) and (ii), this collection of covering spaces is compatible, as defined in Section 2 of [6]. Therefore, by Theorem 2.2 of [6], there exists a finite sheeted covering space

$$p : \widetilde{M} \to M$$

where $\widetilde{M}$ is the graph product of a graph whose vertex spaces are copies of $\widetilde{M}_1$ and $\widetilde{M}_2$. Let $T$ be the torus in $M$ where $T_1 \subset M_1$ and $T_2 \subset M_2$ are identified, let $\widetilde{T}_*$ be a component of $p^{-1}(T)$, and fix a basepoint $* \in \widetilde{T}_* \subset \widetilde{M}$. Let $K$ be the subgroup of $\pi_1(M)$ corresponding to $(\widetilde{M}, *)$. By condition (iii), $\langle \alpha, \beta \rangle \subset K$. However, since $g \notin H_1$, $g \notin K$.

*Case* 2:

$$g = a_1 b_1 a_2 b_2 \cdots a_n b_n c, \quad \text{where}$$
$$a_i \in \pi_1(M_1) \setminus C, \quad b_i \in \pi_1(M_2) \setminus C, \quad c \in C \quad \text{and}$$
$$a_1 \notin \{\alpha^m \gamma \mid m \in \mathbb{Z}, \gamma \in C\}.$$

By Lemmas 3.1 and 3.2, there exist subgroups $H_1$ and $H_1'$ of finite index in $\pi_1(M_1)$, subgroups $H_2$ and $H_2'$ of finite index in $\pi_1(M_2)$, and a positive integer $m$ such that:

(1) If $G$ is a maximal parabolic subgroup of $\pi_1(M_1)$, then
$$G/(G \cap H_1) \cong G/(G \cap H_1') \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z};$$
(2) If $G$ is a maximal parabolic subgroup of $\pi_1(M_2)$, then
$$G/(G \cap H_2) \cong G/(G \cap H_2') \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z};$$
(3) $\alpha \in H_1$, $\beta \in H_2$;
(4) $H_1 \cap \{a_1\gamma \mid \gamma \in C\} = \emptyset$;
(5) $H_1' \cap \{a_i\gamma \mid \gamma \in C\} = \emptyset$, for all $2 \leqslant i \leqslant n$; and
(6) $H_2' \cap \{b_i\gamma \mid \gamma \in C\} = \emptyset$, for all $1 \leqslant i \leqslant n$.

Let $p_1 : \widetilde{M}_1 \to M_1$ be the covering space corresponding to $H_1$, and $p_2 : \widetilde{M}_2 \to M_2$ be the covering space corresponding to $H_2$. By conditions (1) and (2), this collection of covering spaces is compatible, as defined in Section 2 of [6]. Therefore, by Theorem 2.2 of [6], there exists a finite sheeted covering space

$$p : \widetilde{M} \to M$$

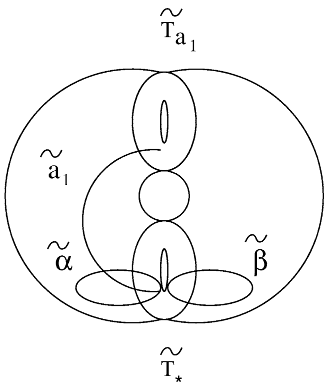


Fig. 2. The covering space $\widetilde{M}$.

where $\widetilde{M}$ is the graph product of a graph whose vertex spaces are copies of $\widetilde{M}_1$ and $\widetilde{M}_2$. Let $T$ be the torus in $M$ where $T_1 \subset M_1$ and $T_2 \subset M_2$ are identified, let $\widetilde{T}_*$ be a component of $p^{-1}(T)$, and fix a basepoint $* \in \widetilde{T}_* \subset \widetilde{M}$. By condition (3), the lift of $\alpha$ with initial point $*$ is a loop and the lift of $\beta$ with initial point $*$ is a loop. Let $\tilde{a}_1$ denote the lift of $a_1$ with initial point $*$, and let $\widetilde{T}_{a_1}$ denote the component of $p^{-1}(T)$ containing the terminal point of $\tilde{a}_1$. By condition (4), $\widetilde{T}_{a_1} \neq \widetilde{T}_*$. (See Fig. 2.)

Let $q_1 : \widetilde{M}_1' \to M_1$ be the covering space corresponding to $H_1'$, and $q_2 : \widetilde{M}_2' \to M_2$ be the covering space corresponding to $H_2'$. By conditions (1) and (2), this collection of covering spaces is compatible. Therefore, by Theorem 2.2 of [6], there exists a finite sheeted covering space

$$q : \widetilde{M}' \to M$$

where $\widetilde{M}'$ is the graph product of a graph whose vertex spaces are copies of $\widetilde{M}_1'$ and $\widetilde{M}_2'$. Let $T$ be the torus in $M$ where $T_1 \subset M_1$ and $T_2 \subset M_2$ are identified, let $\widetilde{T}_*'$ be a component of $q^{-1}(T)$, and fix a basepoint $* \in \widetilde{T}_*' \subset \widetilde{M}'$. For $2 \leqslant i \leqslant n$, let $\tilde{a}_i'$ denote the lift of $a_i$ with initial point $*$. For $1 \leqslant i \leqslant n$, let $\tilde{b}_i'$ denote the lift of $b_i$ with initial point $*$. Let $\widetilde{T}_{a_i}'$ and $\widetilde{T}_{b_i}'$ denote the components of $q^{-1}(T)$ containing the terminal points of $\tilde{a}_i'$ and $\tilde{b}_i'$, respectively. By conditions (5) and (6), $\widetilde{T}_{a_i}' \neq \widetilde{T}_*'$ and $\widetilde{T}_{b_i}' \neq \widetilde{T}_*'$. (See Fig. 3 for the picture when $n = 2$. For simplicity, in Fig. 3 we set $\widetilde{T}_{a_2}' = \widetilde{T}_{b_1}' = \widetilde{T}_{b_2}'$. However, this need not be the case in general.)

We define spaces $\widetilde{M}(j)$, $1 \leqslant j \leqslant 2n$, as follows. Let $\widetilde{M}(1)$ be the space obtained by cutting $\widetilde{M}$ along $\widetilde{T}_* \cup \widetilde{T}_{a_1}$. For each $j \in \{2i - 1 \mid 2 \leqslant i \leqslant n\}$, let $\widetilde{M}(j)$ be the space obtained by cutting a copy of $\widetilde{M}'$ along $\widetilde{T}_*' \cup \widetilde{T}_{a_i}'$. For each $j \in \{2i \mid 1 \leqslant i \leqslant n\}$, let $\widetilde{M}(j)$ be the space obtained by cutting a copy of $\widetilde{M}'$ along $\widetilde{T}_*' \cup \widetilde{T}_{b_i}'$. Label four boundary components of $\widetilde{M}(1)$ as follows. Let $\widetilde{T}_*^1(1)$ and $\widetilde{T}_*^2(1)$ be the boundary components of $\widetilde{M}(1)$ created by cutting along $\widetilde{T}_*$, where $\widetilde{T}_*^1(1)$ bounds a copy of $\widetilde{M}_1$ and $\widetilde{T}_*^2(1)$ bounds a copy of $\widetilde{M}_2$. Let $\widetilde{T}_{a_1}^1(1)$ and $\widetilde{T}_{a_1}^2(1)$ be the boundary components of $\widetilde{M}(1)$ created by cutting along $\widetilde{T}_{a_1}$, where $\widetilde{T}_{a_1}^1(1)$ bounds a copy of $\widetilde{M}_1$ and $\widetilde{T}_{a_1}^2(1)$ bounds a copy of $\widetilde{M}_2$. For $j > 1$, label four boundary components of $\widetilde{M}(j)$ as follows. For each $2 \leqslant j \leqslant 2n$, let $\widetilde{T}_*^1(j)$ and $\widetilde{T}_*^2(j)$ be the boundary components of $\widetilde{M}(j)$ created by cutting along $\widetilde{T}_*'$, where $\widetilde{T}_*^1(j)$ bounds a copy

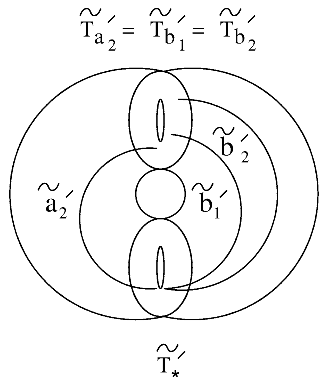


Fig. 3. The covering space $\widetilde{M}'$.

of $\widetilde{M}'_1$ and $\widetilde{T}^2_*(j)$ bounds a copy of $\widetilde{M}'_2$. For $j \in \{2i-1 \mid 2 \leqslant i \leqslant n\}$, let $\widetilde{T}^1_{a_i}(j)$ and $\widetilde{T}^2_{a_i}(j)$ be the boundary components of $\widetilde{M}(j)$ created by cutting along $\widetilde{T}'_{a_i}$, where $\widetilde{T}^1_{a_i}(j)$ bounds a copy of $\widetilde{M}'_1$ and $\widetilde{T}^2_{a_i}(j)$ bounds a copy of $\widetilde{M}'_2$. For $j \in \{2i \mid 1 \leqslant i \leqslant n\}$, let $\widetilde{T}^1_{b_i}(j)$ and $\widetilde{T}^2_{b_i}(j)$ be the boundary components of $\widetilde{M}(j)$ created by cutting along $\widetilde{T}'_{b_i}$, where $\widetilde{T}^1_{b_i}(j)$ bounds a copy of $\widetilde{M}'_1$ and $\widetilde{T}^2_{b_i}(j)$ bounds a copy of $\widetilde{M}'_2$. (See Fig. 4 for the picture when $n = 2$.)

Construct a finite sheeted covering space $\overline{M}$ of $M$ as follows. Start with the collection

$$\big\{\widetilde{M}(1), \widetilde{M}(2), \ldots, \widetilde{M}(2n)\big\}$$

and identify

$$\begin{array}{ll}
\widetilde{T}^1_*(1) & \text{with } \widetilde{T}^2_*(1),\\
\widetilde{T}^1_{a_i}(2i-1) & \text{with } \widetilde{T}^2_*(2i),\ 1 \leqslant i \leqslant n,\\
\widetilde{T}^2_{a_i}(2i-1) & \text{with } \widetilde{T}^1_*(2i)\ 1 \leqslant i \leqslant n,\\
\widetilde{T}^1_{b_i}(2i) & \text{with } \widetilde{T}^2_*(2i+i),\ 1 \leqslant i < n,\\
\widetilde{T}^2_{b_i}(2i) & \text{with } \widetilde{T}^1_*(2i+1),\ 1 \leqslant i < n, \text{ and}\\
\widetilde{T}^1_{b_n}(2n) & \text{with } \widetilde{T}^2_{b_n}(2n).
\end{array}$$

The covering map

$$\overline{M} \to M$$

is induced by $p$ restricted to $\widetilde{M} \setminus \{\widetilde{T}_* \cup \widetilde{T}_{a_1}\}$ on $\widetilde{M}(1)$, $q$ restricted to $\widetilde{M}' \setminus \{\widetilde{T}'_* \cup \widetilde{T}'_{a_i}\}$ on $\widetilde{M}(j)$, for $j > 1$ odd, and $q$ restricted to $\widetilde{M}' \setminus \{\widetilde{T}'_* \cup \widetilde{T}'_{b_i}\}$ on $\widetilde{M}(j)$, for $j$ even. Set the basepoint of $\overline{M}$ to be the basepoint in $\widetilde{T}^1_*(1) \subset \widetilde{M}(1)$, viewed as a subset of $\overline{M}$. Let $K$ be the subgroup of $\pi_1(M)$ corresponding to $\overline{M}$. Since $\overline{M}$ is a finite sheeted covering space of $M$, $K$ has finite index in $\pi_1(M)$. Since the lifts of $\alpha$ and $\beta$ with initial point $*$ are loops in $\overline{M}$, $\alpha, \beta \in K$. By construction, the terminal point of the lift of $g$ with initial point $*$ lies in $\widetilde{T}^2_{b_n}(2n)$. Hence $g$ does not lift to a loop in $\overline{M}$. We conclude that $g \notin K$, as required.

The proofs of Cases 3, 4 and 5 are similar to the proof of Case 2. □

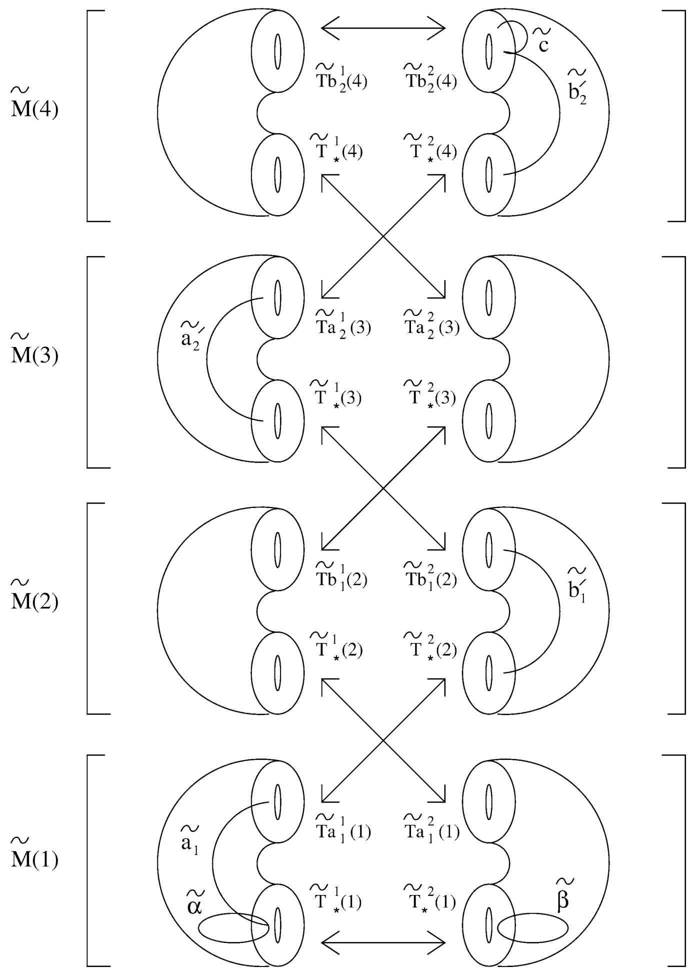


Fig. 4. The covering space $\overline{M}$.

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