Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Pure and Applied Algebra 182 (2003) 65–78
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Ascending HNN extensions of polycyclic groups are residually finite

Tim Hsu[a,*,1], Daniel T. Wise[b,1,2]

[a] Department of Mathematics, San Jose State University, San Jose, CA 95192, USA
[b] Department of Mathematics and Statistics, McGill University, Montreal, Que., Canada H3A 2K6

Received 6 August 2002; received in revised form 15 October 2002
Communicated by Y. Rhodes

Abstract

We prove that every ascending HNN extension of a polycyclic-by-finite group is residually finite. We also give a criterion for the residual finiteness of an ascending HNN extension of a residually nilpotent group, and apply this criterion to recover a result of Moldavanskiĭ on the residual finiteness of certain ascending HNN extensions of free groups.

MSC: 20E26; 20F18; 20E06

1. Introduction

Let P be a group, and let $\varphi : P \to P$ be a monomorphism. The *ascending HNN extension*, or *mapping torus*, corresponding to φ is the group

$$P_\varphi = \langle P, t \mid p^t = \varphi(p) \rangle, \tag{1}$$

where x^y denotes $y^{-1}xy$.

Ascending HNN extensions are an interesting and well-studied class of groups. For example, Feighn and Handel [5] have recently shown that ascending HNN

* Corresponding author.
E-mail addresses: hsu@math.sjsu.edu (T. Hsu), daniwise@math.mcgill.ca (D.T. Wise).
[1] We thank the anonymous referee for his or her helpful remarks.
[2] Partially supported by NSF Grant No. DMS-9971511.

0022-4049/03/$ - see front matter
doi:10.1016/S0022-4049(02)00310-9

extensions of free groups are *coherent* (that is, their f.g. subgroups are finitely presented).

Recall that a group G is *residually finite* if each nontrivial element of G has a nontrivial image in some finite quotient of G. Now, the normal forms of an ascending HNN extension behave much like the normal forms of a split extension, and as observed by Malćev, a split extension $P \rtimes_{\varphi} \mathbb{Z}$ of a f.g. residually finite group P is residually finite [13, III.A, Theorem 7]. One might therefore hope that an ascending HNN extension of a f.g. residually finite group is also residually finite. However, there exist examples of ascending HNN extensions P_{φ} of f.g. residually finite groups P where P_{φ} has very few finite quotients, or where P_{φ} even fails to be Hopfian [17]. It is therefore interesting to see how the nature of the base group governs the residual finiteness of the ascending HNN extension.

In particular, because polycyclic groups (see Section 3) are among the most residually finite of infinite groups, one might hope that their ascending HNN extensions are well-behaved. This class of groups is known to have many interesting properties; most notably, the class of ascending HNN extensions of polycyclic groups is precisely the class of f.g. coherent solvable groups [4,8]. For a comprehensive survey of finitely presented solvable groups, in which ascending HNN extensions of polycyclic groups play a key role, see [19].

Our main result is

Theorem 1.1. *Let $\varphi : P \to P$ be a monomorphism of a polycyclic-by-finite group. Then P_{φ} is residually finite.*

Note that several cases of Theorem 1.1 are already known. Specifically, when P is f.g. and abelian, P_{φ} is a f.g. metabelian group, and therefore residually finite by a theorem of P. Hall ([9] or [15, 15.4.1]); and when P is a f.g. free nilpotent group, P_{φ} is residually finite by a theorem of Moldavanskiĭ [14]. (Unfortunately, we were unable to obtain the English translation of Ref. [14] until after we had completed this work; as it turns out, our Theorems 4.2 and 8.2 are duplications of Moldavanskiĭ's results. We have retained these results in our paper, both for clarity of exposition, and also because Ref. [14] may not be widely accessible.)

We now give a brief overview of this paper. In Sections 2 and 3, we summarize some necessary background material, first from general group theory, and then from the theory of nilpotent and polycyclic groups. The main content of the paper begins in Section 4, with a characterization of f.g. groups whose ascending HNN extensions are all residually finite. Next, in Section 5, we show that ascending HNN extensions of torsion-free nilpotent groups are residually finite in a particularly strong sense, and in Section 6, we show that the class of groups whose ascending HNN extensions are residually finite is closed under certain extensions. We then combine the results of Sections 4–6 to prove Theorem 1.1 in Section 7. Finally, in Section 8, we provide a criterion for the residual finiteness of ascending HNN extensions of f.g. residually nilpotent groups, and apply this criterion to certain ascending HNN extensions of free groups and right-angled Artin groups.

2. General background

In this section, for the sake of completeness, we establish some conventions and review some well-known facts from group theory. The first is a slight extension of a theorem of Marshall Hall (see Lyndon and Schupp [11, Theorem 4.5]).

Definition 2.1. A subgroup H of a group G is *fully invariant* if $\varphi(H) \subseteq H$ for every endomorphism φ of G.

Lemma 2.2. *Let G be a group, let n be a natural number, and let N be the intersection of all subgroups of G of index $\leqslant n$. Then N is fully invariant, and if G is f.g., then N has finite index in G.*

Note that as a consequence, any finite index subgroup of a f.g. group G contains a fully invariant finite index subgroup of G.

Proof. Let φ be an endomorphism of G, and let $N = \bigcap_{[G:H]\leqslant n} H$. Note that

$$\varphi^{-1}(N) = \varphi^{-1}\left(\bigcap_{[G:H]\leqslant n} H\right) = \bigcap_{[G:H]\leqslant n} \varphi^{-1}(H). \tag{2}$$

Now, since the preimage $\varphi^{-1}(H)$ of a subgroup H of index $\leqslant n$ also has index $\leqslant n$, we see that $\varphi^{-1}(N)$ is itself the intersection of subgroups of index $\leqslant n$ (though not necessarily all subgroups of index $\leqslant n$). It follows that $N \leqslant \varphi^{-1}(N)$, or in other words, that $\varphi(N) \leqslant N$. Furthermore, if G is f.g., there are only finitely many homomorphisms $G \to S_k$ for $k \leqslant n$, and therefore, only finitely many subgroups of index $\leqslant n$. In that case, since the intersection of two finite index subgroups has finite index, we see that N has finite index. □

Lemma 2.3. *Let φ be an endomorphism of a group G, let K be a φ-invariant normal subgroup of G, and let $\bar{\varphi} : (G/K) \to (G/K)$ be the map induced by φ. If both $\bar{\varphi}$ and the restriction of φ to K are isomorphisms, then φ is an isomorphism.*

Proof. For $x \in \ker \varphi$, since $\bar{\varphi}$ is injective, we must have $x \in K$, and since φ restricted to K is injective, we must have $x=1$. For $y \in G$, since $\bar{\varphi}$ is surjective, there exists some $x_0 \in G$ such that $\varphi(x_0) = yk$ for some $k \in K$, and since φ restricted to K is surjective, there exists some $x_1 \in K$ such that $\varphi(x_1) = k^{-1}$, in which case $\varphi(x_0 x_1) = y$. □

Definition 2.4. Let G be a group. For $x, y \in G$, we define $[x, y] = xyx^{-1}y^{-1}$, and for subgroups H, K of G, we define $[H, K] = \langle [x, y], x \in H, y \in K \rangle$.

Definition 2.5. For a group G and an integer n, we define $G^n = \langle x^n, x \in G \rangle$ and $G' = [G, G]$. These are examples of *verbal subgroups* of G; see Magnus et al. [12, Section 2.2] for a full definition and other information on verbal subgroups.

Lemma 2.6. *Let G be a group, let $\psi : G \to G/K$ be a quotient with kernel K, let $V(G)$ be a verbal subgroup of G, and let $V(G/K)$ be the analogous verbal subgroup of G/K. Then $V(G)$ is a fully invariant subgroup of G, $V(G/K) = \psi(V(G))$, and $(G/K)/V(G/K) \cong G/(KV(G))$.*

Proof. See Magnus et al. [12, p. 74; ex. 11, p. 80]. □

Finally, we recall the following (easily proven) property of subgroups:

Lemma 2.7. *Let A, B, C be subgroups of a group G such that $C \leqslant A$. Then $(A \cap B)C = A \cap BC$.*

3. Nilpotent and polycyclic groups

In this section, we establish more notation, and assemble some well-known material in the form that we need. We refer the reader to Baumslag [1], Robinson [15], and Segal [18] for a more detailed discussion of this material.

Definition 3.1. Let $Z(K)$ denote the center of a group K. The *upper central series* of G is defined inductively by letting $Z_0(G) = 1$ and letting $Z_{i+1}(G)$ be the preimage of $Z(G/Z_i(G))$ in G, and the *lower central series* of G is defined inductively by letting $\gamma_0(G) = G$ and letting $\gamma_{i+1}(G) = [G, \gamma_i(G)]$. The quotients $Z_{i+1}(G)/Z_i(G)$ are the *upper central quotients* of G, and the quotients $\gamma_i(G)/\gamma_{i+1}(G)$ are the *lower central quotients* of G. We say that G is *nilpotent* if $Z_i(G) = G$ for some i, or equivalently, if $\gamma_i(G) = 1$ for some i (see Robinson [15, 5.1.9]). The *nilpotence class* of G is defined to be the smallest i such that these equalities hold.

Recall that subgroups and quotients of nilpotent groups are nilpotent, and that the torsion elements of a nilpotent group G form a subgroup of G called the *torsion subgroup* of G (see Robinson [15, 5.2.7]). We also recall the following facts.

Lemma 3.2. *Let G be a group. Then $Z_j(G/Z_k(G)) = Z_{j+k}(G)/Z_k(G)$.*

Proof. See Robinson [15, 5.1.11(iv)]. □

Lemma 3.3. *Let G be a nilpotent group. The following are equivalent*:

(1) *G is torsion-free.*
(2) *$Z_1(G)$ is torsion-free.*
(3) *All upper central quotients of G are torsion-free.*

Proof. Condition 1 implies condition 2 a fortiori. Condition 2 implies condition 3 by Robinson [15, 5.2.19]. Finally, if condition 3 holds, then any nontrivial $x \in G$ has nontrivial image in some upper central quotient. However, a torsion element can only have a trivial image in a torsion-free group. The lemma follows. □

Corollary 3.4. *If G is a torsion-free nilpotent group, then $G/Z_1(G)$ is also torsion-free.*

Proof. By Lemma 3.2, $Z_1(G/Z_1(G)) = Z_2(G)/Z_1(G)$, which is torsion-free by Lemma 3.3. Therefore, $G/Z_1(G)$ is also torsion-free, by another application of Lemma 3.3. □

We will also need the following fact about the verbal subgroup G^p (Definition 2.5) of a nilpotent group.

Lemma 3.5. *Let H be a subgroup of a f.g. nilpotent group G. Then $H \cap G^p = H^p$ for all sufficiently large primes p.*

Proof. See Baumslag [1, Proposition 2.2]. □

Finally, we recall the following fact about nilpotent groups.

Lemma 3.6. *If G is a f.g. nilpotent group with finite abelianization G/G', then G is finite.*

Proof. See Segal [18, p. 13, Corollary 9]. □

Definition 3.7. A group G is *polycyclic* if it has a subnormal series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ whose factors G_{i+1}/G_i are cyclic (that is, a *polycyclic series*).

Recall that subgroups and quotients of polycyclic groups are polycyclic; in fact, a group is polycyclic if and only if it is solvable (has a subnormal series with abelian factors) and all of its subgroups are f.g. (see Robinson [15, 5.4.12]). We also recall the following facts.

Lemma 3.8. *Let G be a polycyclic group. The number of infinite cyclic factors in a polycyclic series for G is invariant.*

Proof. See Robinson [15, 5.4.13]. □

Definition 3.9. The number of infinite cyclic factors in a polycyclic series for a polycyclic group G is the *Hirsch length* of G.

Lemma 3.10. *Let H be a subgroup of a polycyclic group G. Then H and G have the same Hirsch length if and only if H has finite index in G.*

Proof. See Robinson [15, Ex. 5.4.10]. □

Corollary 3.11. *If G is polycyclic, and $\varphi : G \to G$ is a monomorphism, then $\varphi(G)$ has finite index in G.*

Proof. Since $\varphi(G)$ is isomorphic to G, it has the same Hirsch length. □

It also follows immediately from Lemma 3.10 that:

Corollary 3.12. *If G is torsion-free polycyclic, and φ is an endomorphism of G such that $\varphi(G)$ has finite index in G, then φ is a monomorphism.*

We will also need some standard structural results on nilpotent and polycyclic groups.

Lemma 3.13. *Every f.g. nilpotent group has a torsion-free subgroup of finite index.*

Proof. This follows a fortiori from Robinson [15, 5.4.15.(i)]. □

Definition 3.14. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be group properties (e.g., finiteness, nilpotence, polycyclicity). We say that a group G is $\mathscr{P}_1$*-by-*$\mathscr{P}_2$ if G has a normal subgroup N with property $\mathscr{P}_1$ such that G/N has property $\mathscr{P}_2$.

Lemma 3.15. *Every f.g. finite-by-abelian group is abelian-by-finite.*

Proof. Let J be a f.g. group with a finite normal subgroup K such that J/K is abelian, and let $C = C_J(K)$ be the centralizer of K in J. Since K is normal and finite, C has finite index in J. Also, since $C/(C \cap K) \cong CK/K$ is abelian, and $C \cap K$ is central in C, we see that C is nilpotent. Therefore, by Lemma 3.13, C has a finite index torsion-free subgroup H, which is abelian, since $H' \leqslant H \cap K = 1$. □

Lemma 3.16 (Malćev). *Every polycyclic-by-finite group has a subgroup of finite index whose commutator subgroup is f.g. nilpotent.*

Proof. This follows easily from Robinson [15, 15.1.6]. □

4. Ascending residual finiteness

Definition 4.1. A group J is *ascending residually finite* (ARF) if, for any monomorphism $\varphi : J \to J$ and any nontrivial $x \in J$, there exists a finite index normal subgroup W of J such that the following conditions hold:

(1) $x \notin W$,
(2) $\varphi(W) \subseteq W$, and
(3) the map $\bar{\varphi} : J/W \to J/W$ induced by φ is an isomorphism.

If W can always be chosen to be fully invariant, we say that J is FIARF.

We will see momentarily that a group is ARF if and only if all of its ascending HNN extensions are residually finite (Theorem 4.2). Consequently, although an ARF group must be residually finite, the examples of [17] show that the converse need not hold, even for f.g. residually finite groups. More importantly, Theorem 1.1 will then follow from the fact that polycyclic-by-finite groups are ARF, which we will prove in Section 7.

Theorem 4.2. *Let J be a group. Then every ascending HNN extension of J is residually finite if and only if J is ARF.*

Proof. Suppose every ascending HNN extension of J is residually finite. Let $\varphi : J \to J$ be a monomorphism, and let x be a nontrivial element of J. Since the ascending HNN extension $J_\varphi = \langle J, t \mid j^t = \varphi(j) \rangle$ is residually finite, there exists a finite quotient $\rho : J_\varphi \to Q$ such that $\rho(x) \neq 1$. Let $W = J \cap \ker \rho$. We claim that W satisfies the conditions of Definition 4.1. First, since $\rho(x) \neq 1$, we have $x \notin W$. Next, if $j \in W$, then since $\rho(j) = 1$, we have

$$\rho(\varphi(j)) = \rho(j^t) = \rho(t)^{-1}\rho(j)\rho(t) = 1, \tag{3}$$

which means that $\varphi(j) \in J \cap \ker \rho = W$. Finally, if $\bar{\varphi} : J/W \to J/W$ is the map induced by φ, we observe that

$$\bar{\varphi}(\rho(j)) = \rho(\varphi(j)) = \rho(t)^{-1}\rho(j)\rho(t). \tag{4}$$

In other words, $\bar{\varphi}$ is conjugation by $\rho(t)$, which is an isomorphism.

Conversely, suppose that J is ARF. Let $\varphi : J \to J$ be a monomorphism, and let a be a nontrivial element of the ascending HNN extension J_φ (as above). By the normal form theorem, $a = t^r x t^{-s}$ for some r, $s \geqslant 0$, $x \in J$. On the one hand, if $r - s \neq 0$, then a has nontrivial image in the quotient $\rho : J_\varphi \to \mathbb{Z}_{r+s+1}$ induced by sending all elements of J to 0 and sending t to 1. On the other hand, suppose $r - s = 0$. In that case, after conjugating, we may as well assume that $a = x$. Then, since J is ARF, there is a normal φ-invariant subgroup W of J such that $x \notin W$, J/W is finite, and $\bar{\varphi} : J/W \to J/W$ is an isomorphism. Writing the image of $j \in J$ in J/W as $\bar{j}$, we see that the quotient

$$\overline{J_\varphi} = \langle J/W, t \mid \bar{j}^t = \bar{\varphi}(\bar{j}) \rangle \tag{5}$$

is finite-by-cyclic (since $\bar{\varphi}$ is an isomorphism), and that $\bar{x} \neq 1$ in $\overline{J_\varphi}$. Therefore, since finite-by-cyclic groups are residually finite, there exists a finite quotient of $\overline{J_\varphi}$ in which $\bar{x}$ has nontrivial image. □

5. Torsion-free nilpotent groups

In this section, we prove the following theorem.

Theorem 5.1. *Every f.g. torsion-free nilpotent group is FIARF.*

Since the subgroup G^p is fully invariant in any group G (Lemma 2.6), it suffices to prove the following theorem.

Theorem 5.2. *Let G be a f.g. torsion-free nilpotent group, let $\varphi : G \to G$ be a monomorphism, and let x be a nontrivial element of G. Then for all sufficiently* *large primes* p*, the quotient* G/G^p *is a finite* p*-group,* $x \notin G^p$*, and the induced map* $\varphi: G/G^p \to G/G^p$ *is an isomorphism.*

In fact, it is well-known that for G f.g. torsion-free nilpotent, every nontrivial $x \in G$ survives in the quotient G/G^p for sufficiently large primes p (see Baumslag [1, Ch. 2]). The point of Theorem 5.2 is that sufficiently large p also agree with a given monomorphism $\varphi: G \to G$.

For the rest of this section, let G be a f.g. torsion-free nilpotent group, let $\varphi: G \to G$ be a monomorphism, and let $Z_i = Z_i(G)$. We begin by showing that φ behaves well with respect to upper central quotients.

Lemma 5.3. *For all* i*,* $\varphi(Z_i) \leqslant Z_i$ *and* $\varphi: (G/Z_i) \to (G/Z_i)$ *is injective.*

Proof. Since G/Z_1 is also torsion-free nilpotent (Corollary 3.4), by Lemma 3.2 and an easy induction, it is enough to show that the lemma holds for $i = 1$. In fact, it is enough to show that

$$\varphi(Z_1) = Z(\varphi(G)) = Z_1 \cap \varphi(G). \tag{6}$$

For then, $\varphi(Z_1)$ is certainly a subgroup of Z_1, and if $\varphi(x) \in Z_1$ for some $x \in G$, then $\varphi(x) \in (Z_1 \cap \varphi(G)) = \varphi(Z_1)$ implies that $x \in Z_1$, since φ is injective.

To prove (6), we first observe that since φ is an embedding, the center of $\varphi(G)$ is precisely the image of the center of G, or in other words, $Z(\varphi(G)) = \varphi(Z_1)$. Also, since φ is an embedding, and G is torsion-free, we see that $Z(\varphi(G))$ and Z_1 are free abelian groups of the same rank. We next observe that $[Z_1 : Z_1 \cap \varphi(G)] \leqslant [G : \varphi(G)]$, which is finite by Corollary 3.11. Therefore, Z_1 and $Z_1 \cap \varphi(G)$ are also free abelian groups of the same rank, which means that $Z_1 \cap \varphi(G)$ has finite index in $Z(\varphi(G))$ as well, since $Z_1 \cap \varphi(G)$ is certainly a subgroup of $Z(\varphi(G))$. Finally, since every element of $Z_1 \cap \varphi(G)$ is central in $\varphi(G)$, we have

$$Z_1 \cap \varphi(G) = Z_1 \cap Z(\varphi(G)). \tag{7}$$

Combining (7) with the second isomorphism theorem, we then have

$$Z(\varphi(G))/(Z_1 \cap \varphi(G)) = Z(\varphi(G))/(Z_1 \cap Z(\varphi(G))) \cong Z(\varphi(G))Z_1/Z_1. \tag{8}$$

It follows that $Z(\varphi(G))Z_1/Z_1$ is a finite subgroup of the torsion-free group G/Z_1 (Corollary 3.4), and is therefore trivial. Consequently, $Z(\varphi(G)) = Z_1 \cap \varphi(G)$, and the lemma follows.

In particular, $\varphi: G \to G$ induces a monomorphism $\varphi_i: (Z_i/Z_{i-1}) \to (Z_i/Z_{i-1})$ on each upper central quotient Z_i/Z_{i-1}. Moreover, each upper central quotient Z_i/Z_{i-1} is free abelian, so each φ_i is a monomorphism of a free abelian group. We may therefore define

$$d_i = \det(\varphi_i) \tag{9}$$

to be the *upper central determinants* of φ. Note that each d_i is a nonzero integer. □

It will be convenient to have the following definition in the sequel.

Definition 5.4. Let p be a prime. We say that G *p-reduces* if

$$Z_i \cap G^p = Z_i^p \tag{10}$$

for all i.

Lemma 5.5. *Let p be a prime. If G p-reduces, then G/Z_1 p-reduces.*

Proof. Let $\psi : G \to G/Z_1$ be the natural map. By Lemmas 2.6 and 3.2, we have $Z_{i-1}(G/Z_1) = \psi(Z_i)$, $(G/Z_1)^p = \psi(G^p)$, and $Z_{i-1}(G/Z_1)^p = \psi(Z_i)^p = \psi(Z_i^p)$. But then, since Lemma 2.7 implies that

$$\psi(Z_i^p) = \psi(Z_i \cap G^p) = (Z_i \cap G^p)Z_1 = Z_i \cap (G^p Z_1) = \psi(Z_i) \cap \psi(G^p), \tag{11}$$

we see that $Z_{i-1}(G/Z_1) \cap (G/Z_1)^p = Z_{i-1}(G/Z_1)^p$. The lemma follows. □

Lemma 5.6. *Let p be a prime. If G p-reduces, then $Z_1 G^p/G^p \cong Z_1/Z_1^p$. More generally, if G p-reduces then each upper central quotient of G/G^p is the corresponding upper central quotient of G reduced modulo p.*

Proof. If G p-reduces we have

$$Z_1 G^p/G^p \cong Z_1/(Z_1 \cap G^p) = Z_1/Z_1^p \tag{12}$$

by the second isomorphism theorem and (10), respectively. The second assertion follows from the first by Lemma 5.5 and an easy induction. □

Lemma 5.7. *Suppose that G p-reduces, where p is a prime that does not divide any of the upper central determinants of φ. Then the induced map $\varphi : (G/G^p) \to (G/G^p)$ is an isomorphism.*

Proof. Proceeding by induction on the nilpotence class n of G, for $n = 0$, both G and the lemma are trivial, so assume $n > 0$. In that case, G/Z_1 is f.g. torsion-free (Corollary 3.4) nilpotent of class $n-1$, G/Z_1 p-reduces (Lemma 5.5), and p does not divide any of the upper central determinants of $\varphi : (G/Z_1) \to (G/Z_1)$, so by induction, the induced map $\varphi : (G/Z_1)/(G/Z_1)^p \to (G/Z_1)/(G/Z_1)^p$ is an isomorphism. In fact, since $(G/Z_1)/(G/Z_1)^p \cong G/(Z_1 G^p)$ (Lemma 2.6), the induced map $\varphi : G/(Z_1 G^p) \to G/(Z_1 G^p)$ is also an isomorphism.

Next, since p does not divide the determinant of $\varphi_1 : Z_1 \to Z_1$, the induced map $\varphi_1 : (Z_1/Z_1^p) \to (Z_1/Z_1^p)$ is an isomorphism. Therefore, since $(Z_1 G^p)/G^p \cong Z_1/Z_1^p$ (Lemma 5.6), the induced map $\varphi : (Z_1 G^p)/G^p \to (Z_1 G^p)/G^p$ is an isomorphism. It then follows by Lemma 2.3 that, since the induced maps $\varphi : G/(Z_1 G^p) \to G/(Z_1 G^p)$ and $\varphi : (Z_1 G^p)/G^p \to (Z_1 G^p)/G^p$ are both isomorphisms, the induced map $\varphi : G/G^p \to G/G^p$ is also an isomorphism. The lemma follows. □

Proof Theorem 5.2. First, for any prime p, G/G^p is a f.g. nilpotent torsion group, and is therefore finite (Lemma 3.6), which means that G^p has finite index in G. Next, by applying Lemma 3.5 for each term Z_i of the upper central series, we see that

G p-reduces for sufficiently large p. Furthermore, since each nontrivial $x \in G$ has a nontrivial image in some upper central quotient of G, by Lemma 5.6, for sufficiently large p, the element x has nontrivial image in one of the upper central quotients of G/G^p. Finally, by Lemma 5.7, for sufficiently large p, the induced map $\varphi:(G/G^p) \to (G/G^p)$ is an isomorphism. The theorem follows. □

6. Stabilization

We begin with the following observation, which leads to a definition.

Theorem 6.1. *Let J be a group, let $\varphi: J \to J$ be a homomorphism, let V be a φ-invariant normal subgroup of J, and let*

$$W = \bigcup_{n=0}^{\infty} \varphi^{-n}(V) = \{x \in J \mid \varphi^n(x) \in V \text{ for some } n\}. \tag{13}$$

Then W is a φ-invariant normal subgroup of J, and the induced map $\bar{\varphi}: J/W \to J/W$ is injective.

Proof. For $x \in J$, if $\varphi^n(x) \in V$ for some n, then

$$\varphi^n(\varphi(x)) = \varphi(\varphi^n(x)) \in V, \tag{14}$$

since V is φ-invariant. Therefore, W is φ-invariant. Furthermore, W is an increasing union of normal subgroups of J, and is therefore itself a normal subgroup of J. Finally, suppose that for $x \in J$, $\varphi(x) \in W$. In that case, for some n, $\varphi^n(\varphi(x)) = \varphi^{n+1}(x)$ is an element of V, which means that $x \in W$. The theorem follows. □

Definition 6.2. Let J, φ, V, W, and $\bar{\varphi}$ be as in Theorem 6.1. In that case, we say that W is the *stable kernel of φ relative to V*, and we say that $\bar{\varphi}$ is the *stabilization* of φ relative to V.

The idea of stabilization is the key to the following theorem.

Theorem 6.3. *Let J be a group, and let G be a fully invariant subgroup of J. Suppose that:*

(1) *G is* FIARF, *and*
(2) *every quotient of a finite-by-(J/G) group is* ARF.

Then J is ARF.

Proof. Let $\varphi: J \to J$ be a monomorphism, let K be the stable kernel of φ relative to G, and let x be a nontrivial element of J. We then have two cases:

(1) If $x \in K$, then $\varphi^n(x) \in G$ for some n, and since φ is a monomorphism, $\varphi^n(x) \neq 1$. Then, by hypothesis 1, there is a finite index fully invariant subgroup V_0 of G

such that $\varphi^n(x) \notin V_0$ and the restriction of φ to G induces an isomorphism $\varphi: G/V_0 \to G/V_0$. Let V be the stable kernel of φ relative to V_0. Note that since $\varphi: G/V_0 \to G/V_0$ is injective, $V \cap G = V_0$. It follows that $\varphi^n(x) \notin V$, since $\varphi^n(x) \notin V_0$ and $\varphi^n(x) \in G$. Therefore, since the induced endomorphism $\bar{\varphi}: J/V \to J/V$ is injective (Theorem 6.1), $x \notin V$.

(2) If $x \notin K$, let $V_0 = G$, and let $V = K$.

In either case, it now follows that J/V is a quotient of the finite-by-(J/G) group J/V_0, that the induced endomorphism $\varphi: J/V \to J/V$ is injective (Theorem 6.1), and that $x \neq 1$ in J/V. Therefore, by hypothesis 2, there exists a finite index φ-invariant subgroup $U \lhd J/V$ such that $x \notin U$ and $\varphi: (J/V)/U \to (J/V)/U$ is an isomorphism. If we then let W be the preimage of U in J, we see that W is normal of finite index in J, $x \notin W$, W is φ-invariant, and the induced map $\varphi: J/W \to J/W$ is an isomorphism. The theorem follows. □

7. Proof of Theorem 1.1

Having previously established the torsion-free nilpotent case, we now use Theorem 6.3 to expand the class of groups we know to be ARF until we obtain Theorem 1.1. We start with a trivial case.

Lemma 7.1. *Every finite group is ARF.*

Proof. Every monomorphism from a finite group to itself is an isomorphism. □

Lemma 7.2. *Every f.g. nilpotent-by-finite group is ARF.*

Proof. Let J be a f.g. group with a nilpotent normal subgroup N such that J/N is finite. By Lemma 3.13, there exists a torsion-free subgroup N_0 of finite index in N, which means that, by Lemma 2.2, there exists a fully invariant torsion-free nilpotent subgroup N_1 of finite index in J. Therefore, since N_1 is FIARF (Theorem 5.1), and since quotients of finite-by-(J/N_1) groups are finite, and therefore ARF (Lemma 7.1), Theorem 6.3 implies that J is ARF. □

Proof of Theorem 1.1. Let J be a polycyclic-by-finite group. By Lemma 3.16, there exists a finite index subgroup H of J such that H' is f.g. nilpotent. In fact, by Lemma 2.2, we may assume that H is fully invariant in J, which means that H' is also fully invariant in J, by Lemma 2.6. Next, by Lemma 3.13, H' has a torsion-free nilpotent subgroup G of finite index, which, by Lemma 2.2, we may also take to be fully invariant in H', and therefore, in J. We then see that, since H'/G is finite, H/H' is abelian, J/H is finite, and all subgroups are fully invariant in J, J/G is f.g. abelian-by-finite, by Lemma 3.15. It follows that G is FIARF, by Theorem 5.1, and that any quotient of a finite-by-(J/G) group is ARF, by Lemmas 3.15 and 7.2. Therefore, by Theorem 6.3, J is ARF. □

8. Ascending HNN extensions of other groups

In [6], it was conjectured that:

Conjecture 8.1. *Every ascending HNN extension F_φ of a f.g. free group F is residually finite.*

In this section, we use Theorem 1.1 to investigate Conjecture 8.1 and related problems. More specifically, let G_{ab} denote the abelianization G/G' of a group G. Proceeding in the direction of Conjecture 8.1, we recover the following result of Moldavanskiĭ.

Theorem 8.2. *Let G be a f.g. residually torsion-free nilpotent group, let $\varphi : G \to G$ be a monomorphism, and suppose that $[G_{\mathrm{ab}} : \varphi_{\mathrm{ab}}(G_{\mathrm{ab}})]$ is finite. Then G_φ is residually finite.*

Most notably, since free groups are residually free nilpotent (see Magnus et al. [12, Section 5.5]), Theorem 8.2 immediately implies the following result, first obtained by Moldavanskiĭ [14, Corollary 3], and later obtained independently in unpublished work of Sapir [16].

Theorem 8.3. *If F is a f.g. free group, and $\varphi : F \to F$ is a monomorphism such that the induced map $\varphi_{\mathrm{ab}} : F_{\mathrm{ab}} \to F_{\mathrm{ab}}$ is also a monomorphism, then F_φ is residually finite.*

Note that Theorem 8.2 can also be applied in many other situations. For example, a *right-angled Artin group* is a group with a presentation of the form

$$\langle a_1, \ldots, a_n | [a_i, a_j], (i,j) \in S \rangle, \tag{15}$$

where S is a subset of $\{(i,j) \mid 1 \leqslant i < j \leqslant n\}$. By Green [15], every right-angled Artin group is residually nilpotent. Furthermore, there are many monomorphisms of right-angled Artin groups that satisfy the criterion of Theorem 8.2; for instance, take the monomorphism induced by $a_i \mapsto a_i^{m_i}$, where each m_i is a nonzero integer. (One can verify that this is a monomorphism by applying the normal form theorem for right-angled Artin groups [7,10].)

We begin the proof of Theorem 8.2 with the following lemma.

Lemma 8.4. *Let G be f.g. nilpotent, let $\alpha : G \to G/G'$ be the abelianization of G, and let H be a subgroup of G such that $\alpha(H)$ has finite index in $\alpha(G)$. Then H has finite index in G.*

Proof. We first observe that every conjugate of H has image $\alpha(H)$ in $\alpha(G)$. Therefore, by replacing H with the intersection of all of its conjugates, we may assume without loss of generality that H is normal in G. In that case, since Lemma 2.6 implies that G/H is a f.g. nilpotent group with finite abelianization, Lemma 3.6 implies that G/H is finite. The lemma follows. □

Proof of Theorem 8.2. Let x be a nontrivial element of G, and choose n such that $x \notin \gamma_n(G)$. Let $Q = G/\gamma_n(G)$, and consider the projection of $\varphi : G \to G$ to $\bar{\varphi}: Q \to Q$. Since the abelianization of φ, and therefore, $\bar{\varphi}$, has nonzero determinant, Lemma 8.4 implies that $\bar{\varphi}(Q)$ has finite index in Q. Since Q is torsion-free polycyclic, Corollary 3.12 implies that $\bar{\varphi}$ is a monomorphism. Therefore, $Q_{\bar{\varphi}}$ is an ascending HNN extension, and x survives in the quotient $G_\varphi \to Q_{\bar{\varphi}}$. However, by Theorem 1.1, $Q_{\bar{\varphi}}$ is residually finite, so x survives in some finite quotient of G_φ. □

Remark 8.5. Baumslag [3] gave an example of an infinitely generated free-by-cyclic group with no noncyclic finite quotient. Therefore, Conjecture 8.1 does not hold if F is not f.g., even if φ is an isomorphism. However, in contrast, Baumslag has also shown that a f.g. free-by-cyclic group is residually finite [2]. It therefore seems likely that if residual finiteness holds for the class of ascending HNN extensions of f.g. free groups, then residual finiteness will also hold for the somewhat more general class of f.g. ascending HNN extensions of free groups.

Remark 8.6. Another family of properly ascending HNN extensions was shown to be residually finite in [20, Example 2.3]. The approach there is of an entirely different nature, and includes examples where $\varphi(F) \leqslant F'$, making φ_{ab} singular.

References

[1] G. Baumslag, Lecture Notes on Nilpotent Groups, CBMS Regional Conference Series in Mathematics, Vol. 2, American Mathematical Society, Providence, RI, 1971.

[2] G. Baumslag, Finitely generated cyclic extensions of free groups are residually finite, Bull. Austral. Math. Soc. 5 (1971) 87–94.

[3] G. Baumslag, A non-cyclic, locally free, free-by-cyclic group all of whose finite factor groups are cyclic, Bull. Austral. Math. Soc. 6 (1972) 313–314.

[4] R. Bieri, R. Strebel, Soluble groups with coherent group rings, in: Homological Group Theory, Proceedings of a Symposium, Durham, 1977, Cambridge University Press, Cambridge, 1979, pp. 235–240.

[5] M. Feighn, M. Handel, Mapping tori of free group automorphisms are coherent, Ann. Math. (2) 149 (3) (1999) 1061–1077.

[6] R. Geoghegan, M.L. Mihalik, M. Sapir, D.T. Wise, Ascending HNN extensions of finitely generated free groups are Hopfian, Bull. London Math. Soc. 33 (3) (2001) 292–298.

[7] E.R. Green, Graph products of groups, Ph.D. Thesis, University of Leeds, 1990.

[8] J.R.J. Groves, Soluble groups in which every finitely generated subgroup is finitely presented, J. Austral. Math. Soc. Ser. A 26 (1) (1978) 115–125.

[9] M. Hall Jr., Coset representations in free groups, Trans. Amer. Math. Soc. 67 (1949) 421–432.

[10] T. Hsu, D.T. Wise, On linear and residual properties of graph products, Michigan Math. J. 46 (2) (1999) 251–259.

[11] R.C. Lyndon, P.E. Schupp, Combinatorial Group Theory, Springer, Berlin, 1977, ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[12] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations, Interscience Publishers, New York, London, Sydney, 1966.

[13] C.F. Miller III, On Group-Theoretic Decision Problems and Their Classification, Annals of Mathematics Studies, No. 68, Princeton University Press, Princeton, NJ, 1971.

[14] D.I. Moldavanskiĭ, Residual finiteness of descending HNN-extensions of groups, Ukraïn. Mat. Zh. 44 (6) (1992) 842–845.

[15] D.J.S. Robinson, A Course in the Theory of Groups, 2nd Edition, Graduate Texts in Mathematics, Vol. 80, Springer, New York, 1996.
[16] M. Sapir, Residually finite hyperbolic groups and dynamics of polynomial maps over Z_p, lecture in: International Conference on Non-Positive Curvature in Group Theory, Topology, and Geometry, Vanderbilt University, May 1998.
[17] M. Sapir, D.T. Wise, Ascending HNN extensions of residually finite groups can be non-Hopfian and can have very few finite quotients, J. Pure Appl. Algebra 166 (2002) 191–202.
[18] D. Segal, Polycyclic Groups, Cambridge University Press, Cambridge, 1983.
[19] R. Strebel, Finitely presented soluble groups, in: K. Gruenberg, J. Roseblade (Eds.), Group Theory: Essays for Philip Hall, Academic Press, London, 1984, pp. 257–314.
[20] D.T. Wise, A residually finite version of Rips's construction, Bull. London Math. Soc. 35 (2003) 23–29.