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On Generalized Free Products of Residually Finite p-Groups

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We derive necessary and sufficient conditions for generalized free products of free groups or finitely generated torsion-free nilpotent groups, amalgamating a cycle, to be residually finite p-groups. Using this, we characterize the residually finite p-group property of tree products of finitely generated torsion-free nilpotent groups amalgamating cycles.

Key Words: generalized free products; free or nilpotent groups; residually finite p-groups

1. INTRODUCTION

A group G is a *residually finite p-group* ($\mathscr{RF}p$) if, for each $1 \neq g \in G$, there exists a normal subgroup N of p-power index ($N \triangleleft_p G$) in G such that $g \notin N$.

Free groups [6] and finitely generated torsion-free nilpotent groups [4] are $\mathscr{RF}p$ for all primes p. In [5], Higman considered generalized free products of two finite p-groups. He showed that if the amalgamated subgroup is cyclic, then these groups are $\mathscr{RF}p$. However, if the amalgamated subgroup is not cyclic, then the generalized free products may not be $\mathscr{RF}p$ [7, Example 2.5]. Higman's result was generalized by Kim and McCarron [7], by proving that generalized free products of $\mathscr{RF}p$ groups,

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amalgamating finite cyclic subgroups, are $\mathcal{RF}p$. They showed that amalgamated free powers of $\mathcal{RF}p$ groups are $\mathcal{RF}p$ if and only if the amalgamated subgroups are p-closed in the factor groups. They also showed that certain generalized free products of free or finitely generated torsion-free nilpotent groups, amalgamating a cyclic subgroup, are $\mathcal{RF}p$. In [8], the residually finite p-group property of one-relator groups of the form $\langle a, b: a^{-\alpha} b^{\beta} a^{\alpha} b^{\delta} \rangle$ was characterized. It was shown that $\langle a \rangle \underset{a^n = b^m}{*} \langle b \rangle$ is $\mathcal{RF}p$ if and only if $|n| = 1$ or $|m| = 1$ or both $|m|$ and $|n|$ are p-powers. In this paper, we completely characterize the residually finite p-group property of generalized free products of free groups or finitely generated torsion-free nilpotent groups amalgamating a cycle. We also consider similar properties for tree products of finitely generated torsion-free nilpotent groups, amalgamating cycles. Recently, Doniz [3] showed that generalized free products of infinitely many finitely generated torsion-free nilpotent groups, amalgamating a single isolated cycle, are $\mathcal{RF}p$ if the factors are bounded in their ranks and their nilpotent classes. Our results also extend this result.

We shall use the following notations. We write that G is $\mathcal{RF}p$ if G is a residually finite p-group. We use $N \triangleleft_p G$ to denote that N is a normal subgroup of p-power index in G. Throughout this paper, p, q are always primes. We use $A *_H B$ (or $A \underset{x=y}{*} B$) to denote the generalized free product of A and B amalgamating H (or $\langle x \rangle$ and $\langle y \rangle$). If $G = A *_H B$, then $\|g\|$ denotes the free product length of g in G. We use $|g|$ to denote the order of g in a group G, and $|n|$ to denote the absolute value of the integer n.

2. PRELIMINARIES

We first note that any element of finite order in a $\mathcal{RF}p$ group has p-power order, and generalized free products of $\mathcal{RF}p$ groups, amalgamating finite cyclic subgroups, are always $\mathcal{RF}p$ by [7, Theorem 4.3]. Hence, throughout this paper, we shall consider only generalized free products amalgamating infinite cyclic subgroups. We make use of the following results.

LEMMA 2.1 [7, Corollary 2.3]. *Let G be $\mathcal{RF}p$ and let $c \in G$ be of infinite order. Then, for each positive integer k, there exists $N_k \triangleleft_p G$ such that $N_k \cap \langle c \rangle = \langle c^{p^k} \rangle$.*

DEFINITION 2.2. Let H be a subgroup of G. We say that H is *p-closed* in G if, for each $g \in G \setminus H$, there exists $N \triangleleft_p G$ such that $g \notin NH$.

THEOREM 2.3 [7, Theorem 4.2]. *Let A, B be $\mathcal{RF}p$ and let $\langle c \rangle$ be p-closed in both A and B. Then $A *_{\langle c \rangle} B$ is $\mathcal{RF}p$.*

Let A be $\mathcal{RF}p$. Corollary 3.5 in [7] shows that $A *_H A$ is $\mathcal{RF}p$ if and only if H is p-closed in A. Thus p-closedness plays an important role in the study of the residually finite p-group property of generalized free products.

LEMMA 2.4. *Let A be $\mathcal{RF}p$. If $\langle c \rangle$ is p-closed in A, then $\langle c^{p^\alpha} \rangle$ is p-closed in A for any $\alpha \geq 0$.*

Proof. Let $g \in G \setminus \langle c^{p^\alpha} \rangle$. If $g \notin \langle c \rangle$, then, by assumption, there exists $N \lhd_p A$ such that $g \notin N\langle c \rangle$, whence $g \notin N\langle c^{p^\alpha} \rangle$. So let $g = c^i \notin \langle c^{p^\alpha} \rangle$. By Lemma 2.1, there exists $N \lhd_p A$ such that $N \cap \langle c \rangle = \langle c^{p^\alpha} \rangle$. Thus $g = c^i \notin N = N\langle c^{p^\alpha} \rangle$. Hence $\langle c^{p^\alpha} \rangle$ is p-closed in A. ∎

COROLLARY 2.5. *Let A be finitely generated torsion-free nilpotent and let $\langle a \rangle$ be a maximal cycle in A. Then $\langle a^{p^\alpha} \rangle$ is p-closed in A for each $\alpha \geq 0$.*

Proof. Since $\langle a \rangle$ is a maximal cycle in A, $\langle a \rangle$ is isolated. Therefore, by Theorem 2.5 [2], $\bigcap_{k=1}^{\infty} A^{p^k}\langle a \rangle = \langle a \rangle$ for any p. This means $\langle a \rangle$ is p-closed in A. It follows from Lemma 2.4 that $\langle a^{p^\alpha} \rangle$ is p-closed in A for each $\alpha \geq 0$. ∎

This result was also proved by Doniz [3, Lemma 4], based on the complicated proof of [7, Lemma 4.5]. But the above proof is simpler and more useful. This method also allows us to prove a similar result for a free group (Lemma 3.1). The following results are important to our study.

THEOREM 2.6 [1]. *Let F be a free group and let $1 \neq d \in F$ such that d generates its own centralizer. Then,*

(1) *$G(n) = \langle A, c : d = c^n \rangle$ is residually p-finite for p not dividing n;*

(2) *$G(n)$ is residually torsion-free nilpotent.*

THEOREM 2.7 [8]. *Let $G = \langle a \rangle \underset{a^n = b^m}{*} \langle b \rangle$. Then G is $\mathcal{RF}p$ if and only if either $|n| = 1$ or $|m| = 1$ or $|n| = p^\alpha$ and $|m| = p^\beta$ for $\alpha, \beta > 0$.*

3. ON GENERALIZED FREE PRODUCTS OF FREE GROUPS

Although both free groups and finitely generated torsion-free nilpotent groups are $\mathcal{RF}p$ for all p, their generalized free products behave somewhat differently. In fact, generalized free products of free groups behave slightly better than those of torsion-free nilpotent groups. We shall deal with the free groups first.

LEMMA 3.1. *Let $\langle c\rangle$ be a maximal cycle in a free group A. Then $\langle c^{p^\alpha}\rangle$ is p-closed in A for $\alpha \geq 0$.*

Proof. First, consider $\alpha = 0$. Let $a \in A \setminus \langle c\rangle$. Since $\langle c\rangle$ is maximal, it has its own centralizer. Thus $[a, c] \neq 1$. Since A is $\mathcal{RF}p$, there exists $N \triangleleft_p A$ such that $[a, c] \notin N$. This implies $a \notin N\langle c\rangle$. Hence $\langle c\rangle$ is p-closed in A. It follows from Lemma 2.4 that $\langle c^{p^\alpha}\rangle$ is p-closed in A for $\alpha \geq 0$. ∎

LEMMA 3.2. *Let A be free and $\langle a\rangle$ be a maximal cycle in A. If $C = A \underset{a=b^n}{*} \langle b\rangle$ is $\mathcal{RF}p$, then $\langle b\rangle$ is p-closed in C.*

Proof. If $|n| = 1$, the result is clear. Hence we assume that $|n| > 1$. Let $g \in C \setminus \langle b\rangle$. First, suppose $g \in A$. Then $g \notin \langle a\rangle$. Thus $[g, b] \neq 1$. Since C is $\mathcal{RF}p$, there exists $N \triangleleft_p C$ such that $[g, b] \notin N$, whence $g \notin N\langle b\rangle$. Next, suppose $g \notin A$. Without loss of generality, we can assume $g = a_1 b^{i_1} \cdots b^{i_{k-1}} a_k$, where $a_j \in A \setminus \langle a\rangle$ and $b^{i_j} \notin \langle b^n\rangle$. Thus $[g, b] \neq 1$. As before, there exists $N \triangleleft_p C$ such that $[g, b] \notin N$, whence $g \notin N\langle b\rangle$. This proves that $\langle b\rangle$ is p-closed in C. ∎

THEOREM 3.3. *Let A be free and $\langle a\rangle$ be a maximal cycle in A. Then $G(n) = A \underset{a=b^n}{*} \langle b\rangle$ is $\mathcal{RF}p$ for all primes p.*

Proof. Let $n = p^\alpha m$, where $(p, m) = 1$. Then $G(n) = (A \underset{a=c^m}{*} \langle c\rangle) \underset{c=b^{p^\alpha}}{*} \langle b\rangle$. Let $B = A \underset{a=c^m}{*} \langle c\rangle$. Since $(p, m) = 1$, by Theorem 2.6, B is $\mathcal{RF}p$. By Lemma 3.2, $\langle c\rangle$ is p-closed in B. Since $\langle b^{p^\alpha}\rangle$ is p-closed in $\langle b\rangle$, by Theorem 2.3, $G(n) = B \underset{c=b^{p^\alpha}}{*} \langle b\rangle$ is $\mathcal{RF}p$ for $p|n$. Thus $G(n)$ is $\mathcal{RF}p$ for all primes p by Theorem 2.6. ∎

Applying this result we obtain the following.

THEOREM 3.4. *Let A be a free group and B be $\mathcal{RF}p$. Let $\langle a\rangle$ be a maximal cycle in A and let $b \in B$ such that $\langle b\rangle$ is p-closed in B. Then $H(n) = A \underset{a=b^n}{*} B$ is $\mathcal{RF}p$.*

Proof. By Theorem 3.3, $G(n) = A \underset{a=b^n}{*} \langle b\rangle$ is $\mathcal{RF}p$. Now, by Lemma 3.2, $\langle b\rangle$ is p-closed in $G(n)$. Since $H(n) = G(n) *_{\langle b\rangle} B$ and $\langle b\rangle$ is p-closed in B, by Theorem 2.3, $H(n)$ is $\mathcal{RF}p$. ∎

THEOREM 3.5. *Let A, B be free and $\langle a\rangle, \langle b\rangle$ be maximal cycles in A, B respectively. Then $G(n, m) = A \underset{a^n=b^m}{*} B$ is $\mathcal{RF}p$ if and only if either $|n| = 1$ or $|m| = 1$ or $|n| = p^\alpha$ and $|m| = p^\beta$ for $\alpha, \beta > 0$.*

Proof. Suppose $G(n,m)$ is $\mathscr{RF}p$ and suppose $|n| \neq 1 \neq |m|$. Since $\langle a\rangle \underset{a^n=b^m}{*} \langle b\rangle \subset G(n,m)$, by Theorem 2.7, $|n| = p^\alpha$ and $|m| = p^\beta$ for $\alpha, \beta > 0$.

Conversely, if $n = 1$ (similarly, $n = -1$, $|m| = 1$), then $G(n,m) = A \underset{a=b^m}{*} B$. Since $\langle b\rangle$ is p-closed in B, by Theorem 3.4, $G(n,m)$ is $\mathscr{RF}p$.

If $|n| = p^\alpha$ and $|m| = p^\beta$ then, by Lemma 3.1, $\langle a^n\rangle, \langle b^m\rangle$ are p-closed in A, B respectively. Hence $G(n,m)$ is $\mathscr{RF}p$ by Theorem 2.3. ∎

COROLLARY 3.6. *Let A, B be free and $\langle a\rangle, \langle b\rangle$ be maximal cycles in A, B, respectively. Then $G(n,m) = A \underset{a^n=b^m}{*} B$ is $\mathscr{RF}p$ for all primes p if and only if either $|n| = 1$ or $|m| = 1$.*

4. ON GENERALIZED FREE PRODUCTS OF NILPOTENT GROUPS

For generalized free products of two nilpotent groups amalgamating a cyclic subgroup to be $\mathscr{RF}p$, we need stronger conditions than those of free groups. We begin with a simple observation.

LEMMA 4.1. *Let A be a noncyclic, finitely generated nilpotent group. If $a \in A$ is of infinite order, then there exists $b \in A$ such that $[b,a] = 1$ and $b \notin \langle a\rangle$.*

Proof. Suppose $a \in Z(A)$. Since A is noncyclic, we can choose $b \in A \setminus \langle a\rangle$. So we can assume that $a \notin Z(A)$. If A is finitely generated torsion-free nilpotent, then choose $b \in Z(A)$. Since $A/Z(A)$ is torsion-free and $a \notin Z(A)$, $b \notin \langle a\rangle$. Clearly, $[b,a] = 1$. If A has elements of finite orders, then $Z(A)$ has elements of finite orders. Let $b \in Z(A)$ of finite order. Since a is of infinite order, $b \notin \langle a\rangle$. Then $[b,a] = 1$. ∎

LEMMA 4.2. *Let A be a noncyclic, finitely generated nilpotent group. Let $\langle a\rangle$ be a maximal cycle in A. If $G = A \underset{a=b^n}{*} \langle b\rangle$ is $\mathscr{RF}p$, then $|n|$ is a p-power.*

Proof. Let G be $\mathscr{RF}p$. Suppose there is a prime $q \neq p$ dividing n, say, $n = qn_1$. By Lemma 4.1, there exists $c \in A \setminus \langle a\rangle$ such that $[c,a] = 1$. Then $g = [b^{n_1}, c] \neq 1$, since $\|g\| = 4$. Since G is $\mathscr{RF}p$, there exists $N \triangleleft_p G$ such that $g \notin N$. Then, in $\overline{G} = G/N$, since $p \neq q$ and $|\bar{b}|$ is a p-power, $\langle \bar{b}\rangle = \langle \bar{b}^q\rangle$. This implies $\bar{b}^{n_1} = \bar{b}^{n_1 qs}$ for some s. Hence $\bar{g} = \overline{[b^{n_1}, c]} = [\bar{b}^{n_1 qs}, \bar{c}] = [\bar{a}^s, \bar{c}] = 1$, since $[a,c] = 1$. This is impossible because of our choice of N. Therefore, n has no prime factor other than p, whence $|n|$ is a p-power. ∎

Applying this, we prove the following.

THEOREM 4.3. *Let A be a finitely generated torsion-free nilpotent group and let $\langle a \rangle$ be a maximal cycle in A. Then $B(n) = A \underset{a=b^n}{*} \langle b \rangle$ is $\mathscr{RF}p$ if and only if either* (1) $A = \langle a \rangle$, *or* (2) $|n|$ *is a p-power.*

Proof. If $B(n)$ is $\mathscr{RF}p$, then, by Lemma 4.2, (1) or (2) holds.

Conversely, if (1) holds, that is, $A = \langle a \rangle$, then $B(n) = \langle b \rangle$ is $\mathscr{RF}p$. So suppose that (2) holds, say, $|n| = p^\alpha$. Then, by Corollary 2.5, $\langle b^n \rangle = \langle b^{p^\alpha} \rangle$ is p-closed in $\langle b \rangle$ and $\langle a \rangle$ is p-closed in A. Hence $B(n)$ is $\mathscr{RF}p$ by Theorem 2.3. ∎

From this we can completely characterize residual p-finiteness of generalized free products of finitely generated torsion-free nilpotent groups, amalgamating a cycle, as below.

THEOREM 4.4. *Let A, B be finitely generated torsion-free nilpotent groups and let $\langle a \rangle, \langle b \rangle$ be maximal cycles in A, B, respectively. Then $B(n, m) = A \underset{a^n=b^m}{*} B$ is $\mathscr{RF}p$ if and only if one of the following holds*:

(1) *If* $|n| = 1$, *then either* $A = \langle a \rangle$ *or* $|m| = p^\beta$ *for* $\beta \geq 0$.

(2) *If* $1 \neq |n| \neq p^\alpha$ *for any* $\alpha > 0$, *then* $|m| = 1$ *and* $B = \langle b \rangle$.

(3) *If* $|n| = p^\alpha$ *for some* $\alpha > 0$, *then* $|m| = p^\beta$ *for* $\beta \geq 0$.

Proof. Suppose $B(n, m)$ is $\mathscr{RF}p$. (1) If $|n| = 1$, then $A \underset{a=b^m}{*} \langle b \rangle$ is a subgroup of $B(1, m)$. Therefore, $A \underset{a=b^m}{*} \langle b \rangle$ is $\mathscr{RF}p$. By Theorem 4.3, either $A = \langle a \rangle$ or $|m| = p^\beta$ for $\beta \geq 0$. (2) Suppose $|n|$ is not a p-power. Since $\langle a \rangle \underset{a^n=b^m}{*} \langle b \rangle \subset B(n, m)$, $\langle a \rangle \underset{a^n=b^m}{*} \langle b \rangle$ is $\mathscr{RF}p$. Hence, by Theorem 2.7, $|m| = 1$. Then $\langle a \rangle \underset{a^n=b}{*} B \subset B(n, 1)$, whence $\langle a \rangle \underset{a^n=b}{*} B$ is $\mathscr{RF}p$. Since $|n|$ is not a p-power, by theorem 4.3, $B = \langle b \rangle$. (3) Suppose $|n| = p^\alpha$ ($\alpha > 0$). Since $\langle a \rangle \underset{a^n=b^m}{*} \langle b \rangle \subset B(n, m)$, $|m|$ must be a p-power by Theorem 2.7.

To prove the converse, suppose $|n| = 1$ and $A = \langle a \rangle$. Then $B(n, m) = B$, whence $\mathscr{RF}p$. If $|n| = 1$ and $|m| = p^\beta$ for $\beta \geq 0$ then, by Theorem 2.3, $B(n, m) = A \underset{a=b^{p^\beta}}{*} B$ is $\mathscr{RF}p$. If $|m| = 1$ and $B = \langle b \rangle$, then $B(n, m) = A$, whence $\mathscr{RF}p$. Finally, if $|n| = p^\alpha$ and $|m| = p^\beta$, then, by Theorem 2.3, $B(n, m)$ is $\mathscr{RF}p$. ∎

COROLLARY 4.5. *Let A, B be finitely generated torsion-free nilpotent groups and let $\langle a \rangle, \langle b \rangle$ be maximal cycles in A, B, respectively. Then $B(n, m) = A \underset{a^n=b^m}{*} B$ is $\mathscr{RF}p$ for all p if and only if either $|n| = 1$ and $A = \langle a \rangle$, or $|m| = 1$ and $B = \langle b \rangle$, or $|n| = 1 = |m|$.*

Combining Theorems 3.5 and 4.4, we obtain the following characterization.

THEOREM 4.6. *Let A be free and B be finitely generated torsion-free nilpotent groups. Let $\langle a\rangle, \langle b\rangle$ be maximal cycles in A, B, respectively. Then $H(n,m) = A \underset{a^n=b^m}{*} B$ is $\mathcal{RF}p$ if and only if either*

(1) $|n| = 1$; *or*

(2) *If $|n|$ is not a p-power, then $|m| = 1$ and $B = \langle b\rangle$; or*

(3) *If $|n| = p^\alpha$ $(\alpha > 0)$, then $|m| = p^\beta$ for $\beta \geq 0$.*

Proof. Let $H(n,m)$ be $\mathcal{RF}p$ and $|n| \neq 1$. Suppose $|n|$ is not a p-power. Since $\langle a\rangle \underset{a^n=b^m}{*} \langle b\rangle$ is a subgroup of $H(n,m)$, $\langle a\rangle \underset{a^n=b^m}{*} \langle b\rangle$ is $\mathcal{RF}p$. Therefore, by Theorem 2.7, $|m| = 1$. Moreover, since $\langle a\rangle \underset{a^n=b}{*} B \subset H(n,1)$, whence $\mathcal{RF}p$, we must have $B = \langle b\rangle$ by Theorem 4.3. Hence (2) holds. Finally, suppose $|n| = p^\alpha$ $(\alpha > 0)$. Consider the subgroup $K = \langle a\rangle \underset{a^{p^\alpha}=b^m}{*} \langle b\rangle$ of $H(p^\alpha, m)$. Since $H(p^\alpha, m)$ is $\mathcal{RF}p$, K is $\mathcal{RF}p$. Hence, by Theorem 2.7, $|m| = p^\beta$ with $\beta \geq 0$.

Conversely, if (1) holds, then, by Theorem 3.4, $H(n,m)$ is $\mathcal{RF}p$. If (2) holds, then $H(n,m) = A$, whence $\mathcal{RF}p$. Finally, suppose (3) holds, say, $|n| = p^\alpha$ $(\alpha > 0)$ and $|m| = p^\beta$ $(\beta \geq 0)$. Then, by Lemmas 3.1 and 2.5, $\langle a^n\rangle, \langle b^m\rangle$ are p-closed in A, B, respectively. Hence, by Theorem 2.3, $H(n,m)$ is $\mathcal{RF}p$. ∎

COROLLARY 4.7. *Let A be free and B be finitely generated torsion-free nilpotent groups. Let $\langle a\rangle, \langle b\rangle$ be maximal cycles in A, B respectively. Then $H(n,m) = A \underset{a^n=b^m}{*} B$ is $\mathcal{RF}p$ for all primes p if and only if either* (1) $|n| = 1$, *or* (2) *if* $|n| \neq 1$, *then* $|m| = 1$ *and* $B = \langle b\rangle$.

The above results generalize Theorem 4.6 in [7]. Moreover, applying these results, we obtain the following interesting examples.

EXAMPLE 4.8. (1) $\langle a, b, c\colon [a,b] = 1, a = c^6\rangle$ is not $\mathcal{RF}p$ for any p.

(2) $\langle a, b, c\colon [a,b] = c^n\rangle$ is $\mathcal{RF}p$ for all primes p.

(3) Let A be free and B be finitely generated torsion-free nilpotent groups. If $\langle x\rangle$ is maximal cycle in A, then, for any $b \in B$, $A \underset{x=b}{*} B$ is $\mathcal{RF}p$ for all primes p.

5. ON TREE PRODUCTS

Let T be a tree. To each vertex v of T assign a group G_v. To each edge e assign a group G_e and monomorphisms α_e and β_e, embedding G_e to the vertex groups at the end of e. The *tree product* G of T amalgamating the edge groups is defined to be the group generated by the generators and relations of the vertex groups, together with the additional relations obtained by identifying $g_e \alpha_e$ and $g_e \beta_e$ for each $g_e \in G_e$.

Tree products of $\mathcal{RF}p$ groups are quite complicated. In fact, tree products of groups that are residually finite p-groups for all p may not be $\mathcal{RF}p$ for any p. The following examples illustrate this point.

(1) Let A, B, C be free. Let $B = \langle b, b_1 \rangle$ and let $a \in A$ and $c \in C$ be elements of infinite order. Let $G_1 = A \underset{a^2=b}{*} B \underset{b=c^3}{*} C$. By Corollary 3.6, the subgroups $\langle A, B \rangle$ and $\langle B, C \rangle$ of G_1 are $\mathcal{RF}p$ for all p. But, by Theorem 2.7, the subgroup $\langle a \rangle \underset{a^2=c^3}{*} \langle c \rangle$ of G_1 is not a residually finite p-group for any p. Hence G_1 is not a residually finite p-group for any p. Similarly, $G_2 = A \underset{a^2=b_1^{-1}bb_1}{*} B \underset{b=c^3}{*} C$ contains the subgroup $\langle a \rangle \underset{a^2=d^3}{*} \langle d \rangle$, where $d = b_1^{-1}cb_1$. Hence G_2 is not a residually finite p-group for any p.

(2) Let A, C be $\mathcal{RF}p$ for all p and B, a, c be as above. Let $G_3 = A \underset{a^2=b}{*} B \underset{b_1=c^3}{*} C$. Then $G_3 = A * C$. By Gruenberg [4, Theorem 4.1], G_3 is a residually finite p-group for all p. Similarly, let $G_4 = A \underset{a^2=b_1^{-1}bb_1}{*} B \underset{b_1=c^3}{*} C$ $= A * C$. Then G_4 is a residually finite p-group for all p.

We first make the following observation, which was also proved in [9] by a different method.

LEMMA 5.1. *Let $G = A *_{\langle b \rangle} B$, where A, B are $\mathcal{RF}p$ and let $\langle b \rangle$ be p-closed in A and B. If $\langle h \rangle \subset B$ is p-closed in B, then $\langle h \rangle$ is p-closed in G.*

Proof. Let $g \in G \setminus \langle h \rangle$. Suppose $[g, h] \neq 1$. By Theorem 2.3, G is $\mathcal{RF}p$. Therefore, there exists $N \lhd_p G$ such that $[g, h] \notin N$. Then $g \notin N\langle h \rangle$. So we need only consider the case $[g, h] = 1$.

Case 1. $g \in B$. Since $\langle h \rangle$ is p-closed in B, there exists $M \lhd_p B$ such that $g \notin M\langle h \rangle$. By Lemma 2.1, there exists $L \lhd_p A$ such that $M \cap \langle b \rangle = L \cap \langle b \rangle$. Thus, in $\overline{G} = A/L *_{\langle \bar{b} \rangle} B/M$, we have $\bar{g} \notin \langle \bar{h} \rangle$. Since $\overline{G}$ is $\mathcal{RF}p$ and $\langle \bar{h} \rangle$ is finite, there exists $\overline{N} \lhd_p \overline{G}$ such that $\bar{g} \notin \overline{N}\langle \bar{h} \rangle$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_p G$ and $g \notin N\langle h \rangle$.

Case 2. $g \notin B$. Then $[g, h] = 1$ implies $h \sim_B b^n$ for some n. Let $h = b_1^{-1} b^n b_1$ for $b_1 \in B$. Since $\langle h \rangle$ is p-closed in B, $\langle b^n \rangle$ is p-closed in B. Thus $\langle b^n \rangle$ is p-closed in $\langle b \rangle$. This implies $|n| = p^\alpha$ for some $\alpha \geq 0$. Now $\langle b \rangle$ is p-closed in G by [7, Lemma 4.1]. Hence $\langle b^n \rangle = \langle b^{p^\alpha} \rangle$ is p-closed in G by Lemma 2.4. Therefore, since $h = b_1^{-1} b^n b_1$, $\langle h \rangle$ is p-closed in G. ∎

When all edge groups of a tree product are cyclic, we use

$$\langle A_i;\, a_{ij} = a_{ji},\, i \neq j,\, i, j \in I \rangle$$

to denote the tree product of $A_i, i \in I$, identifying (amalgamating) $a_{ij} \in A_i$ and $a_{ji} \in A_j$ for the edge between A_i and A_j. Repeatly applying Lemma 5.1 and Theorems 3.3, 4.3, and 2.3, we obtain the following result.

THEOREM 5.2. *Let A_i, $i \in I$, be free or finitely generated torsion-free nilpotent groups, where I is a finite set. Let each $\langle u_{ij}\rangle$ be a maximal cycle in A_i. Let $G = \langle A_i;\ u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}},\ i \neq j,\ i, j \in I\rangle$ be the tree product of A_i, $i \in I$, amalgamating $u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}}$.*

(1) *If all of the $|n_{ij}| = |n_{ji}| = 1$, then G is $\mathscr{RF}p$ for all p.*

(2) *If all of the $|n_{ij}|$ and $|n_{ji}|$ are p-powers, then G is $\mathscr{RF}p$.*

We note that (1) above was also proved in [9].

DEFINITION 5.3. Let S be a subtree of a tree T. Let H be the tree product of S. Then H is called a *subtree product* of the tree product G of T. If $G = H$, then we say that the tree product G of T is *contractible* to the subtree product H.

As seen in examples (1) and (2), it is difficult to keep track of the residually finite p-group property of tree products of $\mathscr{RF}p$ groups. The following theorem gives a partial characterization of tree products of finitely generated torsion-free nilpotent groups.

THEOREM 5.4. *Let A_i, $i \in I$, be finitely generated torsion-free nilpotent groups. Let each $\langle u_{ij}\rangle$ be a maximal cycle in A_i. Let $G = \langle A_i;\ u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}}$, $i \neq j$, $i, j \in I\rangle$ be the tree product of A_i, $i \in I$, amalgamating $u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}}$.*

(1) *If G is $\mathscr{RF}p$, then G is contractible to a subtree product of A_i, $i \in J$, amalgamating $u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}}$, where all of the $|n_{ij}|$ and $|n_{ji}|$ are p-powers.*

(2) *If G is contractible to a subtree product of A_i, $i \in J$, where J is finite, amalgamating $u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}}$, where all of the $|n_{ij}|$ and $|n_{ji}|$ are p-powers, then G is $\mathscr{RF}p$.*

Proof. (1) Suppose G is $\mathscr{RF}p$. Suppose A_i, A_j are connected and $|n_{ij}|$ is not a p-power. Since $A_i \underset{u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}}}{*} A_j \subset G$, by Theorem 4.4, $|n_{ji}| = 1$ and $A_j = \langle u_{ji}\rangle$. Hence $\langle A_i, A_j\rangle = A_i \underset{u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}}}{*} A_j = A_i$. Now if A_j and A_k are also connected, then $\langle A_i, A_j, A_k\rangle = A_i \underset{u_{ij}^{n_{ij}n_{jk}} = u_{kj}^{n_{kj}}}{*} A_k$. Since $n_{ij}n_{jk}$ is again not a p-power, as before, by Theorem 4.4, $|n_{kj}| = 1$ and $A_k = \langle u_{kj}\rangle$. Thus $\langle A_i, A_j, A_k\rangle = A_i$. So the tree product of the subtree connected to A_i by the subgroup $\langle u_{ij}^{n_{ij}}\rangle$ is contractible to A_i, if $|n_{ij}|$ is not a p-power. Hence G can be contractible to a subtree product of A_i, $i \in J$, amalgamating $u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}}$, where all of the $|n_{ij}|$ and $|n_{ji}|$ are p-powers.

(2) follows directly from Theorem 5.2 (2). ∎

COROLLARY 5.5. *Let A_i, $i \in I$, be finitely generated torsion-free nilpotent groups, where I is a finite set. Let each $\langle u_{ij} \rangle$ be a maximal cycle in A_i. Then the tree product $G = \langle A_i; u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}}, i \neq j, i, j \in I \rangle$ is $\mathcal{RF}p$ for all p if and only if G is contractible to a subtree product of A_i, $i \in J$, amalgamating $u_{ij}^{\pm 1} = u_{ji}^{\pm 1}$.*

Finally, we consider generalized free products of infinitely many finitely generated torsion-free nilpotent groups, amalgamating a single cycle. For this, we use

$$F = \langle A_i; u_i^{n_i}, i \in I \rangle$$

to denote the generalized free products of the A_i $(i \in I)$, where I might be an infinite set, amalgamating cyclic subgroups $\langle u_i^{n_i} \rangle$.

THEOREM 5.6. *Let A_i, $i \in I$, be finitely generated torsion-free nilpotent groups. Let each $\langle u_i \rangle$ be a maximal cycle in A_i. Suppose that the A_i are of bounded rank and bounded nilpotent class and that the n_i are bounded. Then F is $\mathcal{RF}p$ if and only if either the $|n_j|$ are all p-powers, or $F = A_i$ for some $i \in I$.*

Proof. Suppose F is $\mathcal{RF}p$, and suppose $|n_i|$ is not a p-power. Since $A_i \underset{u_i^{n_i} = u_j^{n_j}}{*} A_j \subset F$, by Theorem 4.4, for each $j \in I \setminus \{i\}$, $|n_j| = 1$ and $A_j = \langle u_j \rangle$. Hence $F = A_i$.

Conversely, if $F = A_i$ then clearly, F is $\mathcal{RF}p$. If all of the $|n_j|$ are p-powers, then F is $\mathcal{RF}p$ by Theorem 5 in [3]. ∎

COROLLARY 5.7. *Let A_i, $i \in I$, be finitely generated torsion-free nilpotent groups. Let each $\langle u_i \rangle$ be a maximal cycle in A_i. Suppose that the A_i are of bounded rank and bounded nilpotent class. Then F is $\mathcal{RF}p$ for all primes p if and only if either all $|n_j| = 1$, or $F = A_i$ for some $i \in I$.*

Above two results can be compared with main results, Theorems 1 and 5, in [3].

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REFERENCES

1. G. Baumslag, On the residual nilpotence of certain one-relator groups, *Comm. Pure Appl. Math.* **21** (1968), 491–506.
2. G. Baumslag, Lecture notes on nilpotent groups, *in* ''C.B.M.S. Regional Conf. Ser. Math.,'' no. 2, Am. Math. Soc., Providence, RI, 1971.
3. D. Doniz, Residual properties of free products of infinitely many nilpotent groups amalgamating cycles, *J. Algebra* **179** (1996), 930–935.
4. K. W. Gruenberg, Residual properties of infinite soluble groups, *Proc. London Math. Soc.* **7** (1957), 29–62.
5. G. Higman, Amalgams of p-groups, *J. Algebra* **1** (1964), 301–305.
6. K. Iwasawa, Einige Statze über freie Gruppen, *Proc. Imp. Acad. Tokyo* **19** (1943), 272–274.
7. G. Kim and J. McCarron, On amalgamated free products of residually p-finite groups, *J. Algebra* **162** (1993), 1–11.
8. G. Kim and J. McCarron, Some residually p-finite one-relator groups, *J. Algebra* **169** (1994), 817–826.
9. P. C. Wong and C. K. Tang, Tree products of residually p-finite groups, *Algebra Colloq.* **2** (1995), 209–212.