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A Criterion for the Conjugacy Separability of Amalgamated Free Products of Conjugacy Separable Groups

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We first prove a criterion for the conjugacy separability of generalized free products of two conjugacy separable groups amalgamating a cyclic subgroup. Applying this result, we show that tree products of a finite number of conjugacy separable, residually finitely generated torsion-free nilpotent groups amalgamating cyclic subgroups are conjugacy separable. From this we derive that tree products of finitely generated torsion-free nilpotent groups, free groups, or surface groups amalgamating cyclic subgroups are conjugacy separable.

1. INTRODUCTION

Let S be a subset of a group G. Then G is said to be S-*separable* if for each $g \notin S$ there exists a finite index normal subgroup N of G with $g \notin NS$. This is equivalent to S being a closed subset in the profinite topology on G. If G is S-separable for every finitely generated subgroup S of G, then G is said to be subgroup separable (or LERF). In the special case of $S = \{1\}$, G is S-separable means G is residually finite. If G is

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S-separable for every conjugacy class S of G, then G is called conjugacy separable. Separability properties of groups are of great interest to both group theorists and three-manifold topologists. In particular, Mostowski [16] showed that: (1) finitely presented residually finite groups have solvable word problem, (2) finitely presented conjugacy separable groups have solvable conjugacy problem, and (3) finitely presented subgroup separable groups have solvable generalized word problem.

It is well known that free groups and surface groups are both subgroup separable and conjugacy separable (Hall [10], Scott [19], Stebe [21]). Thurston [24] showed that the fundamental groups of Haken manifolds are residually finite. Scott asked whether these groups are subgroup separable. Burns, Karrass and Solitar [5] showed that these groups needs not be subgroup separable. The question of whether these groups are always conjugacy separable is still unknown. In particular, whether knot groups are subgroup separable or conjugacy separable is still unknown. Blackburn [4] showed that finitely generated nilpotent groups are conjugacy separable. Subsequently, Formanek [8] (also Remeslennikov [18]) showed that polycyclic-by-finite groups are conjugacy separable. Stebe [21, 22] first considered the conjugacy separability of generalized free products of free groups and that of Fuchsian groups. The former was completed by Dyer [6], who showed that generalized free products of two free groups or finitely generated nilpotent groups amalgamating a cyclic subgroup are conjugacy separable. The latter was completed by Fine and Rosenberger [7] using some results of Stebe [22] and Allenby and Tang [3]. This naturally leads us to ask whether generalized free products of finitely generated Fuchsian groups amalgamating a cyclic subgroup are conjugacy separable. These groups are proved to be conjugacy separable by Kim and Tang [13]. In the same paper we asked whether generalized free products of polycyclic groups amalgamating a cyclic subgroup are conjugacy separable. L. Ribes recently informed us that together with D. Segal and P. A. Zalesskii, they have proved the conjugacy separability of these groups. Thus it is natural to ask whether tree products of conjugacy separable groups amalgamating cyclic subgroups are conjugacy separable. In this paper we show that tree products of a finite number of conjugacy separable residually nilpotent groups are conjugacy separable. The main difficulty in the proof lies in proving the double coset separability of these groups. Other results on conjugacy separability and double coset separability can be found in Allenby [1], Shirvani [20], Niblo [17], and Gitik and Rips [9].

In Section 2 we prove one of our main results, a criterion for the generalized free products of two conjugacy separable groups amalgamating a cyclic subgroup to be conjugacy separable; this is Theorem 2.3. The main body of our proof is in Section 3, where we prove that tree products of residually nilpotent groups amalgamating cyclic subgroups are double coset separable with respect to the subgroups $\langle h \rangle$ and $\langle k \rangle$, where h and k are elements of some vertex groups. This involves a detailed analysis of the various cases. Finally, in Section 4 we prove our main result, Theorem 4.3, that tree products of a finite number of conjugacy separable residually nilpotent groups, amalgamating cyclic subgroups, are conjugacy separable. From this it follows that tree products of a finite number of free groups, finitely generated nilpotent groups, and surface groups, amalgamating cyclic subgroups, are conjugacy separable. At this point it is natural to ask whether tree products of Fuchsian groups amalgamating cyclic subgroups are conjugacy separable. In view of Ribes, Segal, and Zalesskii's result we also ask whether tree products of polycyclic groups amalgamating cyclic subgroups are conjugacy separable.

Throughout this paper we use standard notations and terminology. For convenience we list the following:

The letter G always denotes a group.

$\{x\}^G$ denotes the set of all conjugates of x in G.

$x \sim_G y$ means x, y are conjugate in G.

If $x \in G = A *_H B$, then $\|x\|$ denotes the free product length of x in G.

$N \lhd_f G$ means N is a normal subgroup of finite index in G.

A group G is π_c if, for every cyclic subgroup H of G and every $x \in G \setminus H$, there exists $N \lhd_f G$ such that in $\overline{G} = G/N$, $\bar{x} \notin \overline{H}$.

A group G is *conjugacy separable* if, for each pair of nonconjugate elements $x, y \in G$, there exists $N \lhd_f G$ such that in $\overline{G} = G/N$, $\bar{x} \nsim_{\overline{G}} \bar{y}$.

$C_H(x)$ is the centralizer of $x \in G$ in the subgroup H of G.

Let $H = \langle h \rangle$ be a infinite cyclic subgroup of G. If, for every positive integer n, there exists $N_n \lhd_f G$ such that $N_n \cap H = \langle h^n \rangle$, then G is said to be *H-potent*. A torsion-free group G is said to be *potent* if it is $\langle h \rangle$-potent for all $1 \neq h \in G$.

Let P be a tree. Assign a group G_v to each vertex v and a group G_e to each edge e of P. Let α_e and β_e be monomorphisms which embed G_e as a subgroup of the two vertex groups at the ends of the edge e. The *tree product* of P is defined to be the group generated by the generators and relations of the vertex groups together with the extra relations obtained by identifying $g_e \alpha_e$ and $g_e \beta_e$ for each $g_e \in G_e$.

We use $\mathfrak{N}$ to denote the class of finitely generated torsion-free nilpotent groups and $\mathrm{R}\mathfrak{N}$ to denote the class of residually $\mathfrak{N}$-groups.

We shall make extensive use of the following results.

THEOREM 1.1 [15, Theorem 4.6]. *Let $G = A *_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclically reduced and that $x \sim_G y$.*

(1) *If* $\|x\| = 0$, *then* $\|y\| \leq 1$ *and, if* $y \in A$, *then there is a sequence* $h_1, h_2, \ldots, h_r$ *of elements in* H *such that* $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim h_r = x$.

(2) *If* $\|x\| = 1$, *then* $\|y\| = 1$ *and either* $x, y \in A$ *and* $x \sim_A y$ *or* $x, y \in B$ *and* $x \sim_B y$.

(3) *If* $\|x\| \geq 2$, *then* $\|x\| = \|y\|$ *and* $y \sim_H x^*$, *where* x^* *is a cyclic permutation of* x.

THEOREM 1.2 [6, Theorem 4]. *If* A *and* B *are conjugacy separable and* H *is finite, then* $A *_H B$ *is conjugacy separable.*

2. A CRITERION

In this section we prove a criterion for the conjugacy separability of generalized free products of conjugacy separable groups amalgamating a cyclic subgroup.

LEMMA 2.1. *Let* G *be* $\langle h^\epsilon \rangle x \langle k^\epsilon \rangle$*-separable, where* $h, k \in G$ *are of infinite order. If* $\langle x^{-1}hx \rangle \cap \langle k \rangle = 1$, *then there exists* $N \lhd_f G$ *such that* $\bar{x}^{-1}\bar{h}^i\bar{x} = \bar{k}^j$ *only if* $\epsilon | i, j$, *where* $\overline{G} = G/N$.

Proof. Note $h^{-i_1}xk^{-j_1} \notin \langle h^\epsilon \rangle x \langle k^\epsilon \rangle$ for $0 \leq i_1, j_1 < \epsilon$ except $i_1 = j_1 = 0$. Hence there exists $N \lhd_f G$ such that $h^{-i_1}xk^{-j_1} \notin N\langle h^\epsilon \rangle x \langle k^\epsilon \rangle$ for $0 \leq i_1, j_1 < \epsilon$ except $i_1 = j_1 = 0$. Therefore, if $\bar{x}^{-1}\bar{h}^i\bar{x} = \bar{k}^j$, then $\epsilon | i, j$, where $\overline{G} = G/N$. ∎

Note that the condition $\bar{x}^{-1}\bar{h}^i\bar{x} = \bar{k}^j$ only if $\epsilon | i, j$ always implies $\epsilon \mid |\bar{h}|, |\bar{k}|$.

DEFINITION 2.2. A group G is *cyclic conjugacy separable* for $\langle h \rangle$ if, for each $x \in G$ such that $\{x\}^G \cap \langle h \rangle = \emptyset$, there exists $N \lhd_f G$ such that $\{\bar{x}\}^{\overline{G}} \cap \langle \bar{h} \rangle = \emptyset$, where $\overline{G} = G/N$. If G is cyclic conjugacy separable for every cyclic subgroup $\langle h \rangle$ of G, then G is called *cyclic conjugacy separable*.

THEOREM 2.3. *Let* $G = G_1 *_{\langle h \rangle} G_2$. *Suppose that* G_1, G_2 *are conjugacy separable, cyclic conjugacy separable for* $\langle h \rangle$ *and* $\langle h \rangle$*-potent. If each* G_k $(k = 1, 2)$ *satisfies*:

C1. *if* $h^i \sim_{G_k} h^j$, *then* $i = j$;

C2. *for every* $\epsilon > 0$, G_k *is* $\langle h^\epsilon \rangle x \langle h^\epsilon \rangle$*-separable for* $x \in G_k$;

C3. *for every* $n > 0$, *there exists* $M \lhd_f G_k$ *such that* $|\bar{h}| = n$ *and* $\bar{h}^i \nsim_{\overline{G}_k} \bar{h}^j$ *for* $\bar{h}^i \neq \bar{h}^j$, *where* $\overline{G}_k = G_k/M$,

then G *is conjugacy separable.*

Proof. Let $x, y \in G$ such that $x \nsim_G y$. Without loss of generality we can assume that x and y are of minimal lengths in their conjugacy classes in G. Since G_1, G_2 are $\langle h \rangle$-potent and $\langle h \rangle$-separable by C2, G is residually finite by [2]. Hence, we may assume $x \neq 1 \neq y$. To prove our result, we shall find $M \triangleleft_f G_1$ and $L \triangleleft_f G_2$ such that $M \cap \langle h \rangle = L \cap \langle h \rangle$ and $\bar{x} \nsim_{\overline{G}} \bar{y}$, where $\overline{G} = G_1/M *_{\langle \bar{h} \rangle} G_2/L$. Then, by Theorem 1.2, $\overline{G}$ is conjugacy separable, whence there exists $\overline{K} \triangleleft_f \overline{G}$ such that $\overline{K}\bar{x} \nsim_{\overline{G}/\overline{K}} \overline{K}\bar{y}$. Let K be the preimage of $\overline{K}$ in G. Then we have $K \triangleleft_f G$ and $Kx \nsim_{G/K} Ky$, as required.

Case 1. $\|x\| = 0 = \|y\|$, say, $x = h^t$ and $y = h^s$. Clearly $s \neq t$.

(i) $|t| \neq |s|$. Since G_1, G_2 are potent, there exist $M \triangleleft_f G_1$ and $L \triangleleft_f G_2$ such that $M \cap \langle h \rangle = \langle h^{st} \rangle = L \cap \langle h \rangle$. Then, in $\overline{G} = G_1/M *_{\langle \bar{h} \rangle} G_2/L$, we have $|\bar{x}| = |s| \neq |t| = |\bar{y}|$, hence $\bar{x} \nsim_{\overline{G}} \bar{y}$.

(ii) $t = -s$, where $t > 0$. By C3, there exists $M \triangleleft_f G_1$ such that $|\bar{h}| = 3t$ and $\bar{h}^i \nsim_{\overline{G}_1} \bar{h}^j$ for $\bar{h}^i \neq \bar{h}^j$, where $\overline{G}_1 = G_1/M$. Similarly, there exists $L \triangleleft_f G_2$ such that $|\bar{h}| = 3t$ and $\bar{h}^i \nsim_{\overline{G}_2} \bar{h}^j$ for $\bar{h}^i \neq \bar{h}^j$, where $\overline{G}_2 = G_2/L$. Then, in $\overline{G} = G_1/M *_{\langle \bar{h} \rangle} G_2/L$, if $\bar{h}^t \sim_{\overline{G}} \bar{h}^{-t}$ then, by Theorem 1.1, $\bar{h}^t = \bar{h}^{-t}$. Since $|\bar{h}| = 3t$, it follows that $\bar{x} \nsim_{\overline{G}} \bar{y}$.

Case 2. $\|x\| = 0$ and $\|y\| = 1$ (or $\|y\| = 0$ and $\|x\| = 1$), say, $y \in G_1 \setminus \langle h \rangle$ and $x \in \langle h \rangle$. Since y is of minimal length 1 in its conjugacy class, we have $\{y\}^{G_1} \cap \langle h \rangle = \emptyset$. Moreover, since G_1 is cyclic conjugacy separable for $\langle h \rangle$, there exists $M \triangleleft_f G_1$ such that $\{\bar{y}\}^{\overline{G}_1} \cap \langle \bar{h} \rangle = \emptyset$, where $\overline{G}_1 = G_1/M$. Now, by the potency of G_2, there exists $L \triangleleft_f G_2$ such that $M \cap \langle h \rangle = L \cap \langle h \rangle$. Let $\overline{G} = G_1/M *_{\langle \bar{h} \rangle} G_2/L$. Then, by Theorem 1.1, $\bar{y}$ is of minimal length 1 in its conjugacy class in $\overline{G}$. Hence $\bar{x} \nsim_{\overline{G}} \bar{y}$, as required.

Case 3. $\|x\| \neq \|y\|$ and $\|x\| \geq 2$ (or $\|y\| \geq 2$). Since x is of minimal length in its conjugacy class, it is cyclically reduced. Let $x = a_1 b_1 \cdots a_n b_n$ (say, other cases being similar), where $a_i \in G_1 \setminus \langle h \rangle$ and $b_i \in G_2 \setminus \langle h \rangle$. Since G_1, G_2 are $\langle h \rangle$-separable (by C2) and $\langle h \rangle$-potent, we can find $M \triangleleft_f G_1$ and $L \triangleleft_f G_2$ such that $M \cap \langle h \rangle = L \cap \langle h \rangle$ and $\|\bar{x}\| = \|x\|$, $\|\bar{y}\| = \|y\|$, where $\overline{G} = G_1/M *_{\langle \bar{h} \rangle} G_2/L$. Thus $\bar{x}$ is cyclically reduced and is of minimal length $2n$ in its conjugacy class. Therefore $\|\bar{x}\| \neq \|\bar{y}\|$. Hence, by Theorem 1.1, $\bar{x} \nsim_{\overline{G}} \bar{y}$, as required.

Case 4. $\|x\| = 1 = \|y\|$.

(i) $x, y \in G_1 \setminus \langle h \rangle$ (or $x, y \in G_2 \setminus \langle h \rangle$). Since x, y are of minimal length 1 in their conjugacy classes, $\{x\}^G \cap \langle h \rangle = \emptyset$ and $\{y\}^G \cap \langle h \rangle = \emptyset$. Now G_1 is conjugacy separable and cyclic conjugacy separable for $\langle h \rangle$. Therefore, there exists $M \triangleleft_f G_1$ such that $\bar{x} \nsim_{\overline{G}_1} \bar{y}$, $\{\bar{x}\}^{\overline{G}_1} \cap \langle \bar{h} \rangle = \emptyset$, and $\{\bar{y}\}^{\overline{G}_1} \cap \langle \bar{h} \rangle = \emptyset$, where $\overline{G}_1 = G_1/M$. Since G_2 is potent, we can choose $L \triangleleft_f G_2$ such that $M \cap \langle h \rangle = L \cap \langle h \rangle$. Thus, in $\overline{G} = G_1/M *_{\langle \bar{h} \rangle} G_2/L$, $\bar{x}, \bar{y}$ are of minimal length 1 in their conjugacy classes, and $\bar{x} \nsim_{\overline{G}_1} \bar{y}$. Hence, by Theorem 1.1, $\bar{x} \nsim_{\overline{G}} \bar{y}$, as required.

(ii) Suppose $x \in G_1 \setminus \langle h \rangle$ and $y \in G_2 \setminus \langle h \rangle$ (or $x \in G_2 \setminus \langle h \rangle$ and $y \in G_1 \setminus \langle h \rangle$). As in (i) above, there exist $M \triangleleft_f G_1$ and $L \triangleleft_f G_2$ such that $M \cap \langle h \rangle = L \cap \langle h \rangle$, $\{\bar{x}\}^{\overline{G}_1} \cap \langle \bar{h} \rangle = \emptyset$, and $\{\bar{y}\}^{\overline{G}_2} \cap \langle \bar{h} \rangle = \emptyset$, where $\overline{G}_1 = G_1/M$ and $\overline{G}_2 = G_2/L$. Let $\overline{G} = G_1/M *_{\langle \bar{h} \rangle} G_2/L$. Then $\bar{x}, \bar{y}$ are of minimal length 1 in their conjugacy classes with $\bar{x} \in \overline{G}_1$ and $\bar{y} \in \overline{G}_2$. Hence, by Theorem 1.1, $\bar{x} \nsim_{\overline{G}} \bar{y}$, as required.

Case 5. $\|x\| = \|y\| = r \geq 2$. Let $x = u_1 u_2 \cdots u_r$ and $y = v_1 v_2 \cdots v_r$ be cyclically reduced. Since $x \nsim_G y$, by Theorem 1.1, $x \nsim_{\langle h \rangle} y^*$ for any cyclic permutation y^* of y. Thus, for each i, the equation

$$I(i)\colon u_1 u_2 \cdots u_r = h^{-\alpha_i} v_i \cdots v_r v_1 \cdots v_{i-1} h^{\alpha_i} \tag{1}$$

has no integer solution α_i. Hence, for each i, we shall find $M_i' \lhd_f G_1$ and $L_i' \lhd_f G_2$ such that $M_i' \cap \langle h \rangle = L_i' \cap \langle h \rangle$. Moreover, in $\overline{G} = G_1/M_i' *_{\langle \bar{h} \rangle} G_2/L_i'$, $\|\bar{x}\| = \|x\|$, $\|\bar{y}\| = \|y\|$, and the equation

$$\overline{I(i)}\colon \overline{u_1 u_2 \cdots u_r} = \bar{h}^{-\alpha_i} \overline{v_i \cdots v_r v_1 \cdots v_{i-1}} \bar{h}^{\alpha_i} \tag{2}$$

has no integer solution α_i. Let $M = \bigcap_{i=1}^r M_i'$, $L = \bigcap_{i=1}^r L_i'$, and $\overline{G} = G_1/M *_{\langle \bar{h} \rangle} G_2/L$. Then $\bar{x}$ and $\bar{y}$ are cyclically reduced and $\bar{x} \nsim_{\langle \bar{h} \rangle} \bar{y}^*$ for all cyclic permutation $\bar{y}^*$ of $\bar{y}$. Therefore, by Theorem 1.1, we have $\bar{x} \nsim_{\overline{G}} \bar{y}$, as required. Since all the cases are similar, we shall only consider the case $i = 1$.

Since equation $I(1)$ has no solution α_1, we have $x = u_1 u_2 \cdots u_r \nsim_{\langle h \rangle} v_1 v_2 \cdots v_r = y$. If u_i and v_i are not in the same factor of G, then we can easily find $\overline{G} = G_1/M_1' *_{\langle \bar{h} \rangle} G_2/L_1'$ such that $\|\bar{x}\| = \|x\|$ and $\|\bar{y}\| = \|y\|$. Clearly equation $\overline{I(1)}$ has no solution, since $\overline{u_i}$ and $\overline{v_i}$ are not in the same factor of $\overline{G}$. Thus we need only consider the case $x = c_1 d_1 \cdots c_n d_n$ and $y = a_1 b_1 \cdots a_n b_n$, where $a_i, c_i \in G_1 \setminus \langle h \rangle$ and $b_i, d_i \in G_2 \setminus \langle h \rangle$. We proceed by finding $M \lhd_f G_1$ and $L \lhd_f G_2$ such that $M \cap \langle h \rangle = L \cap \langle h \rangle$ so that, in $\overline{G} = G_1/M *_{\langle \bar{h} \rangle} G_2/L$, $\|\bar{x}\| = 2n = \|\bar{y}\|$. Moreover, equation $\overline{I(1)}$ has no integer solution α_1 in $\overline{G}$, whence $\bar{x} \nsim_{\langle \bar{h} \rangle} \bar{y}$. Since G_1, G_2 are $\langle h \rangle$-separable and $\langle h \rangle$-potent, there exist $M_1 \lhd_f G_1$ and $L_1 \lhd_f G_2$ such that $M_1 \cap \langle h \rangle = L_1 \cap \langle h \rangle$, $a_i, c_i \notin M_1 \langle h \rangle$, and $b_i, d_i \notin L_1 \langle h \rangle$ for all i.

(1) There exists i such that $c_i \notin \langle h \rangle a_i \langle h \rangle$ (or $d_i \notin \langle h \rangle b_i \langle h \rangle$). By C2, there exists $M_2 \lhd_f G_1$ such that $c_i \notin M_2 \langle h \rangle a_i \langle h \rangle$. Now, by the potency of G_2, we can choose $L_2 \lhd_f G_2$ such that $M_2 \cap \langle h \rangle = L_2 \cap \langle h \rangle$. Let $M = M_1 \cap M_2$ and $L = L_1 \cap L_2$. Then, in $\overline{G} = G_1/M *_{\langle \bar{h} \rangle} G_2/L$, $\bar{x}$ and $\bar{y}$ are cyclically reduced and $\overline{c_1 d_1 \cdots c_n d_n} \notin \langle \bar{h} \rangle \overline{a_1 b_1 \cdots a_n b_n} \langle \bar{h} \rangle$. Hence $\bar{x} \nsim_{\langle \bar{h} \rangle} \bar{y}$, as required.

(2) There exists i such that $c_1 d_1 \cdots c_i = h^{-\alpha} a_1 b_1 \cdots a_i h^{\beta}$, $d_i = h^{\lambda} b_i h^{\delta}$, and that $c_1 d_1 \cdots c_i d_i \notin \langle h \rangle a_1 b_1 \cdots a_i b_i \langle h \rangle$. This implies $a_1 b_1 \cdots a_i h^{\beta+\lambda} b_i \notin \langle h \rangle a_1 b_1 \cdots a_i b_i \langle h \rangle$. For convenience, we write $w = a_1 b_1 \cdots a_i$. Suppose $C_{\langle h \rangle}(w) = \langle h^t \rangle$ and $C_{\langle h \rangle}(b_i) = \langle h^s \rangle$, where $s, t \geq 0$.

(a) $t \neq 0 \neq s$. Then clearly $h^{\beta+\lambda} \notin \langle h^t\rangle\langle h^s\rangle = \langle h^d\rangle$, where $d = \gcd(t, s)$. Now $w^{-1}h^j w \notin \langle h\rangle$ for $1 \le j < t$ and $b_i^{-1}h^j b_i \notin \langle h\rangle$ for $1 \le j < s$ by C1. Since G_1, G_2 are $\langle h\rangle$-separable and $\langle h\rangle$-potent, G is $\langle h\rangle$-separable [2]. Thus there exists $N \lhd_f G$ such that $w^{-1}h^j w \notin N\langle h\rangle$ for all $1 \le j < t$ and $b_i^{-1}h^j b_i \notin N\langle h\rangle$ for all $1 \le j < s$. Since G_1, G_2 are potent, there exist $M_2 \lhd_f G_1$ and $L_2 \lhd_f G_2$ such that $M_2 \cap \langle h\rangle = \langle h^d\rangle = L_2 \cap \langle h\rangle$. Moreover $h^{\beta+\lambda} \notin \langle h^d\rangle$ implies $h^{\beta+\lambda} \notin M_2\langle h^d\rangle$. Let $M = M_1 \cap M_2 \cap N$ and $L = L_1 \cap L_2 \cap N$. Then, in $\overline{G} = G_1/M *_{\langle\overline{h}\rangle} G_2/L$, $\bar{x}$ and $\bar{y}$ are cyclically reduced and $\overline{a_1b_1 \cdots a_ih^{\beta+\lambda}b_i} \notin \langle\overline{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle\overline{h}\rangle$. For, if not, $\overline{w}\overline{h}^{\beta+\lambda}\overline{b}_i = \overline{h}^{-\mu_0}\overline{w}\overline{b}_i\overline{h}^{\mu_i}$ for some μ_0, μ_i. Then $\overline{w} = \overline{h}^{-\mu_0}\overline{w}\overline{h}^{\lambda_1}$ and $\overline{h}^{\beta+\lambda}\overline{b}_i = \overline{h}^{-\lambda_1}\overline{b}_i\overline{h}^{\mu_i}$ for some λ_1. By the choice of N, $t \mid \mu_0$, whence $\overline{h}^{\mu_0} = \overline{h}^{\lambda_1} \in \langle\overline{h}^t\rangle$, and $s \mid \beta + \lambda + \lambda_1$. Hence $\overline{h}^{\beta+\lambda} \in \langle\overline{h}^t\rangle\langle\overline{h}^s\rangle = \langle\overline{h}^d\rangle$, contradicting the choice of M_2. This shows that $\overline{a_1b_1 \cdots a_ih^{\beta+\lambda}b_i} \notin \langle\overline{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle\overline{h}\rangle$. Thus $\overline{c_1d_1 \cdots c_id_i} \notin \langle\overline{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle\overline{h}\rangle$. Hence $\bar{x} \nsim_{\langle\overline{h}\rangle} \bar{y}$, as required.

(b) $t \neq 0$ and $s = 0$. Then clearly $h^{\beta+\lambda} \notin \langle h^t\rangle$. As in (a), there exists $N \lhd_f G$ such that $w^{-1}h^j w \notin N\langle h\rangle$ for $1 \le j < t$ and there exists $M_2 \lhd_f G_1$ such that $h^{\beta+\lambda} \notin M_2\langle h^t\rangle$. Because of C1, $C_{\langle h\rangle}(b_i) = 1$ implies $\langle b_i^{-1}hb_i\rangle \cap \langle h\rangle = 1$. Hence, by Lemma 2.1, for $\epsilon = t$, there exists $M_3 \lhd_f G_1$ such that if $M_3h^j = M_3b_i^{-1}h^kb_i$, then $\epsilon = t \mid k, j$. As before, we can find $M \lhd_f G_1$ and $L \lhd_f G_2$ such that $M \cap \langle h\rangle = L \cap \langle h\rangle$, $M \subset M_1 \cap M_2 \cap M_3 \cap N$, and $L \subset L_1 \cap N$. Then, in $\overline{G} = G_1/M *_{\langle\overline{h}\rangle} G_2/L$, $\bar{x}$ and $\bar{y}$ are cyclically reduced, and $\overline{a_1b_1 \cdots a_ih^{\beta+\lambda}b_i} \notin \langle\overline{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle\overline{h}\rangle$. For, if not, $\overline{w}\overline{h}^{\beta+\lambda}\overline{b}_i = \overline{h}^{-\mu_0}\overline{w}\overline{b}_i\overline{h}^{\mu_i}$ for some μ_0, μ_i. This implies $\overline{w} = \overline{h}^{-\mu_0}\overline{w}\overline{h}^{\lambda_1}$ and $\overline{h}^{\beta+\lambda}\overline{b}_i = \overline{h}^{-\lambda_1}\overline{b}_i\overline{h}^{\mu_i}$ for some λ_1. By the choice of N, we have $t \mid \mu_0$, whence $\overline{h}^{\mu_0} = \overline{h}^{\lambda_1} \in \langle\overline{h}^t\rangle$, and $\epsilon = t \mid \beta + \lambda + \lambda_1$. Hence $\overline{h}^{\beta+\lambda} \in \langle\overline{h}^t\rangle$, contradicting the choice of M_2. This shows that $\overline{a_1b_1 \cdots a_ih^{\beta+\lambda}b_i} \notin \langle\overline{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle\overline{h}\rangle$. Therefore, $\overline{c_1d_1 \cdots c_id_i} \notin \langle\overline{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle\overline{h}\rangle$. Hence $\bar{x} \nsim_{\langle\overline{h}\rangle} \bar{y}$, as required.

(c) $t = 0$ and $s \neq 0$. Then clearly $h^{\beta+\lambda} \notin \langle h^s\rangle$. Since $C_{\langle h\rangle}(w) = 1$, there exists a_k (or b_k) with $k \le i$ such that $\langle h\rangle \cap \langle a_k^{-1}ha_k\rangle = 1$ by C1. Thus, by Lemma 2.1, there exists $M_2 \lhd_f G_1$ such that if $M_2h^r = M_2a_k^{-1}h^ja_k$, then $\epsilon = s \mid r, j$. Since G_1 is $\langle h^s\rangle$-separable, we can find $M_3 \lhd_f G_1$ such that $h^{\beta+\lambda} \notin M_3\langle h^s\rangle$. Also we can find $N \lhd_f G$ such that $b_i^{-1}h^jb_i \notin N\langle h\rangle$ for $1 \le j < s$. Let $M \lhd_f G_1$ and $L \lhd_f G_2$ such that $M \cap \langle h\rangle = L \cap \langle h\rangle$, $M \subset M_1 \cap M_2 \cap M_3 \cap N$, and $L \subset L_1 \cap N$. Then, in $\overline{G} = G_1/M *_{\langle\overline{h}\rangle} G_2/L$, $\bar{x}$ and $\bar{y}$ are cyclically reduced and $\overline{a_1b_1 \cdots a_ih^{\beta+\lambda}b_i} \notin \langle\overline{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle\overline{h}\rangle$. For, if $\overline{a_1b_1 \cdots a_ih^{\beta+\lambda}b_i} = \overline{h}^{\mu_0}\overline{a_1b_1 \cdots a_ib_i}\,\overline{h}^{\mu_i}$ for some μ_0, μ_i, then $\bar{a}_1 = \overline{h}^{\mu_0}\bar{a}_1\overline{h}^{\lambda_1}$, $\overline{b}_1 = \overline{h}^{-\lambda_1}\overline{b}_1\overline{h}^{\mu_1}, \ldots, \bar{a}_k = \overline{h}^{-\mu_{k-1}}\bar{a}_k\overline{h}^{\lambda_k}, \ldots, \bar{a}_i = \overline{h}^{-\mu_{i-1}}\bar{a}_i\overline{h}^{\lambda_i}$, and $\overline{h}^{\beta+\lambda}\overline{b}_i = \overline{h}^{-\lambda_i}\overline{b}_i\overline{h}^{\mu_i}$ for some λ_j, μ_j. Thus, by the choice of M_2, $\epsilon = s \mid \mu_{k-1}, \lambda_k$. Since $\epsilon \mid |\overline{h}|$, $\epsilon \mid \lambda_i$. Now, by the choice of N, $s \mid \beta + \lambda + \lambda_i$. Hence $\epsilon = s \mid \beta + \lambda$, thus $\overline{h}^{\beta+\lambda} \in \langle\overline{h}^s\rangle$, a contradiction. This shows that $\overline{a_1b_1 \cdots a_ih^{\beta+\lambda}b_i} \notin$

$\langle \bar{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle \bar{h}\rangle$. Therefore, $\overline{c_1d_1 \cdots c_id_i} \notin \langle \bar{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle \bar{h}\rangle$. Hence $\bar{x} \nsim_{\langle \bar{h}\rangle} \bar{y}$, as required.

(d) $s = 0$ and $t = 0$. Then clearly we have $\langle h\rangle \cap \langle b_i^{-1}hb_i\rangle = 1$. Moreover, since $C_{\langle h\rangle}(w) = 1$, there exists a_k (or b_k) with $k \le i$ such that $\langle h\rangle \cap \langle a_k^{-1}ha_k\rangle = 1$. By Lemma 2.1, for $\epsilon = 2|\beta + \lambda|$, there exist $M_2 \lhd_f G_1$ and $L_2 \lhd_f G_2$ such that if $M_2h^i = M_2a_k^{-1}h^ja_k$, then $\epsilon | i, j$, and that if $L_2h^j = L_2b_i^{-1}h^kb_i$, then $\epsilon | j, k$. As before, we can find $M_3 \lhd_f G_1$ such that $h^{\beta+\lambda} \notin M_3\langle h^{2(\beta+\lambda)}\rangle$. Again, as before, we can find $M \lhd_f G_1$ and $L \lhd_f G_2$ such that $M \cap \langle h\rangle = L \cap \langle h\rangle$, $M \subset M_1 \cap M_2 \cap M_3$, and $L \subset L_1 \cap L_2$. Thus, in $\bar{G} = G_1/M *_{\langle \bar{h}\rangle} G_2/L$, $\bar{x}$ and $\bar{y}$ are cyclically reduced, and $\overline{a_1b_1 \cdots a_ih^{\beta+\lambda}b_i} \notin \langle \bar{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle \bar{h}\rangle$. For, if $\overline{a_1b_1 \cdots a_ih^{\beta+\lambda}b_i} = \bar{h}^{\mu_0}\overline{a_1b_1 \cdots a_ib_ih^{\mu_i}}$ for some μ_0, μ_i, then $\bar{a}_1 = \bar{h}^{\mu_0}\bar{a}_1\bar{h}^{\lambda_1}$, $\bar{b}_1 = \bar{h}^{-\lambda_1}\bar{b}_1\bar{h}^{\mu_1}, \ldots, \bar{a}_k = \bar{h}^{-\mu_{k-1}}\bar{a}_k\bar{h}^{\lambda_k}, \ldots, \bar{a}_i = \bar{h}^{-\mu_{i-1}}\bar{a}_i\bar{h}^{\lambda_i}$, and $\bar{h}^{\beta+\lambda}\bar{b}_i = \bar{h}^{-\lambda_i}\bar{b}_i\bar{h}^{\mu_i}$ for some λ_j, μ_j. Then, by the choice of M_2, $\epsilon | \mu_{k-1}, \lambda_k$. Since $\epsilon \| \bar{h}|$, $\epsilon | \lambda_i$. Again, by the choice of L_2, $\epsilon | \beta + \lambda + \lambda_i$. Hence $\epsilon | \beta + \lambda$. This means $\bar{h}^{\beta+\lambda} \in \langle \bar{h}^{2(\beta+\lambda)}\rangle$, a contradiction. This shows that $\overline{a_1b_1 \cdots a_ih^{\beta+\lambda}b_i} \notin \langle \bar{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle \bar{h}\rangle$. Therefore, $\overline{c_1d_1 \cdots c_id_i} \notin \langle \bar{h}\rangle\overline{a_1b_1 \cdots a_ib_i}\langle \bar{h}\rangle$. Hence $\bar{x} \nsim_{\langle \bar{h}\rangle} \bar{y}$, as required.

(3) $c_1d_1 \cdots c_nd_n = h^\alpha a_1b_1 \cdots a_nb_nh^\beta$ and $h^{\alpha+\beta} \neq 1$. Choose $M_2 \lhd_f G_1$ such that $h^{\alpha+\beta} \notin M_2$. Let $M_1 \cap \langle h\rangle = \langle h^s\rangle = L_1 \cap \langle h\rangle$ and $M_2 \cap \langle h\rangle = \langle h^t\rangle$. By C3, there exist $M_3 \lhd_f G_1$ and $L_3 \lhd_f G_2$ such that $M_3 \cap \langle h\rangle = \langle h^{st}\rangle = L_3 \cap \langle h\rangle$ and $M_3h^i \nsim_{G_1/M_3} M_3h^j$ for $M_3h^i \neq M_3h^j$ and $L_3h^i \nsim_{G_2/L_3} L_3h^j$ for $L_3h^i \neq L_3h^j$. Let $M = M_1 \cap M_2 \cap M_3$, $L = L_1 \cap L_3$, and $\bar{G} = G_1/M *_{\langle \bar{h}\rangle} G_2/L$. Then $\bar{x}$ and $\bar{y}$ are cyclically reduced and $\bar{x} \nsim_{\langle \bar{h}\rangle} \bar{y}$. For, if $\bar{x} \sim_{\langle \bar{h}\rangle} \bar{y}$, then $\bar{x} = \bar{h}^\alpha\bar{y}\bar{h}^\beta = \bar{h}^{-\lambda}\bar{y}\bar{h}^\lambda$ for some λ. This implies that $\bar{h}^\alpha\bar{a}_1 = \bar{h}^{-\lambda}\bar{a}_1\bar{h}^{\lambda_1}$, $\bar{b}_1 = \bar{h}^{-\lambda_1}\bar{b}_1\bar{h}^{\mu_1}, \ldots, \bar{a}_k = \bar{h}^{-\mu_{k-1}}\bar{a}_k\bar{h}^{\lambda_k}, \ldots, \bar{a}_n = \bar{h}^{-\mu_{n-1}}\bar{a}_n\bar{h}^{\lambda_n}$, and $\bar{b}_n\bar{h}^\beta = \bar{h}^{-\lambda_n}\bar{b}_n\bar{h}^\lambda$ for some λ_j, μ_j. Thus, by the choice of M_3, L_3, we have $\bar{h}^{\alpha+\lambda} = \bar{h}^{\lambda_1}$, $\bar{h}^{\lambda_1} = \bar{h}^{\mu_1}, \ldots, \bar{h}^{\mu_{k-1}} = \bar{h}^{\lambda_k}, \ldots, \bar{h}^{\mu_{n-1}} = \bar{h}^{\lambda_n}$, and $\bar{h}^{\lambda_n} = \bar{h}^{\lambda-\beta}$. This means $\bar{h}^{\alpha+\lambda} = \bar{h}^{\lambda-\beta}$, whence $\bar{h}^{\alpha+\beta} = 1$, contradicting the choice of M_2. Hence $\bar{x} \nsim_{\langle \bar{h}\rangle} \bar{y}$, as required.

This completes our proof that there exist $M_1' \lhd_f G_1$ and $L_1' \lhd_f G_2$ such that equation $\overline{I(1)}$ has no solution in $\tilde{G} = G_1/M_1' *_{\langle \tilde{h}\rangle} G_2/L_1'$. Similarly, for each i, we can find $M_i' \lhd_f G_1$ and $L_i' \lhd_f G_2$ such that equation $\overline{I(i)}$ has no solution in $\hat{G} = G_1/M_i' *_{\langle \bar{h}\rangle} G_2/L_i'$. Let $M = \bigcap_{i=1}^r M_i'$ and $L = \bigcap_{i=1}^r L_i'$. Then $M \lhd_f G_1$, $L \lhd_f G_2$, and, in $\bar{G} = G_1/M *_{\langle \bar{h}\rangle} G_2/L$, $\overline{I(i)}$ has no solution for each i. Hence $\bar{x} \nsim_{\langle \bar{h}\rangle} \bar{y}^*$ for any cyclic permutation $\bar{y}^*$ of $\bar{y}$. Thus it follows from Theorem 1.1 that $\bar{x} \nsim_{\bar{G}} \bar{y}$, as required. ∎

It is easy to see that $R\mathfrak{N}$-groups are potent and they satisfy C1, C2, and C3 (see Lemmas 3.2 and 3.4 and [23, Corollary 2.2]). Since, by [6] and [23], free groups, finitely generated nilpotent groups and surface groups are cyclic conjugacy separable, we have:

COROLLARY 2.4 [6, 23]. *Let G_i $(i = 1, 2)$ be a finitely generated torsion-free nilpotent or free or surface group. Then $G = G_1 *_{\langle c \rangle} G_2$ is conjugacy separable.*

A generalized free product of two free groups amalgamating an infinite cyclic subgroup is called a *cyclically pinched one-relator group*. Hence Corollary 2.4 implies in particular that cyclically pinched one-relator groups are conjugacy separable.

It was also known by [12] that generalized free products of two free or surface groups, amalgamating a maximal cyclic subgroup, satisfy C1 and C2 in Theorem 2.3. Thus, by Lemmas 4.1, 4.2, and 3.1, we have:

COROLLARY 2.5 [12]. *Let A_i be free or surface groups and let $\langle u \rangle$ and $\langle v \rangle$ be maximal cyclic subgroups of A_1, A_2 and A_3, A_4, respectively, such that $A_1 \cap A_2 = \langle u \rangle$ and $A_3 \cap A_4 = \langle v \rangle$. If $A_2 \cap A_3 = \langle c \rangle$, then $A_1 *_{\langle u \rangle} A_2 *_{\langle c \rangle} A_3 *_{\langle v \rangle} A_4$ and $A_1 *_{\langle u \rangle} A_2 *_{\langle c \rangle} A_3$ are conjugacy separable.*

3. DOUBLE COSET SEPARABILITY OF CERTAIN TREE PRODUCTS

Let G be a tree product of a finite number of $R\mathfrak{N}$-groups, amalgamating cyclic subgroups. Suppose G has r vertices. Then, either G is a $R\mathfrak{N}$-group when G has one vertex or $G = A *_{\langle c \rangle} B$, where B is a vertex group, A is a subtree product of $r - 1$ vertices, and c is in a vertex of A. In the latter case, if h is in a vertex of G, then either h is in a vertex of A or $h \in B$. Throughout this section, we shall use the above description for G and show that G is $\langle h \rangle x \langle k \rangle$-separable for $x \in G$, where h, k are in the vertices of G.

LEMMA 3.1. *Let G be a tree product of a finite number of $R\mathfrak{N}$-groups amalgamating cyclic subgroups. If h is in a vertex group of G, then G is $\langle h \rangle$-potent.*

Proof. Clearly a $R\mathfrak{N}$-group is potent. Let $G = A *_{\langle c \rangle} B$ as described above. By induction, we can assume that A is $\langle u \rangle$-potent for any u in a vertex of A. Then A, B are $\langle c \rangle$-potent. Hence, by Lemma 3.2 in [2], G is $\langle h \rangle$-potent for any h in a vertex of G. ∎

LEMMA 3.2. *Let G be a tree product of a finite number of $R\mathfrak{N}$-groups amalgamating cyclic subgroups. Let h be in a vertex group of G. If $h^i \sim_G h^j$, then $i = j$.*

Proof. Clearly a $R\mathfrak{N}$-group has the property. Hence we let $G = A *_{\langle c\rangle} B$ as above and, by induction, we suppose that A has the property that $u^i \sim_A u^j$ only if $i = j$ for u in a vertex of A. Let h be in a vertex of G and $h^i \sim_G h^j$.

Suppose $h \in A$. If $\{h^i\}^G \cap \langle c\rangle = \emptyset$, then, by Theorem 1.1, $h^i \sim_A h^j$. Thus, by induction, $i = j$. Hence we assume $h^i \sim_G c^n$ for some n. Then, by Theorem 1.1, $h^i \sim_A c^{\epsilon_1} \sim_B c^{\epsilon_2} \sim_A \cdots \sim c^{\epsilon_r} = c^n$ for some ϵ_k. This implies $\epsilon_1 = \epsilon_2 = \cdots \epsilon_r = n$, since c is in B and is in a vertex of A. Thus $h^i \sim_A c^n$. Similarly $h^j \sim_A c^n$, since $h^j \sim_G h^i \sim_A c^n$. It follows that $h^i \sim_A h^j$. Since h is in a vertex of A, by induction $i = j$.

The case of $h \in B$ is similar. ∎

LEMMA 3.3. *Let G be a tree product of a finite number of $R\mathfrak{N}$-groups amalgamating cyclic subgroups. Then G is π_c.*

Proof. A $R\mathfrak{N}$-group is π_c by [13, Lemma 2.6]. So let $G = A *_{\langle c\rangle} B$ as above. By induction, we assume that A is π_c. Since A, B are $\langle c\rangle$-potent, G is π_c by Corollary 2.2 in [11]. ∎

LEMMA 3.4. *Let G be a $R\mathfrak{N}$-group. Let $h, k \in G$. Then G is $\langle h\rangle\langle k\rangle$-separable.*

Proof. Let $g \notin \langle h\rangle\langle k\rangle$, where $g, h, k \in G$.

Case 1. $\langle h\rangle \cap \langle k\rangle = \langle h^n\rangle \neq 1$, where $h^n = k^m$ and $n > 0$. Then $h^{-i}gk^{-j} \notin \langle h^n\rangle\langle k^m\rangle = \langle h^n\rangle$ for all $0 \le i < n$ and $0 \le j < |m|$. Since G is π_c by Lemma 3.3, there exists $N \lhd_f G$ such that $h^{-i}gk^{-j} \notin N\langle h^n\rangle$ for all $0 \le i < n$ and $0 \le j < |m|$. Then clearly $g \notin N\langle h\rangle\langle k\rangle$.

Case 2. $\langle h\rangle \cap \langle k\rangle = 1$. First, we shall find $N_1 \lhd G$ such that $h, k \notin N_1$ and $\langle N_1 h\rangle \cap \langle N_1 k\rangle = N_1$ with $G/N_1 \in \mathfrak{N}$. Since G is $R\mathfrak{N}$, there exists $M_1 \lhd G$ such that $h, k \notin M_1$ and $G/M_1 \in \mathfrak{N}$. If $\langle M_1 h\rangle \cap \langle M_1 k\rangle = M_1$, then let $N_1 = M_1$. Otherwise, let $M_1 h^s = M_1 k^t$, where $h^s k^{-t} \neq 1$. Therefore, there exists $M_2 \lhd G$ such that $h^s k^{-t} \notin M_2$ and $G/M_2 \in \mathfrak{N}$. Let $N_1 = M_1 \cap M_2$. Since G/N_1 can be imbedded in a direct product $G/M_1 \times G/M_2 \in \mathfrak{N}$, $G/N_1 \in \mathfrak{N}$. We note that $\langle N_1 h\rangle \cap \langle N_1 k\rangle = N_1$. For, if $N_1 h^n = N_1 k^m$, then $N_1 h^{ns} = N_1 k^{ms}$. This implies $M_1 k^{nt} = M_1 h^{ns} = M_1 k^{ms}$. Since $G/M_1 \in \mathfrak{N}$ and $|M_1 k| = \infty$, $nt = ms$. This means $N_1 h^{ns} = N_1 k^{ms} = N_1 k^{nt}$. Now $G/N_1 \in \mathfrak{N}$ has the unique root property. Therefore $N_1 h^s = N_1 k^t$, contradicting the choice of M_2. Hence $h, k \notin N_1$ and $\langle N_1 h\rangle \cap \langle N_1 k\rangle = N_1$.

Since $G/N_1 \in \mathfrak{N}$, G/N_1 is $\langle N_1 h\rangle\langle N_1 k\rangle$-separable. Therefore, if $N_1 g \notin \langle N_1 h\rangle\langle N_1 k\rangle$, then we can find $N \lhd_f G$ such that $g \notin N\langle h\rangle\langle k\rangle$. So we can assume $N_1 g = N_1 h^s N_1 k^t$ for some s, t. Since $\langle N_1 h\rangle \cap \langle N_1 k\rangle = N_1$, $N_1 g = N_1 h^s N_1 k^t$ is unique. Now choose $N_2 \lhd G$ such that $g^{-1}h^s k^t \notin N_2$

and $G/N_2 \in \mathfrak{N}$. Let $M = N_1 \cap N_2$. Then, as before, $G/M \in \mathfrak{N}$. Moreover $Mg \notin \langle Mh\rangle\langle Mk\rangle$. Then we can find $N \lhd_f G$ such that $g \notin N\langle h\rangle\langle k\rangle$. ∎

LEMMA 3.5. *Let G be a tree product of a finite number of* $\mathrm{R}\mathfrak{N}$*-groups amalgamating cyclic subgroups. Then G is $\langle u\rangle\langle v\rangle$-separable for u, v in vertices of G.*

Proof. By Lemma 3.4, the result holds for tree products with one vertex. Hence we assume that $G = A *_{\langle c\rangle} B$, where c is in a vertex of the subtree product A and B is a $\mathrm{R}\mathfrak{N}$-group. By induction, we can assume that A is $\langle h\rangle\langle k\rangle$-separable for h, k in vertices of A. Let $g \notin \langle u\rangle\langle v\rangle$, where $g \in G$ and u, v are elements of vertices of G. We shall find $M \lhd_f A$ and $L \lhd_f B$ such that $M \cap \langle c\rangle = L \cap \langle c\rangle$ and $\bar{g} \notin \langle\bar{u}\rangle\langle\bar{v}\rangle$, where $\overline{G} = A/M *_{\langle c\rangle} B/L$. Since $\overline{G}$ is residually finite and $|\langle\bar{u}\rangle\langle\bar{v}\rangle| < \infty$, $\overline{G}$ is $\langle\bar{u}\rangle\langle\bar{v}\rangle$-separable. Therefore, there exists $\overline{N} \lhd_f \overline{G}$ such that $\bar{g} \notin \overline{N}\langle\bar{u}\rangle\langle\bar{v}\rangle$. Let N be the preimage of $\overline{N}$. Then $N \lhd_f G$ and $g \notin N\langle u\rangle\langle v\rangle$.

Case 1. $u, v \in A$ (or $u, v \in B$)

(1) $g \in A$. By the induction hypothesis, there exists $M \lhd_f A$ such that $\bar{g} \notin \langle\bar{u}\rangle\langle\bar{v}\rangle$, where $\overline{A} = A/M$. Since B is potent, there exists $L \lhd_f B$ such that $M \cap \langle c\rangle = L \cap \langle c\rangle$. Then, in $\overline{G} = A/M *_{\langle\bar{c}\rangle} B/L$, $\bar{g} \notin \langle\bar{u}\rangle\langle\bar{v}\rangle$.

(2) $g \notin A$. By Lemmas 3.3 and 3.1, A, B are π_c and $\langle c\rangle$-potent. Therefore, there exist $M \lhd_f A$ and $L \lhd_f B$ such that $M \cap \langle c\rangle = L \cap \langle c\rangle$ and $\|\bar{g}\| = \|g\|$, where $\overline{G} = A/M *_{\langle\bar{c}\rangle} B/L$. Clearly $\bar{g} \notin \langle\bar{u}\rangle\langle\bar{v}\rangle$.

Case 2. $u \in A$ and $v \in B$ (or $u \in B$ and $v \in A$). If $\langle u\rangle \cap \langle c\rangle = \langle c^s\rangle \neq 1$ (similarly, if $\langle v\rangle \cap \langle c\rangle \neq 1$), say, $c^s = u^t$, then $u^i g \notin \langle u^t\rangle\langle v\rangle = \langle c^s\rangle\langle v\rangle$ for $0 \le i < |t|$. Since $c^s, v \in B$, by Case 1, there exists $N \lhd_f G$ such that $u^i g \notin N\langle c^s\rangle\langle v\rangle = N\langle u^t\rangle\langle v\rangle$ for $0 \le i < |t|$. It follows that $g \notin N\langle u\rangle\langle v\rangle$. So we assume that $\langle u\rangle \cap \langle c\rangle = 1 = \langle v\rangle \cap \langle c\rangle$.

(1) $g \in \langle c\rangle$. Let $g = c^\alpha \notin \langle u\rangle\langle v\rangle$. Since $\langle u\rangle \cap \langle c\rangle = 1 = \langle v\rangle \cap \langle c\rangle$, $c^i \notin \langle u\rangle\langle c^{2\alpha}\rangle$ and $c^i \notin \langle c^{2\alpha}\rangle\langle v\rangle$ for all $1 \le i < 2|\alpha|$. By induction, A is $\langle u\rangle\langle c^{2\alpha}\rangle$-separable. Also, by Lemma 3.4, B is $\langle c^{2\alpha}\rangle\langle v\rangle$-separable. Therefore, there exist $M \lhd_f A$ and $L \lhd_f B$ such that $c^\alpha \notin M\langle c^{2\alpha}\rangle$, $c^i \notin M\langle u\rangle\langle c^{2\alpha}\rangle$, and $c^i \notin L\langle c^{2\alpha}\rangle\langle v\rangle$ for all $1 \le i < 2|\alpha|$. Since A, B are $\langle c\rangle$-potent, we can also assume $M \cap \langle c\rangle = L \cap \langle c\rangle$. Then, in $\overline{G} = A/M *_{\langle\bar{c}\rangle} B/L$, $\bar{g} = \bar{c}^\alpha \notin \langle\bar{u}\rangle\langle\bar{v}\rangle$. For, if $\bar{c}^\alpha = \bar{u}^i\bar{v}^j$, then $\bar{u}^i = \bar{c}^{\delta_1}$ and $\bar{v}^j = \bar{c}^{\delta_2}$. Thus, from the choice of M, L, $2\alpha | \delta_1, \delta_2$. This implies $\bar{c}^\alpha = c^{\delta_1}c^{\delta_2} \in \langle\bar{c}^{2\alpha}\rangle$, contradicting the choice of M. Hence, in $\overline{G} = A/M *_{\langle\bar{c}\rangle} B/L$, $\bar{g} = \bar{c}^\alpha \notin \langle\bar{u}\rangle\langle\bar{v}\rangle$.

(2) $g \in A \setminus \langle c\rangle$ (or $g \in B \setminus \langle c\rangle$). If $g \notin \langle u\rangle\langle c\rangle$, then, by induction, there exists $M \lhd_f A$ such that $g \notin M\langle u\rangle\langle c\rangle$. Since B is $\langle c\rangle$-potent, there exists $L \lhd_f B$ such that $L \cap \langle c\rangle = M \cap \langle c\rangle$. Then $\bar{g} \notin \langle\bar{u}\rangle\langle\bar{v}\rangle$ in $\overline{G} = A/M *_{\langle\bar{c}\rangle} B/L$. If $g = u^i c^\alpha$, then we have $c^\alpha \notin \langle u\rangle\langle v\rangle$. Thus, as in (1),

there exist $M \lhd_f A$ and $L \lhd_f B$ such that $\bar{c}^\alpha \notin \langle \bar{u} \rangle \langle \bar{v} \rangle$ in $\overline{G} = A/M *_{\langle \bar{c} \rangle} B/L$. Hence $\bar{g} \notin \langle \bar{u} \rangle \langle \bar{v} \rangle$.

(3) $g = ab$, where $a \in A \setminus \langle c \rangle$ and $b \in B \setminus \langle c \rangle$. If $a \notin \langle u \rangle \langle c \rangle$ (or, similarly, $b \notin \langle c \rangle \langle v \rangle$), then there exists $M \lhd_f A$ and $L \lhd_f B$ with $M \cap \langle c \rangle = L \cap \langle c \rangle$ such that $\bar{a} \notin \langle \bar{u} \rangle \langle \bar{c} \rangle$ and $\bar{b} \in \overline{B} \setminus \langle \bar{c} \rangle$ in $\overline{G} = A/M *_{\langle \bar{c} \rangle} B/L$. Then $\bar{g} \notin \langle \bar{u} \rangle \langle \bar{v} \rangle$. So assume $a = u^i c^\alpha$ and $b = c^\beta v^j$. Since $g = ab \notin \langle u \rangle \langle v \rangle$, $c^{\alpha+\beta} \notin \langle u \rangle \langle v \rangle$. As in (1), there exist $M \lhd_f A$ and $L \lhd_f B$ such that $\bar{c}^{\alpha+\beta} \notin \langle \bar{u} \rangle \langle \bar{v} \rangle$ in $\overline{G} = A/M *_{\langle \bar{c} \rangle} B/L$. Hence $\bar{g} \notin \langle \bar{u} \rangle \langle \bar{v} \rangle$.

(4) $g = ba$ or $\|g\| \geq 3$, where $b \in B \setminus \langle c \rangle$ and $a \in A \setminus \langle c \rangle$. As in Case 1, we can find $M \lhd_f A$ and $L \lhd_f B$ such that $\|g\| = \|\bar{g}\|$, where $\overline{G} = A/M *_{\langle \bar{c} \rangle} B/L$. Then $\bar{g} \notin \langle \bar{u} \rangle \langle \bar{v} \rangle$. ∎

Recall that a tree product of r vertices of R$\mathfrak{N}$-groups ($r > 1$), amalgamating cyclic subgroups, can be written as $G = A *_{\langle c \rangle} B$, where A is a subtree product of $r - 1$ vertices, B is a R$\mathfrak{N}$-group, and c is in a vertex of A. For the rest of this section, we shall show that G is $\langle h \rangle x \langle k \rangle$-separable for $x \in G$ and h, k in some vertices of G. Applying induction, we can assume that A is $\langle u \rangle a \langle v \rangle$-separable for $a \in A$ whenever u, v are in some vertices of A.

LEMMA 3.6. *Let G, A, B, c be as above and h, k be in the vertices of G. We can find $M \lhd_f A$ and $L \lhd_f B$ with $M \cap \langle c \rangle = L \cap \langle c \rangle$ such that, in $\overline{G} = A/M *_{\langle \bar{c} \rangle} B/L$,*

(1) *if $xc^\alpha \notin \langle h \rangle x \langle k \rangle$, where $x, h \in A$, $k \in B$, and $\langle k \rangle \cap \langle c \rangle = 1$, then $\overline{xc^\alpha} \notin \langle \bar{h} \rangle \bar{x} \langle \bar{k} \rangle$;*

(2) *if $xc^\alpha \notin \langle h \rangle x \langle k \rangle$, where $x, h \in B$, $k \in A$, and $\langle k \rangle \cap \langle c \rangle = 1$, then $\overline{xc^\alpha} \notin \langle \bar{h} \rangle \bar{x} \langle \bar{k} \rangle$.*

Proof. (1)

Case 1. There exists a minimal integer $m > 0$ such that $x^{-1}h^m x = c^s$, that is, $h^m x = xc^s$. Clearly $c^\alpha \notin \langle c^s \rangle \langle k \rangle$. By Lemma 3.5, we can find $\overline{G} = A/M *_{\langle \bar{c} \rangle} B/L$ with $M \cap \langle c \rangle = L \cap \langle c \rangle$ such that $\bar{c}^\alpha \notin \langle \bar{c}^s \rangle \langle \bar{k} \rangle$, $\overline{x^{-1}h^i x} \notin \langle \bar{c} \rangle$ for all $0 < i < m$. If $\overline{xc^\alpha} = \bar{h}^i \bar{x} \bar{k}^j$ for some i, j, then $\bar{k}^j = \bar{c}^\delta$ for some δ. Thus $\bar{x}^{-1} \bar{h}^i \bar{x} = \bar{c}^{\alpha-\delta}$. This implies $m | i$. It follows that $\bar{c}^{\alpha-\delta} \in \langle \bar{c}^s \rangle$, thus $\bar{c}^\alpha \in \langle \bar{c}^s \rangle \langle \bar{k} \rangle$, a contradiction. Therefore, $\overline{xc^\alpha} \notin \langle \bar{h} \rangle \bar{x} \langle \bar{k} \rangle$.

Case 2. $\langle x^{-1}hx \rangle \cap \langle c \rangle = 1$. Since $\alpha \neq 0$ and $\langle k \rangle \cap \langle c \rangle = 1$, $c^\alpha \notin \langle c^{2\alpha} \rangle \langle k \rangle$. By induction, A is $\langle h^\epsilon \rangle x \langle c^\epsilon \rangle$-separable. Therefore, by Lemmas 2.1 and 3.5, we can find $\overline{G} = A/M *_{\langle \bar{c} \rangle} B/L$ such that $\overline{x^{-1}h^{i_1}x} = \bar{c}^{i_2}$ only if $\epsilon = 2\alpha | i_1, i_2$ and $\bar{c}^\alpha \notin \langle \bar{c}^{2\alpha} \rangle \langle \bar{k} \rangle$. If $\overline{xc^\alpha} = \bar{h}^i \bar{x} \bar{k}^j$ for some i, j, then $\bar{k}^j = \bar{c}^\delta$ for some δ. Thus $\bar{x}^{-1} \bar{h}^i \bar{x} = \bar{c}^{\alpha-\delta}$. This implies $\epsilon | i, \alpha - \delta$, whence $\bar{c}^{\alpha-\delta} \in \langle \bar{c}^{2\alpha} \rangle$. This means $\bar{c}^\alpha \in \langle \bar{c}^{2\alpha} \rangle \langle \bar{k} \rangle$, a contradiction. Therefore, $\overline{xc^\alpha} \notin \langle \bar{h} \rangle \bar{x} \langle \bar{k} \rangle$.

(2) is similar to (1). ∎

LEMMA 3.7. *Let G, A, B, c be as in Lemma* 3.6 *and let h, k be in the vertices of G. Then G is $\langle h\rangle x\langle k\rangle$-separable for $x \in A \cup B$.*

Proof. We only consider $x \in A$, the case $x \in B$ being similar. Let $g \in G$ and $g \notin \langle h\rangle x\langle k\rangle$, where h, k are in the vertices of G. We shall find $\overline{G} = A/M *_{\langle\bar{c}\rangle} B/L$ such that $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$, where $M \lhd_f A$, $L \lhd_f B$, and $M \cap \langle c\rangle = L \cap \langle c\rangle$. Then, since $\overline{G}$ is residually finite and $|\bar{h}|, |\bar{k}| < \infty$, we can easily find $N \lhd_f G$ such that $\tilde{g} \notin \langle\tilde{h}\rangle\tilde{x}\langle\tilde{k}\rangle$, where $\tilde{G} = G/N$.

Case 1. $h, k \in A$. If $g \in A$ then, by induction, there exists $M \lhd_f A$ such that $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$ in $\overline{A} = A/M$. Since B is $\langle c\rangle$-potent, there exists $L \lhd_f B$ such that $M \cap \langle c\rangle = L \cap \langle c\rangle$. Then, in $\overline{G} = A/M *_{\langle\bar{c}\rangle} B/L$, we have $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. If $g \in B \setminus \langle c\rangle$ or $\|g\| \geq 2$, then we can find $\overline{G} = A/M *_{\langle\bar{c}\rangle} B/L$ such that $\|\bar{g}\| = \|g\|$. Therefore $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$.

Case 2. $h \in A$ and $k \in B$ or $h \in B$ and $k \in A$. Since $g \notin \langle h\rangle x\langle k\rangle$ is equivalent to $g^{-1} \notin \langle k\rangle x^{-1}\langle h\rangle$, it suffices to consider $h \in A$ and $k \in B$. If $\langle k\rangle \cap \langle c\rangle = \langle k^t\rangle \neq 1$, where $k^t = c^s$, then $gk^{-i} \notin \langle h\rangle x\langle k^t\rangle = \langle h\rangle x\langle c^s\rangle$ for $0 \leq i < t$. Thus, by Case 1, we can find $\overline{G}$ such that $\overline{gk^{-i}} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{c}^s\rangle$ for $0 \leq i < t$. This implies $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. Therefore we may assume $\langle k\rangle \cap \langle c\rangle = 1$.

(a) $g \in A$. If $g \notin \langle h\rangle x\langle c\rangle$, then, by Case 1, we can find $\overline{G}$ such that $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{c}\rangle$. Since $k \in B$, this implies $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. So let $g = h^i x c^\alpha$ for some i, α. Then $xc^\alpha \notin \langle h\rangle x\langle k\rangle$. By Lemma 3.6, we can find $\overline{G}$ such that $\overline{xc}^\alpha \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. Hence $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$.

(b) $g \in B \setminus \langle c\rangle$. If $x \notin \langle h\rangle\langle c\rangle$, then, by Lemma 3.5, we can find $\overline{G}$ such that $\bar{x} \notin \langle\bar{h}\rangle\langle\bar{c}\rangle$ and $\bar{g} \notin \langle\bar{c}\rangle$. This implies $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. So let $x = h^i c^\alpha$ for some i, α. Then $g \notin \langle h\rangle c^\alpha\langle k\rangle$. This means $gc^{-\alpha} \notin \langle h\rangle\langle c^\alpha k c^{-\alpha}\rangle$. Since $c^\alpha k c^{-\alpha} \in B$, by Lemma 3.5, we can find $\overline{G}$ such that $\overline{gc^{-\alpha}} \notin \langle\bar{h}\rangle\langle\overline{c^\alpha k c^{-\alpha}}\rangle$. Hence $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$.

(c) $g = a_1 b_1$, where $a_1 \in A \setminus \langle c\rangle$ and $b_1 \in B \setminus \langle c\rangle$. If $a_1 \notin \langle h\rangle x\langle c\rangle$ or $b_1 \notin \langle c\rangle\langle k\rangle$, then, by induction and Lemma 3.5, we can find $\overline{G}$ such that $\|\bar{g}\| = \|g\|$ and $\bar{a}_1 \notin \langle\bar{h}\rangle\bar{x}\langle\bar{c}\rangle$ or $\bar{b}_1 \notin \langle\bar{c}\rangle\langle\bar{k}\rangle$. Since $k \in B$, we have $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. So let $a_1 = h^i x c^\alpha$ and $b_1 = c^\beta k^j$ for some i, j, α, β. This implies $xc^{\alpha+\beta} \notin \langle h\rangle x\langle k\rangle$. Thus, by Lemma 3.6, we can find $\overline{G}$ such that $\overline{xc}^{\alpha+\beta} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. Hence $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$.

(d) $g = b_1 a_1$ or $\|g\| \geq 3$, where $a_1 \in A \setminus \langle c\rangle$ and $b_1 \in B \setminus \langle c\rangle$. We can find $\overline{G}$ such that $\|\bar{g}\| = \|g\|$. Then $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$.

Case 3. $h, k \in B$. In this case, since G is $\langle h\rangle\langle c^\epsilon k c^{-\epsilon}\rangle$-separable by Lemma 3.5, which is equivalent to $\langle h\rangle c^\epsilon\langle k\rangle$-separable, we may assume $x \in A \setminus \langle c\rangle$. Suppose $\langle h\rangle \cap \langle c\rangle = \langle h^m\rangle$, where $m > 0$. Let $h^m = c^s$. Then $h^{-i}g \notin \langle h^m\rangle x\langle k\rangle = \langle c^s\rangle x\langle k\rangle$ for all $0 \leq i < m$. Thus, by Case 2, we can find $\overline{G}$ such that $\overline{h^{-i}g} \notin \langle\overline{c^s}\rangle\bar{x}\langle\bar{k}\rangle$ for all $0 \leq i < m$. This means $\tilde{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. Therefore, we can assume $\langle h\rangle \cap \langle c\rangle = 1$ and similarly $\langle k\rangle \cap \langle c\rangle = 1$.

(a) $g \in A$. Suppose $g \notin \langle c\rangle x\langle c\rangle$. Then, by Case 1, we can find $\overline{G}$ such that $\bar{g} \notin \langle\bar{c}\rangle\bar{x}\langle\bar{c}\rangle$. This means $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. So let $g = c^{\alpha}xc^{\beta}$. Moreover, we can assume $\alpha \neq 0$ (or $\beta \neq 0$).

(i) $C_{\langle c\rangle}(x) = 1$. By Lemma 3.2, $\langle x^{-1}cx\rangle \cap \langle c\rangle = 1$. Since, by Case 1, G is $\langle c^{\epsilon}\rangle x\langle c^{\epsilon}\rangle$-separable, we can find $\overline{G}$ such that $\bar{x}^{-1}\bar{c}^{i_1}\bar{x} = \bar{c}^{i_2}$ only if $\epsilon = 2\alpha | i_1, i_2$; $\bar{x} \notin \langle\bar{c}\rangle$ and $\overline{c^{\alpha}} \notin \langle\bar{h}\rangle\langle\bar{c}^{2\alpha}\rangle$ by Lemmas 2.1 and 3.5. Since $h,k \in B$ and $\bar{x} \in \bar{A} \setminus \langle\bar{c}\rangle$, if $\bar{g} = \overline{c^{\alpha}xc^{\beta}} = \bar{h}^{i}\bar{x}\bar{k}^{j}$ for some i, j, then $\bar{h}^{i} = \bar{c}^{\delta_1}$ and $\bar{k}^{j} = \bar{c}^{\delta_2}$. Hence $\bar{c}^{\alpha}\overline{xc}^{\beta} = \bar{c}^{\delta_1}\overline{xc}^{\delta_2}$. This implies $\epsilon | \alpha - \delta_1, \beta - \delta_2$. Thus $\bar{c}^{\alpha-\delta_1} \in \langle\bar{c}^{\epsilon}\rangle$. This means $\bar{c}^{\alpha} = \langle\bar{h}\rangle\langle\bar{c}^{\epsilon}\rangle$, a contradiction. Therefore $\bar{g} = \overline{c^{\alpha}xc^{\beta}} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$.

(ii) $C_{\langle c\rangle}(x) = \langle c^{t}\rangle \neq 1$. If $c^{\alpha} \notin \langle h\rangle\langle c^{t}\rangle$ (or similarly $c^{\alpha} \notin \langle c^{t}\rangle\langle k\rangle$), then we can find $\overline{G}$ such that $\bar{x} \notin \langle\bar{c}\rangle$, $\bar{x}^{-1}\bar{c}^{i}\bar{x} \notin \langle\bar{c}\rangle$ for all $0 < i < t$, and $\overline{c^{\alpha}} \notin \langle\bar{h}\rangle\langle\bar{c}^{t}\rangle$ by Lemma 3.5. If $\bar{g} = \overline{c^{\alpha}xc^{\beta}} = \bar{h}^{i}\bar{x}\bar{k}^{j}$ for some i, j, then $\bar{h}^{i} = \bar{c}^{\delta_1}$ and $\bar{k}^{j} = \bar{c}^{\delta_2}$. This implies $t | \alpha - \delta_1$. Thus $\bar{c}^{\alpha} \in \langle\bar{h}\rangle\langle\bar{c}^{t}\rangle$, a contradiction. Hence $\bar{g} = \overline{c^{\alpha}xc^{\beta}} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. Therefore we can assume $c^{\alpha} \in \langle h\rangle\langle c^{t}\rangle \cap \langle c^{t}\rangle\langle k\rangle$. Since $\langle h\rangle \cap \langle c\rangle = 1$, $c^{\alpha} = c^{n_1 t}$. Similarly, $c^{\beta} = c^{n_2 t}$. This means $\alpha + \beta \neq 0$, since $c^{t} \in C_{\langle c\rangle}(x)$. Thus we can find $\overline{G}$ such that $\bar{x} \notin \langle\bar{c}\rangle$, $\bar{c}^{\alpha+\beta} \notin \langle\bar{c}^{\epsilon}\rangle$ for $\epsilon = 2(\alpha + \beta)$, $\bar{x}^{-1}\bar{c}^{i}\bar{x} \notin \langle\bar{c}\rangle$ for all $0 < i < t$, $\bar{h}^{i_1} = \bar{c}^{i_2}$ only if $\epsilon | i_1, i_2$, and $\bar{k}^{j_1} = \bar{c}^{j_2}$ only if $\epsilon | j_1, j_2$ by Lemmas 3.3 and 2.1. If $\bar{g} = \overline{c^{\alpha}xc^{\beta}} = \bar{h}^{i}\bar{x}\bar{k}^{j}$ for some i, j, then $\bar{h}^{i} = \bar{c}^{\delta_1}$ and $\bar{k}^{j} = \bar{c}^{\delta_2}$. Thus $\epsilon | \delta_1, \delta_2$. Since $t | \alpha - \delta_1$, this implies $\bar{c}^{\alpha+\beta} = \bar{c}^{\delta_1+\delta_2} \in \langle\bar{c}^{\epsilon}\rangle$, a contradiction. Therefore $\bar{g} = \overline{c^{\alpha}xc^{\beta}} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$.

(b) $g \in B$. Since $g \notin \langle h\rangle x\langle k\rangle$, we have $x \notin \langle h\rangle g\langle k\rangle$, where $g, h, k \in B$. Therefore, as in Case 1, we can find $\overline{G}$ such that $\bar{x} \notin \langle\bar{h}\rangle\bar{g}\langle\bar{k}\rangle$. Hence $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$.

(c) $g = b_1a_1$ or $g = a_1b_2$ or $g = b_1a_1b_2$, where $a_1 \in A \setminus \langle c\rangle$ and $b_i \in B \setminus \langle c\rangle$. We consider only the case $g = b_1a_1b_2$, other cases being similar. If $b_1 \notin \langle h\rangle\langle c\rangle$ or $b_2 \notin \langle c\rangle\langle k\rangle$, then, by Lemma 3.5, we can find $\overline{G}$ such that $\|\bar{g}\| = \|g\|$, $\bar{x} \notin \langle\bar{c}\rangle$, and $\bar{b}_1 \notin \langle\bar{h}\rangle\langle\bar{c}\rangle$ or $\bar{b}_2 \notin \langle\bar{c}\rangle\langle\bar{k}\rangle$. This implies $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. Therefore, we assume $b_1 = h^{i}c^{\alpha}$ and $b_2 = c^{\beta}k^{j}$. Thus $h^{-i}gk^{-j} = c^{\alpha}a_1c^{\beta} \notin \langle h\rangle x\langle k\rangle$. Hence, by (a), we can find $\overline{G}$ such that $\overline{c^{\alpha}xc^{\beta}} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. Therefore $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$.

(d) $g = a_1b_1a_2$ or $\|g\| \geq 4$, where $a_i \in A \setminus \langle c\rangle$ and $b_1 \in B \setminus \langle c\rangle$. We choose $\overline{G}$ such that $\|\bar{g}\| = \|g\|$. Then $\bar{g} \notin \langle\bar{h}\rangle\bar{x}\langle\bar{k}\rangle$. ∎

LEMMA 3.8. *Let G, A, B, c be as in Lemma* 3.6. *Let h, k be in vertices of G. For $a_i \in A \setminus \langle c\rangle$ and $b_i \in B \setminus \langle c\rangle$, we can find $M \lhd_f A$ and $L \lhd_f B$ with $M \cap \langle c\rangle = L \cap \langle c\rangle$ such that, in $\overline{G} = A/M *_{\langle\bar{c}\rangle} B/L$.*

(1) *if $a_1c^{\alpha}b_1 \cdots a_nb_n \notin \langle h\rangle a_1b_1 \cdots a_nb_n\langle k\rangle$, then $\overline{a_1c^{\alpha}b_1 \cdots a_nb_n} \notin \langle\bar{h}\rangle\overline{a_1b_1 \cdots a_nb_n}\langle\bar{k}\rangle$;*

(2) *if $a_1c^{\alpha}b_1 \cdots b_{n-1}a_n \notin \langle h\rangle a_1b_1 \cdots b_{n-1}a_n\langle k\rangle$, then $\overline{a_1c^{\alpha}b_1 \cdots b_{n-1}a_n} \notin \langle\bar{h}\rangle\overline{a_1b_1 \cdots b_{n-1}a_n}\langle\bar{k}\rangle$;*

(3) *if* $b_1c^{\alpha}a_1 \cdots b_na_n \notin \langle h\rangle b_1a_1 \cdots b_na_n\langle k\rangle$, *then* $\overline{b_1c^{\alpha}a_1 \cdots b_na_n} \notin \langle \bar{h}\rangle \overline{b_1a_1 \cdots b_na_n}\langle \bar{k}\rangle$;

(4) *if* $b_1c^{\alpha}a_1 \cdots a_{n-1}b_n \notin \langle h\rangle b_1a_1 \cdots a_{n-1}b_n\langle k\rangle$, *then* $\overline{b_1c^{\alpha}a_1 \cdots a_{n-1}b_n} \notin \langle \bar{h}\rangle \overline{b_1a_1 \cdots a_{n-1}b_n}\langle \bar{k}\rangle$.

Proof. We shall only prove (1), since the others are similar. Let $u = b_1a_2 \cdots b_{n-1}a_n$ and $C_{\langle c\rangle}(u) = \langle c^t\rangle$, where $t \geq 0$.

Case 1. $t = 0$. By Lemma 3.2, there exists b_r (or a_r) in u such that $\langle b_r^{-1}cb_r\rangle \cap \langle c\rangle = 1$.

Subcase 1. There exists a minimal integer $m > 0$ such that $a_1^{-1}h^ma_1 = c^s$. Thus $a_1^{-1}h^{i_1}a_1 \notin \langle c\rangle$ for $0 < i_1 < m$. Clearly $c^{\alpha} \notin \langle c^s\rangle$. Let $u = b_1a_2 \cdots b_{n-1}a_n$. Choose $M \lhd_f A$ and $L \lhd_f B$ with $M \cap \langle c\rangle = L \cap \langle c\rangle$ such that, in $\overline{G} = A/M *_{\langle \bar{c}\rangle} B/L$, $\|\overline{a_1ub_n}\| = \|a_1ub_n\|$, $\bar{c}^{\alpha} \notin \langle \bar{c}^s\rangle$, $\overline{a_1^{-1}h^{i_1}a_1} \notin \langle \bar{c}\rangle$ for $0 < i_1 < m$, and, moreover, by Lemma 3.7, since G is $\langle c^{\epsilon}\rangle b_r\langle c^{\epsilon}\rangle$-separable, we can also have $\bar{b}_r^{-1}\bar{c}^{i_2}\bar{b}_r = \bar{c}^{i_3}$ only if $\epsilon = s|i_2, i_3$ (by Lemma 2.1). If $\overline{a_1c^{\alpha}ub_n} = \bar{h}^i\overline{a_1ub_nk}^j$ for some i, j, then $\bar{a}_1 = \bar{h}^i\bar{a}_1\bar{c}^{\lambda_1}$, $\bar{c}^{\alpha}\bar{b}_1 = \bar{c}^{-\lambda_1}\bar{b}_1\bar{c}^{\mu_1}, \ldots, \bar{b}_r = \bar{c}^{-\lambda_r}\bar{b}_r\bar{c}^{\mu_r}, \ldots, \bar{b}_n = \bar{c}^{-\lambda_n}\bar{b}_n\bar{k}^j$ for some λ_j, μ_j. This implies $m|i$, whence $\bar{c}^{\lambda_1} \in \langle \bar{c}^s\rangle$. Moreover, $\epsilon = s|\lambda_r, \mu_r$. Thus $\epsilon|\mu_1, \alpha + \lambda_1$. It follows that $\bar{c}^{\alpha+\lambda_1} \in \langle \bar{c}^s\rangle$, which means $\bar{c}^{\alpha} \in \langle \bar{c}^s\rangle$, a contradiction. Therefore, $\overline{a_1c^{\alpha}ub_n} \notin \langle \bar{h}\rangle \overline{a_1ub_n}\langle \bar{k}\rangle$.

Subcase 2. $\langle a_1^{-1}ha_1\rangle \cap \langle c\rangle = 1$. Clearly $\alpha \neq 0$. Let $\epsilon = 2\alpha$. We can find $\overline{G}$ such that $\|\overline{a_1ub_n}\| = \|a_1ub_n\|$, $\bar{c}^{\alpha} \notin \langle \bar{c}^{\epsilon}\rangle$, and, moreover, by Lemma 3.7, since G is $\langle h^{\epsilon}\rangle a_1\langle c^{\epsilon}\rangle$-separable and $\langle c^{\epsilon}\rangle b_r\langle c^{\epsilon}\rangle$-separable, we can also have $\overline{a_1^{-1}h^{i_1}a_1} = \bar{c}^{i_2}$ only if $\epsilon|i_1, i_2$ and $\bar{b}_r^{-1}\bar{c}^{i_3}\bar{b}_r = \bar{c}^{i_4}$ only if $\epsilon|i_3, i_4$ (by Lemma 2.1). If $\overline{a_1c^{\alpha}ub_n} = \bar{h}^i\overline{a_1ub_nk}^j$ for some i, j, then $\bar{a}_1 = \bar{h}^i\bar{a}_1\bar{c}^{\lambda_1}$, $\bar{c}^{\alpha}\bar{b}_1 = \bar{c}^{-\lambda_1}\bar{b}_1\bar{c}^{\mu_1}, \ldots, \bar{b}_r = \bar{c}^{-\lambda_r}\bar{b}_r\bar{c}^{\mu_r}, \ldots, \bar{b}_n = \bar{c}^{-\lambda_n}\bar{b}_n\bar{k}^j$ for some λ_j, μ_j. This implies $\epsilon|i, \lambda_1$ and $\epsilon|\lambda_r, \mu_r$. It follows $\epsilon|\mu_1, \alpha + \lambda_1$. Hence $\epsilon = 2\alpha|\alpha$, a contradiction. Therefore, $\overline{a_1c^{\alpha}ub_n} \notin \langle \bar{h}\rangle \overline{a_1ub_n}\langle \bar{k}\rangle$.

Case 2. $C_{\langle c\rangle}(u) = \langle c^t\rangle \neq 1$.

Subcase 1. There exists a minimal integer $m > 0$ such that $a_1^{-1}h^ma_1 = c^s$.

(a) There exists a minimal integer $m' > 0$ such that $b_nk^{m'}b_n^{-1} = c^{s'}$. Thus $b_nk^{i_1}b_n^{-1} \notin \langle c\rangle$ for $0 < i_1 < m'$. Clearly $c^{\alpha} \notin \langle c^s\rangle(\langle c^t\rangle \cap \langle c^{s'}\rangle)$. Let $\langle c^t\rangle \cap \langle c^{s'}\rangle = \langle c^{dt}\rangle$. Choose $\overline{G}$ such that $\|\overline{a_1ub_n}\| = \|a_1ub_n\|$, $\bar{c}^{\alpha} \notin \langle \bar{c}^s\rangle\langle \bar{c}^{dt}\rangle$, $\bar{c}^{i_1t} \notin \langle \bar{c}^{s'}\rangle$ for $0 < i_1 < d$, $\overline{u^{-1}c^{i_2}u} \notin \langle \bar{c}\rangle$ for $0 < i_2 < t$, $\overline{a_1^{-1}h^{i_3}a_1} \notin \langle \bar{c}\rangle$ for $0 < i_3 < m$, and $\overline{b_nk^{i_4}b_n^{-1}} \notin \langle \bar{c}\rangle$ for $0 < i_4 < m'$. If $\overline{a_1c^{\alpha}ub_n} = \bar{h}^i\overline{a_1ub_nk}^j$ for some i, j, then $\bar{a}_1 = \bar{h}^i\bar{a}_1\bar{c}^{\lambda_1}$, $\bar{c}^{\alpha}\bar{u} = \bar{c}^{-\lambda_1}\overline{uc}^{\lambda_n}$, and $\bar{b}_n = \bar{c}^{-\lambda_n}\bar{b}_n\bar{k}^j$ for some λ_1, λ_n. Hence we have $m|i$, $t|\alpha + \lambda_1$, and $m'|j$. This implies $\bar{c}^{\lambda_1} \in \langle \bar{c}^s\rangle$, $\bar{c}^{\alpha+\lambda_1} = \bar{c}^{\lambda_n} \in \langle \bar{c}^t\rangle$, and $\bar{c}^{\lambda_n} \in \langle \bar{c}^{s'}\rangle$. Thus $\bar{c}^{\alpha+\lambda_1} \in \langle \bar{c}^t\rangle \cap \langle \bar{c}^{s'}\rangle = \langle \bar{c}^{dt}\rangle$. This means $\bar{c}^{\alpha} \in \langle \bar{c}^s\rangle\langle \bar{c}^{dt}\rangle$, a contradiction. Therefore, $\overline{a_1c^{\alpha}ub_n} \notin \langle \bar{h}\rangle \overline{a_1ub_n}\langle \bar{k}\rangle$.

(b) $\langle b_n k b_n^{-1}\rangle \cap \langle c\rangle = 1$. Then $c^{\alpha} \notin \langle c^s\rangle$. Choose $\overline{G}$ such that $\|\overline{a_1 u b_n}\| = \|a_1 u b_n\|$, $\bar{c}^{\alpha} \notin \langle \bar{c}^s\rangle$, $\overline{u^{-1}c^{i_1}u} \notin \langle \bar{c}\rangle$ for $0 < i_1 < t$, $\overline{a_1^{-1}h^{i_2}a_1} \notin \langle \bar{c}\rangle$ for $0 < i_2 < m$, and $\overline{b_n k^{i_3} b_n^{-1}} = \bar{c}^{i_4}$ only if $\epsilon = s|i_3, i_4$ (by Lemmas 3.7 and 2.1). Applying similar argument in (a), we get $\overline{a_1 c^{\alpha} u b_n} \notin \langle \bar{h}\rangle \overline{a_1 u b_n}\langle \bar{k}\rangle$.

Subcase 2. $\langle a_1^{-1} h a_1\rangle \cap \langle c\rangle = 1$.

(a) There exists a minimal integer $m' > 0$ such that $b_n k^{m'} b_n^{-1} = c^{s'}$. Thus $b_n k^{i_1} b_n^{-1} \notin \langle c\rangle$ for $0 < i_1 < m'$. Clearly $c^{\alpha} \notin \langle c^t\rangle \cap \langle c^{s'}\rangle$. Let $\langle c^t\rangle \cap \langle c^{s'}\rangle = \langle c^{dt}\rangle$. Choose $\overline{G}$ such that $\|\overline{a_1 u b_n}\| = \|a_1 u b_n\|$, $\bar{c}^{\alpha} \notin \langle \bar{c}^{dt}\rangle$, $\bar{c}^{i_1 t} \notin \langle \bar{c}^{s'}\rangle$ for $0 < i_1 < d$, $\overline{u^{-1}c^{i_2}u} \notin \langle \bar{c}\rangle$ for $0 < i_2 < t$, $\overline{a_1^{-1}h^{i_3}a_1} = \bar{c}^{i_4}$ only if $\epsilon = dt|i_3, i_4$, and $\overline{b_n k^{i_5} b_n^{-1}} \notin \langle \bar{c}\rangle$ for $0 < i_5 < m'$. Then, as before, we get $\overline{a_1 c^{\alpha} u b_n} \notin \langle \bar{h}\rangle \overline{a_1 u b_n}\langle \bar{k}\rangle$.

(b) $\langle b_n k b_n^{-1}\rangle \cap \langle c\rangle = 1$. Clearly $c^{\alpha} \notin \langle c^{2\alpha}\rangle$. Choose $\overline{G}$ such that $\|\overline{a_1 u b_n}\| = \|a_1 u b_n\|$, $\bar{c}^{\alpha} \notin \langle \bar{c}^{2\alpha}\rangle$, $\overline{u^{-1}c^{i_1}u} \notin \langle \bar{c}\rangle$ for $0 < i_1 < t$, $\overline{a_1^{-1}h^{i_2}a_1} = \bar{c}^{i_3}$ only if $\epsilon = 2\alpha|i_2, i_3$, and $\overline{b_n k^{i_4} b_n^{-1}} = \bar{c}^{i_5}$ only if $\epsilon = 2\alpha|i_4, i_5$. Again, as before, we get $\overline{a_1 c^{\alpha} u b_n} \notin \langle \bar{h}\rangle \overline{a_1 u b_n}\langle \bar{k}\rangle$. ∎

THEOREM 3.9. *Let G be a tree product of a finite number of* R$\mathfrak{N}$*-groups amalgamating cyclic subgroups. Then G is $\langle h\rangle x\langle k\rangle$-separable for $x \in G$, where h, k are in the vertices of G.*

Proof. If G has only one vertex, then the result follows from Lemma 3.4. So suppose G has r vertices. Then $G = A *_{\langle c\rangle} B$, where A is a subtree product of G consisting of $r - 1$ vertices with B as the remaining vertex and c in a vertex of A. By induction, we can assume that A is $\langle u\rangle a\langle v\rangle$-separable for $a \in A$ and u, v in the vertices of A. We shall prove the theorem by induction on $\|x\|$. By Lemma 3.7, G is $\langle h\rangle x\langle k\rangle$-separable for $x \in A \cup B$. Thus we assume that G is $\langle h\rangle y\langle k\rangle$-separable for any $y \in G$ with $\|y\| < \|x\|$. Let $g \notin \langle h\rangle x\langle k\rangle$, where $g, x \in G$ and h, k are in the vertices of G. We shall find $M \triangleleft_f A$ and $L \triangleleft_f B$ with $M \cap \langle c\rangle = L \cap \langle c\rangle$ such that, in $\overline{G} = A/M *_{\langle \bar{c}\rangle} B/L$, $\|\bar{x}\| = \|x\|$, $\|\bar{g}\| = \|g\|$, and $\bar{g} \notin \langle \bar{h}\rangle \bar{x}\langle \bar{k}\rangle$. Then, since $\overline{G}$ is residually finite and $|\bar{h}|, |\bar{k}| < \infty$, we can easily find $N \triangleleft_f G$ such that $\tilde{g} \notin \langle \tilde{h}\rangle \tilde{x}\langle \tilde{k}\rangle$, where $\tilde{G} = G/N$. Throughout the following, we let $a_i, c_j \in A \setminus \langle c\rangle$ and $b_i, d_j \in B \setminus \langle c\rangle$. Moreover, we shall only consider the case $x = a_1 b_1 \cdots a_n b_n$, other cases being similar.

If $\|g\| > \|x\| + 2$, then we can find $\overline{G}$ such that $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. Then $\bar{g} \notin \langle \bar{h}\rangle \bar{x}\langle \bar{k}\rangle$. If $\|g\| < \|x\|$, then, by induction, we can find $\overline{G}$ such that $\bar{x} \notin \langle \bar{h}\rangle \bar{g}\langle \bar{k}\rangle$. This implies $\bar{g} \notin \langle \bar{h}\rangle \bar{x}\langle \bar{k}\rangle$. Thus we need only consider $g \in G$ with $\|x\| \le \|g\| \le \|x\| + 2$.

Case 1. $h, k \in A$.

(1) $\|g\| = \|x\|$. Suppose $g = d_1 c_1 \cdots d_n c_n$. If $a_1 \notin \langle h\rangle\langle c\rangle$, then, by Lemma 3.5, we can find $\overline{G} = A/M *_{\langle \bar{c}\rangle} B/L$ such that $\|\bar{g}\| = \|g\|$, $\|\bar{x}\| = \|x\|$, and $\bar{a}_1 \notin \langle \bar{h}\rangle\langle \bar{c}\rangle$. Thus $\bar{g} \notin \langle \bar{h}\rangle \bar{x}\langle \bar{k}\rangle$. So let $a_1 = h^i c^{\alpha}$. This implies

$g \notin \langle h\rangle c^\alpha b_1 \cdots a_n b_n \langle k\rangle$. Hence, by induction, there exists $\overline{G}$ such that $\bar g \notin \langle \bar h\rangle \overline{c^\alpha b_1 \cdots a_n b_n}\langle \bar k\rangle$. It follows that $\bar g \notin \langle \bar h\rangle \bar x \langle \bar k\rangle$.

So let $g = c_1 d_1 \cdots c_n d_n$. If $c_1 \notin \langle h\rangle a_1 \langle c\rangle$ or $d_1 c_2 \cdots c_n d_n \notin \langle c\rangle b_1 a_2 \cdots a_n b_n \langle k\rangle$, then, by induction, we can find $\overline{G}$ such that $\|\bar g\| = \|g\|$, $\|\bar x\| = \|x\|$, and $\bar c_1 \notin \langle \bar h\rangle \bar a_1 \langle \bar c\rangle$ or $\overline{d_1 c_2 \cdots c_n d_n} \notin \langle \bar c\rangle \overline{b_1 a_2 \cdots a_n b_n}\langle \bar k\rangle$. This implies $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$. Therefore, let $c_1 = h^i a_1 c^\alpha$ and $d_1 c_2 \cdots c_n d_n = c^\beta b_1 a_2 \cdots a_n b_n k^j$. Then $a_1 c^{\alpha+\beta} b_1 \cdots a_n b_n \notin \langle h\rangle a_1 b_1 \cdots a_n b_n \langle k\rangle$. Hence, by Lemma 3.8, we can find $\overline{G}$ such that $\overline{a_1 c^{\alpha+\beta} b_1 \cdots a_n b_n} \notin \langle \bar h\rangle \overline{a_1 b_1 \cdots a_n b_n}\langle \bar k\rangle$. It follows that $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$.

(2) $\|g\| = \|x\| + 1$. Suppose $g = d_1 c_1 \cdots d_n c_n d_{n+1}$. Then we can find $\overline{G}$ such that $\|\bar g\| = \|g\|$ and $\|\bar x\| = \|x\|$. This implies $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$. So let $g = c_1 d_1 \cdots c_n d_n c_{n+1}$. If $c_{n+1} \notin \langle c\rangle\langle k\rangle$, then we can choose $\overline{G}$ such that $\|\bar g\| = \|g\|$, $\|\bar x\| = \|x\|$, and $\bar c_{n+1} \notin \langle \bar c\rangle\langle \bar k\rangle$. Thus $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$. Therefore, we let $c_{n+1} = c^\alpha k^j$. This means $gk^{-j} = c_1 d_1 \cdots c_n d_n c^\alpha \notin \langle h\rangle x\langle k\rangle$. Hence, by (1), we can find $\overline{G}$ such that $\overline{gk^{-j}} \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$. Thus $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$.

(3) $\|g\| = \|x\| + 2$. We choose $\overline{G}$ such that $\|\bar g\| = \|g\|$ and $\|\bar x\| = \|x\|$. Then $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$.

Case 2. $h \in A$ and $k \in B$.

(1) $\|g\| > \|x\|$. We choose $\overline{G}$ such that $\|\bar g\| = \|g\|$ and $\|\bar x\| = \|x\|$. Then $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$.

(2) $\|g\| = \|x\|$. Suppose $g = d_1 c_1 \cdots d_n c_n$. Then we can find $\overline{G}$ such that $\|\bar g\| = \|g\|$ and $\|\bar x\| = \|x\|$. This implies $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$. So let $g = c_1 d_1 \cdots c_n d_n$. Then we can apply the method of the second part of Case 1(1).

Case 3. $h, k \in B$.

(1) $\|g\| = \|x\|$. Suppose $g = d_1 c_1 \cdots d_n c_n$. If $b_n \notin \langle c\rangle\langle k\rangle$, then, by Lemma 3.5, we can find $\overline{G}$ such that $\|\bar g\| = \|g\|$, $\|\bar x\| = \|x\|$, and $\bar b_n \notin \langle \bar c\rangle\langle \bar k\rangle$. This implies $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$. So suppose $b_n = c^\alpha k^j$. Then $g \notin \langle h\rangle a_1 b_1 \cdots a_n c^\alpha \langle k\rangle$. Hence, by induction, we can find $\overline{G}$ such that $\bar g \notin \langle \bar h\rangle \overline{a_1 b_1 \cdots a_n c^\alpha}\langle \bar k\rangle$. It follows that $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$. So let $g = c_1 d_1 \cdots c_n d_n$. Then we can apply the method of the second part of Case 1(1).

(2) $\|g\| = \|x\| + 1$ or $\|g\| = \|x\| + 2$. Suppose $g = d_1 c_1 \cdots d_n c_n d_{n+1}$. If $d_1 c_1 \notin \langle h\rangle a_1 \langle c\rangle$ or $d_2 c_2 \cdots c_n d_{n+1} \notin \langle c\rangle b_1 a_2 \cdots a_n b_n\langle k\rangle$, then, by induction, we can find $\overline{G}$ such that $\|\bar g\| = \|g\|$, $\|\bar x\| = \|x\|$, and $\overline{d_1 c_1} \notin \langle \bar h\rangle \bar a_1\langle \bar c\rangle$ or $\overline{d_2 c_2 \cdots c_n d_{n+1}} \notin \langle \bar c\rangle \overline{b_1 a_2 \cdots a_n b_n}\langle \bar k\rangle$. This implies $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$. So let $d_1 c_1 = h^i a_1 c^\alpha$ and $d_2 c_2 \cdots c_n d_{n+1} = c^\beta b_1 a_2 \cdots a_n b_n k^j$. Then $a_1 c^{\alpha+\beta} b_1 \cdots a_n b_n \notin \langle h\rangle a_1 b_1 \cdots a_n b_n\langle k\rangle$. Hence, by Lemma 3.8, we can find $\overline{G}$ such that $\overline{a_1 c^{\alpha+\beta} b_1 \cdots a_n b_n} \notin \langle \bar h\rangle \overline{a_1 b_1 \cdots a_n b_n}\langle \bar k\rangle$. It follows that $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$.

Now suppose $g = c_1 d_1 \cdots c_n d_n c_{n+1}$ or $\|g\| = \|x\| + 2$. Then we can choose $\overline{G}$ such that $\|\bar g\| = \|g\|$ and $\|\bar x\| = \|x\|$. This implies $\bar g \notin \langle \bar h\rangle \bar x\langle \bar k\rangle$.

Case 4. $h \in B$ and $k \in A$.

(1) $\|g\| = \|x\|$. Suppose $g = d_1 c_1 \cdots d_n c_n$. Then we can choose $\overline{G}$ such that $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. This implies $\bar{g} \notin \langle \bar{h} \rangle \bar{x} \langle \bar{k} \rangle$. So let $g = c_1 d_1 \cdots c_n d_n$. Then we can apply the method of the second part of Case 1(1).

(2) $\|g\| = \|x\| + 1$. Suppose $g = c_1 d_1 \cdots c_n d_n c_{n+1}$. Again we can apply the method of the second part of Case 1(1). So let $g = d_1 c_1 \cdots d_n c_n d_{n+1}$. In this case, we can apply the method of Case 3(2).

(3) $\|g\| = \|x\| + 2$. Suppose $g = c_1 d_1 \cdots d_n c_{n+1} d_{n+1}$. Then we can choose $\overline{G}$ such that $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. This implies $\bar{g} \notin \langle \bar{h} \rangle \bar{x} \langle \bar{k} \rangle$. So let $g = d_1 c_1 \cdots c_n d_{n+1} c_{n+1}$. Again we can apply the method of Case 3(2). ∎

4. MAIN RESULTS

To prove our main result we need the following lemmas.

LEMMA 4.1. *Let G be a tree product of a finite number of* R$\mathfrak{N}$*-subgroups amalgamating cyclic subgroups. If h is in a vertex group of G, then, for each n, there exists $N \triangleleft_f G$ such that* (1) $|\tilde{h}| = n$ *and* (2) $\tilde{h}^i \nsim_{\tilde{G}} \tilde{h}^j$ *for* $\tilde{h}^i \neq \tilde{h}^j$, *where* $\tilde{G} = G/N$.

Proof. Let G be a tree product of r R$\mathfrak{N}$-groups. We shall prove the lemma by induction on r. By Corollary 2.2 in [23], every R$\mathfrak{N}$-group has properties (1) and (2). Now let $G = A *_{\langle c \rangle} B$, where $B \in$ R$\mathfrak{N}$, A is a subtree product of G with $r-1$ vertices, and c is in a vertex of A. By induction, A has the properties (1) and (2). We shall find $M \triangleleft_f A$ and $L \triangleleft_f B$ such that $M \cap \langle c \rangle = L \cap \langle c \rangle$ and $|\bar{h}| = n$ and $\bar{h}^i \nsim_{\overline{G}} \bar{h}^j$ for $\bar{h}^i \neq \bar{h}^j$ in $\overline{G} = A/M *_{\langle \bar{c} \rangle} B/L$. Since $\overline{G}$ is conjugacy separable by Theorem 1.2, we can find $\overline{N} \triangleleft_f \overline{G}$ such that $\overline{N} \cap \langle \bar{h} \rangle = 1$ and $\overline{N\bar{h}}^i \nsim_{\overline{G}/\overline{N}} \overline{N\bar{h}}^j$ for $\bar{h}^i \neq \bar{h}^j$. Let N be the preimage of $\overline{N}$ in G. Then $N \triangleleft_f G$ and $|\tilde{h}| = n$ and $\tilde{h}^i \nsim_{\tilde{G}} \tilde{h}^j$ for $\tilde{h}^i \neq \tilde{h}^j$, where $\tilde{G} = G/N$.

Suppose $h \in A$ (the case $h \in B$ is similar). Since h is in a vertex of A, by induction there exists $M \triangleleft_f A$ such that $|\bar{h}| = n$ and $\bar{h}^i \nsim_{\overline{A}} \bar{h}^j$ for $\bar{h}^i \neq \bar{h}^j$, where $\overline{A} = A/M$. Let $M \cap \langle c \rangle = \langle c^s \rangle$. Now $B \in$ R$\mathfrak{N}$ implies that there exists $N \triangleleft_f B$ such that $|\bar{c}| = s$ and $\bar{c}^i \nsim_{\overline{B}} \bar{c}^j$ for $\bar{c}^i \neq \bar{c}^j$, where $\overline{B} = B/L$. Therefore, in $\overline{G} = A/M *_{\langle \bar{c} \rangle} B/L$ and $\bar{h}^i \nsim_{\overline{G}} \bar{h}^j$ for $\bar{h}^i \neq \bar{h}^j$ as in the proof of Lemma 3.2. ∎

LEMMA 4.2. *Let G be a tree product of a finite number of conjugacy separable* R$\mathfrak{N}$*-groups amalgamating cyclic subgroups. Let h be in a vertex group of G. If there is $x \in G$ such that $\{x\}^G \cap \langle h \rangle = \emptyset$, then there exists $N \triangleleft_f G$ such that $\{\tilde{x}\}^{\tilde{G}} \cap \langle \tilde{h} \rangle = \emptyset$, where $\tilde{G} = G/N$.*

Proof. By Theorem 3.1 in [14], every conjugacy separable $R\mathfrak{N}$-group is cyclic conjugacy separable. So let $G = A *_{\langle c\rangle} B$ as in the proof of Lemma 4.1. Thus, by induction, A is cyclic conjugacy separable for $\langle u\rangle$, where u is in any vertex of A. Let h be in a vertex group of G and $x \in G$ such that $\{x\}^G \cap \langle h\rangle = \emptyset$. We may assume that x has the minimal length in its conjugacy class in G. We shall find $M \triangleleft_f A$ and $L \triangleleft_f B$ such that $M \cap \langle c\rangle = L \cap \langle c\rangle$ and $\{\bar{x}\}^{\overline{G}} \cap \langle \bar{h}\rangle = \emptyset$, where $\overline{G} = A/M *_{\langle \bar{c}\rangle} B/L$. Since $\overline{G}$ is conjugacy separable and $\langle \bar{h}\rangle$ is finite, there exists $\overline{N} \triangleleft_f \overline{G}$ such that $\{\overline{N}\bar{x}\}^{\overline{G}/\overline{N}} \cap \langle \overline{N}\bar{h}\rangle = \emptyset$. Let N be the preimage of $\overline{N}$ in G. Then $N \triangleleft_f G$ and $\{\tilde{x}\}^{\tilde{G}} \cap \langle \tilde{h}\rangle = \emptyset$ in $\tilde{G} = G/N$. Thus, without loss of generality, we assume h is in a vertex of A.

(1) $\|x\| \geq 2$. Since x is of minimal length in its conjugacy class in G, x is cyclically reduced. By Lemmas 3.1 and 3.3, A, B are $\langle c\rangle$-potent and $\langle c\rangle$-separable. Therefore, there exist $M \triangleleft_f A$ and $L \triangleleft_f B$ such that $M \cap \langle c\rangle = L \cap \langle c\rangle$ and $\|\bar{x}\| = \|x\|$, where $\overline{G} = A/M *_{\langle \bar{c}\rangle} B/L$. Since $\bar{h} \in \overline{A}$ and $\bar{x}$ is cyclically reduced with length ≥ 2 in its conjugacy class in $\overline{G}$, by Theorem 1.1, $\{\bar{x}\}^{\overline{G}} \cap \langle \bar{h}\rangle = \emptyset$.

(2) $x \in A \setminus \langle c\rangle$. Clearly $\{x\}^A \cap \langle c\rangle = \emptyset$ and $\{x\}^A \cap \langle h\rangle = \emptyset$. Since c, h are in vertices of A, by induction, there exists $M \triangleleft_f A$ such that $\{\bar{x}\}^{\overline{A}} \cap \langle \bar{c}\rangle = \emptyset$ and $\{\bar{x}\}^{\overline{A}} \cap \langle \bar{h}\rangle = \emptyset$, where $\overline{A} = A/M$. Since B is $\langle c\rangle$-potent, there exists $L \triangleleft_f B$ such that $M \cap \langle c\rangle = L \cap \langle c\rangle$. Thus, in $\overline{G} = A/M *_{\langle \bar{c}\rangle} B/L$, $\bar{x}$ is of minimal length 1 in its conjugacy class in $\overline{G}$. By Theorem 1.1, $\bar{x} \nsim_{\overline{G}} \bar{h}^i$ for all i. Hence $\{\bar{x}\}^{\overline{G}} \cap \langle \bar{h}\rangle = \emptyset$.

(3) $x \in B \setminus \langle c\rangle$. Clearly $\{x\}^B \cap \langle c\rangle = \emptyset$. Since B is cyclic conjugacy separable by [14, Theorem 3.1], there exists $L \triangleleft_f B$ such that $\{\bar{x}\}^{\overline{B}} \cap \langle \bar{c}\rangle = \emptyset$, where $\overline{B} = B/L$. By Lemma 3.1, A is $\langle c\rangle$-potent. So there exists $M \triangleleft_f A$ such that $M \cap \langle c\rangle = L \cap \langle c\rangle$. Thus, in $\overline{G} = A/M *_{\langle \bar{c}\rangle} B/L$, $\bar{x} \in \overline{B}$ is of minimal length 1 in its conjugacy class in $\overline{G}$. Since $\bar{h} \in \overline{A}$, by Theorem 1.1, $\bar{x} \nsim_{\overline{G}} \bar{h}^i$ for all i. Hence $\{\bar{x}\}^{\overline{G}} \cap \langle \bar{h}\rangle = \emptyset$.

(4) $x \in \langle c\rangle$. Clearly $\{x\}^A \cap \langle h\rangle = \emptyset$. By induction, since h is in a vertex of A and $x \in \langle c\rangle$, there exists $M \triangleleft_f A$ such that $\{\bar{x}\}^{\overline{A}} \cap \langle \bar{h}\rangle = \emptyset$, where $\overline{A} = A/M$. By Lemma 4.1, there exists $L \triangleleft_f B$ such that $M \cap \langle c\rangle = L \cap \langle c\rangle$ and $\bar{c}^i \nsim_{\overline{B}} \bar{c}^j$ for $\bar{c}^i \neq \bar{c}^j$, where $\overline{B} = B/L$. Thus, in $\overline{G} = A/M *_{\langle \bar{c}\rangle} B/L$, $\bar{x} \nsim_{\overline{G}} \bar{h}^i$ for all i by Theorem 1.1. Hence $\{\bar{x}\}^{\overline{G}} \cap \langle \bar{h}\rangle = \emptyset$. ∎

THEOREM 4.3. *Let G be a tree product of a finite number of conjugacy separable* $R\mathfrak{N}$-*groups amalgamating cyclic subgroups. Then G is conjugacy separable.*

Proof. Let G be a tree product of r $R\mathfrak{N}$-groups. Then $G = A *_{\langle h\rangle} B$, where A is a subtree product of $r-1$ vertices of G, B is a conjugacy separable $R\mathfrak{N}$-group, and h is in a vertex of A. By induction, we assume that A is conjugacy separable. Then A, B are conjugacy separable, cyclic conjugacy separable for $\langle h \rangle$ (Lemma 4.2), and $\langle h \rangle$-potent (Lemma 3.1). Moreover, by Lemmas 3.2, 3.9, and 4.1, A, B satisfy C1, C2, and C3 of Theorem 2.3. Hence, by Theorem 2.3, G is conjugacy separable. ∎

Applying Theorem 4.3, we immediately have the following:

THEOREM 4.4. *Let G be a tree product of a finite number of finitely generated torsion-free nilpotent or free or surface groups, amalgamating cyclic subgroups. Then G is conjugacy separable.*

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