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A Criterion for the Conjugacy Separability of Certain HNN Extensions of Groups*

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We first prove the double coset separability of certain HNN extensions with cyclic associated subgroups. Using this we prove a criterion for the conjugacy separability of HNN extensions of conjugacy separable groups with cyclic associated subgroups. Applying this result we show that certain HNN extensions of free products of cycles and certain Baumslag–Solitar groups with cyclic associated subgroups are conjugacy separable.

Key Words: HNN-extension, conjugacy separable, residually finite.

1. INTRODUCTION

A group G is *conjugacy separable* if, for each $x, y \in G$ such that x and y are not conjugate in G, there exists a finite homomorphic image $\overline{G}$ of G such that the images of x and y in $\overline{G}$ are not conjugate in $\overline{G}$.

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In this paper we consider the conjugacy separability of HNN extensions

$$G = \langle A, t \colon t^{-1}ht = k \rangle$$

of a conjugacy separable group A with cyclic associated subgroups $\langle h \rangle$ and $\langle k \rangle$.

Residual properties of HNN extensions are difficult to obtain. They depend very much on the choice of $h, k \in A$. For example, the Baumslag–Solitar group, $\langle a, t \colon t^{-1}a^{\lambda}t = a^{\delta} \rangle$, is an HNN extension of the cyclic group $\langle a \rangle$. It is residually finite [13] if and only if $|\lambda| = 1$ or $|\delta| = 1$ or $\lambda = \pm\delta$.

In [16], Raptis, Talelli, and Varsos studied the Hopficity and the residual finiteness of HNN extensions of Baumslag–Solitar groups with cyclic associated subgroups of the form $\langle t, a, b \colon t^{-1}a^{\nu}t = b^{\xi}, b^{-1}a^{\lambda}b = a^{\mu} \rangle$.

Recently, in [6, Definition 2.4], Kim and Tang introduced the concept of a group to be quasi-regular at $\{h, k\}$. If A is quasi-regular at $\{h, k\}$, then the residual properties of HNN extension G of A are nicely preserved (Theorem 2.7). Using this, they characterized the residual finiteness and cyclic subgroup separability of G when A is a finitely generated abelian group. They also characterized the conjugacy separability of G when A is a finitely generated abelian group [8].

In this paper we prove the double coset separability of HNN extensions of groups which are quasi-regular at the associated subgroups (Lemma 3.4). Using this, we derive a criterion for the conjugacy separability of HNN extensions with cyclic associated subgroups (Theorem 4.5). In general it is difficult to get groups to satisfy double coset separability for associated subgroups. However, if the associated subgroups are in the center of a polycyclic-by-finite group, then we can easily apply the criterion to show that such HNN extensions are conjugacy separable (Corollary 4.6). This gives us an extension of a result in [8] for HNN extensions of abelian groups. We also apply the criterion to show that HNN extensions of free products of cycles with cyclic associated subgroups are conjugacy separable (Theorem 5.4). Finally, we show that certain HNN extensions of Baumslag–Solitar groups with cyclic associated subgroups are conjugacy separable (Theorem 6.7).

2. PRELIMINARIES

Throughout this paper we use standard notations and terminology for this topic [12]. The letter G always denotes a group. In addition: $N \triangleleft_f G$ means that N is a normal subgroup of finite index in G; $x \sim_G y$ means x is conjugate to y in G. We use $\|x\|$ to denote the length of x in HNN-extensions and generalized free products.

We shall make extensive use of the following two results by D. J. Collins.

THEOREM 2.1 [3, Theorem 3]. *Let x and y be cyclically reduced elements of the HNN-extension $G = \langle B, t\colon t^{-1}Ht = K\rangle$. Suppose that $x \sim_G y$. Then $\|x\| = \|y\|$, and one of the following holds.*

(1) $\|x\| = \|y\| = 0$ *and there is a finite sequence* $z_1, z_2, \ldots, z_m$ *of elements in* $H \cup K$ *such that* $x \sim_B z_1 \sim_{B,t^*} z_2 \sim_{B,t^*} \cdots \sim_{B,t^*} z_m \sim_B y$, *where* $u \sim_{B,t^*} v$ *means one of*: (i) $u \sim_B v$, (ii) $u \in H$ *and* $v = t^{-1}ut$ $(\in K)$, *or* (iii) $u \in K$ *and* $v = tut^{-1}$ $(\in H)$.

(2) $\|x\| = \|y\| \geq 1$ *and* $y \sim_{H \cup K} x^*$ *where* x^* *is a cyclic permutation of* x.

A group G is said to be *S-separable* if for each $x \in G \setminus S$ there exists $N \triangleleft_f G$ such that $x \notin NS$. In particular, G is *residually finite* ($\mathscr{RF}$) if G is $\{1\}$-separable. A group G is said to be *cyclic subgroup separable*, briefly π_c, if G is $\langle x\rangle$-separable for each $x \in G$.

A group G is *conjugacy separable* if, for each $x \in G$, G is $\{x\}^G$-separable, where $\{x\}^G$ is the set of all conjugates of x in G.

THEOREM 2.2 [3, Theorem 13]. *If A is conjugacy separable and H, K are finite, then the HNN-extension $\langle A, t\colon t^{-1}Ht = K\rangle$ is conjugacy separable.*

DEFINITION 2.3. Let A be a group and let $h, k \in A$ be of infinite order. Then A is said to be *quasi-regular* at $\{h, k\}$ if, for each integer $\epsilon > 0$, there exist ana integer $\lambda_\epsilon > 0$ and $N_\epsilon \triangleleft_f A$, depending on ϵ, such that $N_\epsilon \cap \langle h\rangle = \langle h^{\epsilon\lambda_\epsilon}\rangle$ and $N_\epsilon \cap \langle k\rangle = \langle k^{\epsilon\lambda_\epsilon}\rangle$.

In [14], Niblo defined that a group A has *regular quotients* at $\{h, k\}$ if there exists a positive integer λ, such that for each positive integer ϵ, there exists $N \triangleleft_f A$ such that $N \cap \langle h\rangle = \langle h^{\lambda\epsilon}\rangle$ and $N \cap \langle k\rangle = \langle k^{\lambda\epsilon}\rangle$. Thus, in the case of regular quotients, the integer λ is independent of ϵ. Hence, if A has regular quotients at $\{h, k\}$, then A is quasi-regular at $\{h, k\}$.

Remark 2.4. Let $G = \langle A, t\colon t^{-1}ht = k\rangle$. For $N \triangleleft_f A$ such that $N \cap \langle h\rangle = \langle h^s\rangle$ and $N \cap \langle k\rangle = \langle k^s\rangle$, there is a natural homomorphism

$$\pi_N\colon \langle A, t\colon t^{-1}ht = k\rangle \to \langle \overline{A}, \tau\colon \tau^{-1}\overline{h}\tau = \overline{k}\rangle$$

by $a\pi_N = \bar{a}$ for $a \in A$, where $\overline{A} = A/N$, and $t_{\pi_N} = \tau$. Then, by Theorem 2.2, $G\pi_N$ is conjugacy separable.

LEMMA 2.5. *Let $G = \langle A, t\colon t^{-1}ht = k\rangle$. Suppose A is $\langle h\rangle$-separable, $\langle k\rangle$-separable, and quasi-regular at $\{h, k\}$. Then, for each $M \triangleleft_f A$, for each $s > 0$ and for each reduced element $x \in G$, there exist $N \triangleleft_f A$ and $\lambda > 0$ such that $N \subset M$, $N \cap \langle h\rangle = \langle h^{s\lambda}\rangle$, $N \cap \langle k\rangle = \langle k^{s\lambda}\rangle$ and $\|\bar{x}\| = \|x\|$ in $\overline{G} = G\pi_N$.*

Proof. Let $x = u_1 t^{\epsilon_1} u_2 t^{\epsilon_2} \cdots t^{\epsilon_n} u_{n+1} \in G$ be reduced where $u_i \in A$ and $\epsilon_i = \pm 1$. Since x is reduced, we have $u_{i+1} \notin \langle h \rangle$ if $\epsilon_i = -\epsilon_{i+1} = -1$ and $u_{i+1} \notin \langle k \rangle$ if $\epsilon_i = -\epsilon_{i+1} = 1$. Since A is $\langle h \rangle$-separable and $\langle k \rangle$-separable, we can find $N_i \lhd_f A$ such that if $\epsilon_i = \epsilon_{i+1}$ then $N_i = A$, or if $\epsilon_i = -\epsilon_{i+1} = -1$ then $u_{i+1} \notin N_i\langle h \rangle$, or if $\epsilon_i = -\epsilon_{i+1} = 1$ then $u_{i+1} \notin N_i\langle k \rangle$. Let $\cap_{i=1}^n N_i \cap M \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $\cap_{i=1}^n N_i \cap M \cap \langle k \rangle = \langle k^{s_2} \rangle$. Let $\epsilon = s_1 s_2 s$. By quasi-regularity, there exist λ_1 and $N_s \lhd_f A$ such that $N_s \cap \langle h \rangle = \langle h^{\epsilon\lambda_1} \rangle$ and $N_s \cap \langle k \rangle = \langle k^{\epsilon\lambda_1} \rangle$. Let $N = \cap_{i=1}^n N_i \cap M \cap N_s$ and $\lambda = s_1 s_2 \lambda_1$. Then $N \lhd_f A, N \subset M, N \cap \langle h \rangle = \langle h^{s_1 s_2 \lambda_1 s} \rangle = \langle h^{\lambda s} \rangle, N \cap \langle k \rangle = \langle k^{s_1 s_2 \lambda_1 s} \rangle = \langle k^{\lambda s} \rangle$ and, in $\overline{G} = G\pi_N$ above, $\bar{x} = \bar{u}_1 \tau^{\epsilon_1} \bar{u}_2 \tau^{\epsilon_2} \cdots \tau^{\epsilon_n} \bar{u}_{n+1}$ is reduced with $\|\bar{x}\| = \|x\|$. ∎

Remark 2.6. Let $G = \langle A, t \colon t^{-1}ht = k \rangle$. Suppose A is quasi-regular at $\{h, k\}$ and $h^i \sim_G h^j$. Let $\epsilon = i$. Then there exist λ and $N \lhd_f A$ such that $N \cap \langle h \rangle = \langle h^{i\lambda} \rangle$ and $N \cap \langle k \rangle = \langle k^{i\lambda} \rangle$. In $\overline{G} = G\pi_N, |\overline{h}| = i\lambda = |\overline{k}|$. Since $\overline{h}^i \sim_G \overline{h}^j$ in $\overline{G} = G\pi_N$, we have $1 = \overline{h}^{i\lambda} \sim_{\overline{G}} \overline{h}^{j\lambda}$. This implies $i\lambda | j\lambda$. Hence $i|j$. Similarly $j|i$. *Therefore, if A is quasi-regular at $\{h, k\}$ and $h^i \sim_G h^j$ then $j = \pm i$. Similarly, if A is quasi-regular at $\{h, k\}$ and $k^i \sim_G k^j$ or $h^i \sim_G k^j$ then $j = \pm i$.*

The following theorem shows that quasi-regularity is closely related to the cyclic subgroup separability of HNN extensions.

THEOREM 2.7 [6]. *Let A be π_c and let $h, k \in A$ be of infinite order. Then $G = \langle A, t \colon t^{-1}ht = k \rangle$ if π_c if and only if A is quasi-regular at $\{h, k\}$.*

In this paper we consider the conjugacy separability of HNN extensions of the type

$$G = \langle A, t \colon t^{-1}ht = k \rangle,$$

where $\langle h \rangle \cap \langle k \rangle = 1$ and A is conjugacy separable. If $|h| = |k|$ is finite then, by Theorem 2.2, G is conjugacy separable. Thus, we need only consider the case when h, k are of infinite order.

3. DOUBLE COSET SEPARABILITY

Throughout this note, we consider the HNN extension $G = \langle A, t \colon t^{-1}ht = k \rangle$ of a base group A with infinite cyclic associated subgroups $\langle h \rangle$ and $\langle k \rangle$, where $\langle h \rangle \cap \langle k \rangle = 1$.

DEFINITION 3.1. A group A is said to be *double coset separable* at $\{h, k\}$ if, for each $u \in A$ and for each integer $\epsilon > 0$, A is $\langle h^\epsilon \rangle u \langle h^\epsilon \rangle$-separable, $\langle h^\epsilon \rangle u \langle k^\epsilon \rangle$-separable, and $\langle k^\epsilon \rangle u \langle k^\epsilon \rangle$-separable.

We note that if A is $\langle h^{\epsilon}\rangle u\langle k^{\epsilon}\rangle$-separable for all $u \in A$ then A is $\langle k^{\epsilon}\rangle u^{-1}\langle h^{\epsilon}\rangle$-separable for all $u \in A$. It follows that if A is double coset separable at $\{h, k\}$ then A is also $\langle k^{\epsilon}\rangle u\langle h^{\epsilon}\rangle$-separable for all $u \in A$. In the case of a free-by-finite or polycyclic-by-finite group G, it was known by [11, 15] that G is $\langle a\rangle x\langle b\rangle$-separable for all $a, b, x \in G$. Hence G is double coset separable at $\{h, k\}$ for all $h, k \in G$.

Clearly if a group G is double coset separable at $\{h, k\}$, then G is $\langle h^{\epsilon}\rangle$-separable, $\langle k^{\epsilon}\rangle$-*separable, and* $\langle h^{\epsilon}\rangle\langle k^{\epsilon}\rangle$-*separable for any* ϵ.

Remark 3.2. If a group G is $\langle h^{\epsilon}\rangle u\langle k^{\epsilon}\rangle$-separable for any ϵ, then G is $\langle h^{\alpha}\rangle u\langle k^{\beta}\rangle$-separable for any α, β. For, if $g \notin \langle h^{\alpha}\rangle u\langle k^{\beta}\rangle$ then $h^{-\alpha i}gk^{-\beta j} \notin \langle h^{l}\rangle u\langle k^{l}\rangle$ for each $0 \le i < \frac{l}{\alpha}$ and $0 \le j < \frac{l}{\beta}$, where l is the least common multiple of α, β. Therefore, there exists $N \triangleleft_f G$ such that $h^{-\alpha i}gk^{-\beta j} \notin N\langle h^{l}\rangle u\langle k^{l}\rangle$ for all $0 \le i < \frac{l}{\alpha}$ and $0 \le j < \frac{l}{\beta}$. This implies that $g \notin N\langle h^{\alpha}\rangle u\langle k^{\beta}\rangle$. Hence G is $\langle h^{\alpha}\rangle u\langle k^{\beta}\rangle$-separable for any α, β. Similarly, if a group G is double coset separable at $\{h, k\}$, then G is $\langle h^{\alpha}\rangle u\langle h^{\beta}\rangle$-separable, $\langle k^{\alpha}\rangle u\langle k^{\beta}\rangle$-separable, and $\langle h^{\alpha}\rangle u\langle k^{\beta}\rangle$-separable for any α, β.

LEMMA 3.3 [9]. *Let G be $\langle a^{\epsilon}\rangle x\langle b^{\epsilon}\rangle$-separable, where $x, a, b \in G$ and a, b are of infinite order. If $\langle x^{-1}ax\rangle \cap \langle b\rangle = 1$, then there exists $N \triangleleft_f G$ such that $\bar{x}^{-1}\bar{a}^{i}\bar{x} = \bar{b}^{j}$ only if $\epsilon | i, j$, where $\overline{G} = G/N$.*

Note that the condition "$\bar{x}^{-1}\bar{a}^{i}\bar{x} = \bar{b}^{j}$ only if $\epsilon | i, j$" always implies $\epsilon \mid |\bar{a}|, |\bar{b}|$.

For convenience, we say $N \in \mathcal{N}$ if $N \triangleleft_f A$ such that $N \cap \langle h\rangle = \langle h^{\epsilon\lambda}\rangle$ and $N \cap \langle k\rangle = \langle k^{\epsilon\lambda}\rangle$. In this case, we can construct $\overline{G} = G\pi_N$ as in Remark 2.4.

LEMMA 3.4. *Let A be quasi-regular and double coset separable at $\{h, k\}$. Then $G = \langle A, t\colon t^{-1}ht = k\rangle$ is double coset separable at $\{h, k\}$.*

Proof. To show that, for example, G is $\langle h^{s}\rangle x\langle k^{s}\rangle$-separable for any $s > 0$, we let $y, x \in G$ such that $y \notin \langle h^{s}\rangle x\langle k^{s}\rangle$. Then, by Lemma 2.5, we can find $N_1 \in \mathcal{N}$ such that $\|\hat{y}\| = \|y\|$ and $\|\hat{x}\| = \|x\|$, in $\widehat{G} = G\pi_{N_1}$. If we can find $N \in \mathcal{N}$ such that $N \subset N_1$ and $\bar{y} \notin \langle \bar{h}^{s}\rangle\bar{x}\langle \bar{k}^{s}\rangle$ in $\overline{G} = G\pi_N$, then there exists $\overline{L} \triangleleft_f \overline{G}$ such that $\bar{y} \notin \overline{L}\langle \bar{h}^{s}\rangle\bar{x}\langle \bar{k}^{s}\rangle$, since $\overline{G}$ is residually finite and $|\bar{h}| = |\bar{k}|$ is finite. Let L be the preimage of $\overline{L}$ in G. Then $L \triangleleft_f G$ and $y \notin L\langle h^{s}\rangle x\langle k^{s}\rangle$. This is the general sketch of our long proof.

We prove the theorem by induction on $\|x\|$. Suppose $\|x\| = 0$, i.e., $x \in A$, we shall show that, for any $s > 0$ and $x \in A, G$ is $\langle h^{s}\rangle x\langle h^{s}\rangle$-separable, $\langle h^{s}\rangle x\langle k^{s}\rangle$-separable, and $\langle k^{s}\rangle x\langle k^{s}\rangle$-separable.

To show that G is $\langle h^{s}\rangle x\langle k^{s}\rangle$-separable for $x \in A$, let $y \in G$ such that $y \notin \langle h^{s}\rangle x\langle k^{s}\rangle$. Suppose $\|y\| = 0$, that is, $y \in A$. Then, by the double coset separability of A, there exists $N_2 \triangleleft_f A$ such that $y \notin N_2\langle h^{s}\rangle x\langle k^{s}\rangle$.

As in Lemma 2.5, we choose $N \in \mathscr{N}$ such that $N \subset N_2$. Thus $\bar{y} \notin \langle \bar{h}^s \rangle \bar{x} \langle \bar{k}^s \rangle$ in $\overline{G} = G\pi_N$. Suppose $\|y\| \geq 1$. Again, by Lemma 2.5, we choose $N \in \mathscr{N}$ such that $\|\bar{y}\| = \|y\|$ in $\overline{G} = G\pi_N$. Since $\|\bar{y}\| \geq 1$ and $x \in A$, $\bar{y} \notin \langle \bar{h}^s \rangle \bar{x} \langle \bar{k}^s \rangle$ in $\overline{G} = G\pi_N$. This proves that G is $\langle h^s \rangle x \langle k^s \rangle$-separable for $x \in A$. Similarly, for any $s > 0$, G is $\langle h^s \rangle x \langle h^s \rangle$-separable and $\langle k^s \rangle x \langle k^s \rangle$-separable for $x \in A$.

Let $\|x\| = n$. By induction we can assume that, for any $s > 0, G$ is $\langle h^s \rangle g \langle h^s \rangle$-separable, $\langle h^s \rangle g \langle k^s \rangle$-separable, and $\langle k^s \rangle g \langle k^s \rangle$-separable for all $g \in G$ such that $\|g\| \leq n - 1$. We need to show that G is $\langle h^s \rangle x \langle h^s \rangle$-separable, $\langle h^s \rangle x \langle k^s \rangle$-separable, and $\langle k^s \rangle x \langle k^s \rangle$-separable for $x \in G$ with $\|x\| = n$.

Case 1. To show G is $\langle h^s \rangle x \langle k^s \rangle$-separable for any $x \in G$ with $\|x\| = n$.

Let $y \in G$ such that $y \notin \langle h^s \rangle x \langle k^s \rangle$. If $\|y\| > n$ or $\|y\| < n$ or $y = v_1 t^{\delta_1} v_2 \cdots t^{\delta_n} v_{n+1}$ and $x = u_1 t^{\epsilon_1} u_2 \cdots t^{\epsilon_n} u_{n+1}$ with $\delta_i \neq \epsilon_i$ for some i then, by Lemma 2.5, we can find $N_1 \in \mathscr{N}$ such that, in $\overline{G} = G\pi_{N_1}$, $\|\bar{y}\| = \|y\|$ and $\|\bar{x}\| = \|x\|$. This implies $\bar{y} \notin \langle \bar{h}^s \rangle \bar{x} \langle \bar{k}^s \rangle$ in $\overline{G} = G\pi_{N_1}$. So we consider $y = v_1 t^{\epsilon_1} v_2 \cdots t^{\epsilon_n} v_{n+1}$ and $x = u_1 t^{\epsilon_1} u_2 \cdots t^{\epsilon_n} u_{n+1}$ where $y \notin \langle h^s \rangle x \langle k^s \rangle$. For convenience, we let $x_{n-1} = u_1 t^{\epsilon_1} u_2 \cdots t^{\epsilon_{n-1}} u_n$ and $y_{n-1} = v_1 t^{\epsilon_1} v_2 \cdots t^{\epsilon_{n-1}} v_n$. By Lemma 2.5, there exists $N_1 \in \mathscr{N}$ such that, in $\overline{G} = G\pi_{N_1}$, $\|\tilde{x}\| = \|\tilde{y}\| = n$. We need to consider the cases $\epsilon_n = 1$ and $\epsilon_n = -1$.

Subcase 1. Suppose $\epsilon_n = 1$. If $y_{n-1} \notin \langle h^s \rangle x_{n-1} \langle h \rangle$ or $v_{n+1} \notin \langle k \rangle u_{n+1} \langle k^s \rangle$, then, by induction and Remark 3.2, we can find $M \triangleleft_f G$ such that $y_{n-1} \notin M(\langle h^s \rangle x_{n-1} \langle h \rangle)$ or $v_{n+1} \notin M\langle k \rangle u_{n+1} \langle k^s \rangle$. By Lemma 2.5, we can find $N \in \mathscr{N}$ such that $N \subset N_1 \cap M$ and $\|\bar{x}\| = \|\bar{y}\| = n$ in $\overline{G} = G\pi_N$. If $\bar{y} = \bar{h}^{si} \bar{x} \bar{k}^{sj}$ for some i, j, then

$$\tau^{-1} \bar{x}_{n-1}^{-1} \bar{h}^{-si} \bar{y}_{n-1} \tau = \bar{u}_{n+1} \bar{k}^{sj} \bar{v}_{n+1}^{-1}. \tag{3.1}$$

This implies

$$\bar{x}_{n-1}^{-1} \bar{h}^{-si} \bar{y}_{n-1} = \bar{h}^{\alpha} \quad \text{and} \quad \bar{k}^{\alpha} = \bar{u}_{n+1} \bar{k}^{sj} \bar{v}_{n+1}^{-1} \tag{3.2}$$

for some α, hence $\bar{y}_{n-1} \in \langle \bar{h}^s \rangle \bar{x}_{n-1} \langle \bar{h} \rangle$ and $\bar{v}_{n+1} \in \langle \bar{k} \rangle \bar{u}_{n+1} \langle \bar{k}^s \rangle$, contradicting the choice of $N \subset M$. Thus $\bar{y} \notin \langle \bar{h}^s \rangle \bar{x} \langle \bar{k}^s \rangle$. Therefore we suppose $y_{n-1} = h^{si_1} x_{n-1} h^{i_2}$ and $v_{n+1} = k^{j_1} u_{n+1} k^{sj_2}$ for some i_1, i_2, j_1, j_2. Then

$$y = y_{n-1} t v_{n+1} = h^{si_1} x_{n-1} h^{i_2} t k^{j_1} u_{n+1} k^{sj_2} \notin \langle h^s \rangle x_{n-1} t u_{n+1} \langle k^s \rangle. \tag{3.3}$$

This implies that $h^{i_2} t k^{j_1} \notin x_{n-1}^{-1} \langle h^s \rangle x_{n-1} t u_{n+1} \langle k^s \rangle u_{n+1}^{-1}$. Hence, we need to find $N \in \mathscr{N}$ such that $\bar{y} \notin \langle \bar{h}^s \rangle \bar{x} \langle \bar{k}^s \rangle$ for each of the following four cases:

1-a. Suppose $\langle x_{n-1}^{-1} h x_{n-1} \rangle \cap \langle h \rangle = 1$ and $\langle u_{n+1}^{-1} k u_{n+1} \rangle \cap \langle k \rangle = 1$.

We note that $i_2 + j_1 \neq 0$, since, in (3.3), $h^{i_2}tk^{j_1} = h^{i_2+j_1}t$. Let $\epsilon > |i_2 + j_1|$. By induction, G is $\langle h^{\epsilon}\rangle x_{n-1}\langle h^{\epsilon}\rangle$-separable and $\langle k^{\epsilon}\rangle u_{n+1}\langle k^{\epsilon}\rangle$-separable. Hence, by Lemma 3.3, there exist $M_1 \triangleleft_f G$ and $M_2 \triangleleft_f G$ such that if $\tilde{x}_{n-1}^{-1}\tilde{h}^i\tilde{x}_{n-1} = \tilde{h}^j$ in G/M_1 then $\epsilon|i,j$ and that if $\hat{u}_{n+1}^{-1}\hat{k}^i\hat{u}_{n+1} = \hat{k}^j$ in G/M_2 then $\epsilon|i,j$. Let $N \in \mathcal{N}$ such that $N \subset M_1 \cap M_2 \cap N_1$ as in Lemma 2.5. In $\overline{G} = G\pi_N$, if $\bar{y} \in \langle \bar{h}^s\rangle\bar{x}\langle\bar{k}^s\rangle$, then $\bar{y} = \bar{h}^{si}\bar{x}\bar{k}^{sj}$. Thus, by the equation (3.2), we have $\bar{y}_{n-1} = \bar{h}^{si}\bar{x}_{n-1}\bar{h}^{\alpha} = \bar{h}^{si_1}\bar{x}_{n-1}\bar{h}^{i_2}$ and $\bar{v}_{n+1} = \bar{k}^{-\alpha}\bar{u}_{n+1}\bar{k}^{sj} = \bar{k}^{j_1}\bar{u}_{n+1}\bar{k}^{sj_2}$ for some α. This implies that

$$\bar{x}_{n-1}^{-1}\bar{h}^{s(i-i_1)}\bar{x}_{n-1} = \bar{h}^{i_2-\alpha} \quad \text{and} \quad \bar{u}_{n+1}^{-1}\bar{k}^{-\alpha-j_1}\bar{u}_{n+1} = \bar{k}^{s(j_2-j)}. \tag{3.4}$$

Thus $\epsilon|i_2 - \alpha$ and $\epsilon| -\alpha - j_1$ by the choice of $N \subset M_1 \cap M_2$. Hence $\epsilon|i_2 + j_1$, contradicting the choice of ϵ. Hence $\bar{y} \notin \langle\bar{h}^s\rangle\bar{x}\langle\bar{k}^s\rangle$.

1-b. Suppose $\langle x_{n-1}^{-1}hx_{n-1}\rangle \cap \langle h\rangle = 1$ and $\langle u_{n+1}^{-1}ku_{n+1}\rangle \cap \langle k\rangle \neq 1$. Let $\alpha_2 > 0$ be the smallest integer such that $u_{n+1}^{-1}k^{\alpha_2}u_{n+1} \in \langle k\rangle$. Then, by Remark 2.6, we have $u_{n+1}^{-1}k^{\alpha_2}u_{n+1} = k^{\pm\alpha_2}$. By the equation (3.3), since $h^{i_2}tk^{j_1} = tk^{i_2+j_1}$, we note that $k^{i_2+j_1} \notin \langle k^{\alpha_2}\rangle \cap \langle k^s\rangle = \langle k^{\alpha_2 s/d_2}\rangle$, where $d_2 = (\alpha_2, s)$. Thus $\alpha_2 s/d_2$ does not divide $i_2 + j_1$. Let $\epsilon = (\alpha_2 s/d_2)l$ and, as before, we can find $N \in \mathcal{N}$ such that $N \subset N_1$ and $\bar{u}_{n+1}^{-1}\bar{k}^i\bar{u}_{n+1} \notin \langle\bar{k}\rangle$ for all $1 \leq i < \alpha_2$, and if $\bar{x}_{n-1}^{-1}\bar{h}^i\bar{x}_{n-1} = \bar{h}^j$ then $\epsilon|i,j$. Thus, in $\overline{G} = G\pi_N$, if $\bar{y} \in \langle\bar{h}^s\rangle\bar{x}\langle\bar{k}^s\rangle$ then equations of (3.4) hold. This implies $\epsilon|i_2 - \alpha$ and $\alpha_2| -\alpha - j_1$. Now $s|\epsilon$ and $\epsilon|\,|\bar{h}| = |\bar{k}|$. Hence $s|\,|\bar{k}|$. This implies $s| -\alpha - j_1$ by the second equation of (3.4). Therefore $(\alpha_2 s/d_2)|i_2 - \alpha - (-\alpha - j_1) = i_2 + j_1$, a contradiction. Hence $\bar{y} \notin \langle\bar{h}^s\rangle\bar{x}\langle\bar{k}^s\rangle$.

1-c. Suppose $\langle x_{n-1}^{-1}hx_{n-1}\rangle \cap \langle h\rangle \neq 1$ and $\langle u_{n+1}^{-1}ku_{n+1}\rangle \cap \langle k\rangle = 1$. Let $\alpha_1 > 0$ be the smallest integer such that $x_{n-1}h^{\alpha_1}x_{n-1}^{-1} \in \langle h\rangle$. Then by Remark 2.6 $x_{n-1}h^{\alpha_1}x_{n-1}^{-1} = h^{\pm\alpha_1}$. Since $h^{i_2}tk^{j_1} = h^{i_2+j_1}t$, from the equation (3.3), we have $h^{i_2+j_1} \notin \langle h\alpha_1\rangle \cap \langle h^s\rangle = \langle h^{\alpha_1 s/d_1}\rangle$, where $d_1 = (\alpha_1, s)$. Thus $\alpha_1 s/d_1$ does not divide $i_2 + j_1$. Hence, as in 1-b above, for $\epsilon = (\alpha_1 s/d_1)l$, we can find $N \in \mathcal{N}$ such that $N \subset N_1$ and $\bar{y} \notin \langle\bar{h}^s\rangle\bar{x}\langle\bar{k}^s\rangle$ in $\overline{G} = G\pi_N$.

1-d. Suppose $\langle x_{n-1}^{-1}hx_{n-1}\rangle \cap \langle h\rangle \neq 1$ and $\langle u_{n+1}^{-1}ku_{n+1}\rangle \cap \langle k\rangle \neq 1$. As in 1-b and 1-c above, let $\alpha_1, \alpha_2 > 0$ be the smallest integer such that $x_{n-1}h^{\alpha_1}x_{n-1}^{-1} \in \langle h\rangle$ and $u_{n+1}^{-1}k^{\alpha_2}u_{n+1} \in \langle k\rangle$. By Eq. (3.3), $h^{i_2+j_1} \notin (\langle h^{\alpha_1}\rangle \cap \langle h^s\rangle)(\langle h^{\alpha_2}\rangle \cap \langle h^s\rangle) = \langle h^{\alpha_1 s/d_1}\rangle\langle h^{\alpha_2 s/d_2}\rangle$, where $d_1 = (\alpha_1, s)$ and $d_2 = (\alpha_2, s)$. Hence, as before, for $\epsilon = (\alpha_1 s/d_1)(\alpha_2 s/d_2)l$ we can find $N \in \mathcal{N}$ such that $N \subset N_1$ and, in $\overline{G} = G\pi_N$, $\bar{x}_{n-1}\bar{h}^i\bar{x}_{n-1}^{-1} \notin \langle\bar{h}\rangle$ for all $1 \leq i < \alpha_1$, $\bar{u}_{n+1}^{-1}\bar{k}^j\bar{u}_{n+1} \notin \langle\bar{k}\rangle$ for all $1 \leq j < \alpha_2$, and, by Remark 3.2, $\bar{h}^{i_2+j_1} \notin \langle\bar{h}^{\alpha_1 s/d_1}\rangle\langle\bar{h}^{\alpha_2 s/d_2}\rangle$. In $\overline{G} = G\pi_N$, if $\bar{y} \in \langle\bar{h}^s\rangle\bar{x}\langle\bar{k}^s\rangle$ then we have the equation (3.4). Thus, from the first equation of (3.4), we have $\alpha_1|i_2 - \alpha$;

hence $\bar{h}^{i_2-\alpha} = \bar{h}^{\pm s(i-i_1)}$. Since $s|\epsilon$ and $\epsilon|\,|\bar{h}|$, $s|i_2-\alpha$. Thus $(\alpha_1 s/d_1)|i_2-\alpha$. Hence $\bar{h}^{i_2-\alpha} \in \langle \bar{h}^{\alpha_1 s/d_1}\rangle$. Similarly, from the second equation of (3.4), $\bar{h}^{-\alpha-j_1} \in \langle \bar{h}^{\alpha_2 s/d_2}\rangle$. Therefore, $\bar{h}^{i_2+j_1} = \bar{h}^{i_2-\alpha}\bar{h}^{\alpha+j_1} \in \langle \bar{h}^{\alpha_1 s/d_1}\rangle\langle \bar{h}^{\alpha_2 s/d_2}\rangle$, a contradiction. Hence $\bar{y} \notin \langle \bar{h}^s\rangle\bar{x}\langle \bar{k}^s\rangle$.

Subcase 2. Suppose $\epsilon_n = -1$. If $y_{n-1} \notin \langle h^s\rangle x_{n-1}\langle k\rangle$ or $v_{n+1} \notin \langle h\rangle u_{n+1}\langle k^s\rangle$ then by induction, as in Subcase 1 above, we can find $N \in \mathcal{N}$ such that, in $\bar{G} = G\pi_N$, $\|\bar{x}\| = \|x\|$, $\|\bar{y}\| = \|y\|$, and $\bar{y}_{n-1} \notin \langle \bar{h}^s\rangle\bar{x}_{n-1}\langle \bar{k}\rangle$ or $\tilde{v}_{n+1} \notin \langle \bar{h}\rangle\bar{u}_{n+1}\langle \bar{k}^s\rangle$. Moreover, if $\bar{y} = \bar{h}^{si}\bar{x}\bar{k}^{sj}$ for some i, j, then $\tau\bar{x}_{n-1}^{-1}\bar{h}^{-si}\bar{y}_{n-1}\tau^{-1} = \bar{u}_{n+1}\bar{k}^{sj}\bar{v}_{n+1}^{-1}$. This implies

$$\bar{x}_{n-1}^{-1}\bar{h}^{si}\bar{y}_{n-1} = \bar{k}^{\alpha} \quad \text{and} \quad \bar{h}^{\alpha} = \bar{u}_{n+1}\bar{k}^{sj}\bar{v}_{n+1}^{-1} \tag{3.5}$$

for some α. This means $\bar{y}_{n-1} \in \langle \bar{h}^s\rangle\bar{x}_{n-1}\langle \bar{k}\rangle$ and $\bar{v}_{n+1} \in \langle \bar{h}\rangle\bar{u}_{n+1}\langle \bar{k}^s\rangle$, contradicting the choice of N. Hence $\bar{y} \notin \langle \bar{h}^s\rangle\bar{x}\langle \bar{k}^s\rangle$. So we can assume $y_{n-1} = h^{si_1}x_{n-1}k^{i_2}$ and $v_{n+1} = h^{j_1}u_{n+1}k^{sj_2}$ for some i_1, i_2, j_1, j_2. This implies

$$y = y_{n-1}t^{-1}v_{n+1} = h^{si_1}x_{n-1}k^{i_2}t^{-1}h^{j_1}u_{n+1}k^{sj_2} \notin \langle h^s\rangle x_{n-1}t^{-1}u_{n+1}\langle k^s\rangle. \tag{3.6}$$

This implies that $k^{i_2}th^{j_1} \notin x_{n-1}^{-1}\langle h^s\rangle x_{n-1}tu_{n+1}\langle k^s\rangle u_{n+1}^{-1}$. Hence, we need to find $N \in \mathcal{N}$ such that $\bar{y} \notin \langle \bar{h}^s\rangle\bar{x}\langle \bar{k}^s\rangle$ for each of the following four cases:

2-a. Suppose $\langle x_{n-1}^{-1}hx_{n-1}\rangle \cap \langle k\rangle = 1$ and $\langle u_{n+1}^{-1}hu_{n+1}\rangle \cap \langle k\rangle = 1$.

Equation (3.6) implies $i_2 + j_1 \neq 0$. Let $\epsilon > |i_2 + j_1|$. By induction, G is $\langle h^\epsilon\rangle x_{n-1}\langle k^\epsilon\rangle$-separable and $\langle h^\epsilon\rangle u_{n+1}\langle k^\epsilon\rangle$-separable. Hence, by Lemma 3.3, there exist $M_1 \lhd_f G$ and $M_2 \lhd_f G$ such that if $\tilde{x}_{n-1}^{-1}\tilde{h}^i\tilde{x}_{n-1} = \tilde{k}^j$ in G/M_1 then $\epsilon|i, j$ and that if $\hat{u}_{n+1}^{-1}\hat{h}^i\hat{u}_{n+1} = \hat{k}^j$ in G/M_2 then $\epsilon|i, j$. Let $N \in \mathcal{N}$ such that $N \subset M_1 \cap M_2 \cap N_1$ as in 1-a of Subcase 1. In $\bar{G} = G\pi_N$, if $\bar{y} \in \langle \bar{h}^s\rangle\bar{x}\langle \bar{k}^s\rangle$, then $\bar{y} = \bar{h}^{si}\bar{x}\bar{k}^{sj}$. Thus, by Eq. (3.5), we have $\bar{y}_{n-1} = \bar{h}^{si}\bar{x}_{n-1}\bar{k}^{\alpha} = \bar{h}^{si_1}\bar{x}_{n-1}\bar{k}^{i_2}$ and $\bar{v}_{n+1} = \bar{h}^{-\alpha}\bar{u}_{n+1}\bar{k}^{sj} = \bar{h}^{j_1}\bar{u}_{n+1}\bar{k}^{sj_2}$ for some α. This implies that $\bar{x}_{n-1}^{-1}\bar{h}^{s(i-i_1)}\bar{x}_{n-1} = \bar{k}^{i_2-\alpha}$ and $\bar{u}_{n+1}^{-1}\bar{h}^{-\alpha-j_1}\bar{u}_{n+1} = \bar{k}^{s(j_2-j)}$. Thus, by the choice of $N \subset M_1 \cap M_2$, $\epsilon|i_2-\alpha$ and $\epsilon|-\alpha-j_1$. Hence $\epsilon|i_2+j_1$, contradicting the choice of ϵ. Hence $\bar{y} \notin \langle \bar{h}^s\rangle\bar{x}\langle \bar{k}^s\rangle$.

The following three cases are also similar to Subcase 1 above.

2-b. Suppose $\langle x_{n-1}^{-1}hx_{n-1}\rangle \cap \langle k\rangle = 1$ and $\langle u_{n+1}^{-1}hu_{n+1}\rangle \cap \langle k\rangle \neq 1$.

2-c. Suppose $\langle x_{n-1}^{-1}hx_{n-1}\rangle \cap \langle k\rangle \neq 1$ and $\langle u_{n+1}^{-1}hu_{n+1}\rangle \cap \langle k\rangle = 1$.

2-d. Suppose $\langle x_{n-1}^{-1}hx_{n-1}\rangle \cap \langle k\rangle \neq 1$ and $\langle u_{n+1}^{-1}hu_{n+1}\rangle \cap \langle k\rangle \neq 1$.

Hence there exists $N \in \mathcal{N}$ such that, in $\bar{G} = G\pi_N$, $\bar{y} \notin \langle \bar{h}^s\rangle\bar{x}\langle \bar{k}^s\rangle$. Therefore, by induction, G is $\langle h^s\rangle x\langle k^s\rangle$-separable for any $x \in G$.

Case 2. G is $\langle h^s \rangle x \langle h^s \rangle$-separable for any $x \in G$ with $\|x\| = n$.

Let $y \in G$ such that $y \notin \langle h^s \rangle x \langle h^s \rangle$. If $\|y\| > n$ or $\|y\| < n$ or $y = v_1 t^{\delta_1} v_2 \cdots t^{\delta_n} v_{n+1}$ and $x = u_1 t^{\epsilon_1} u_2 \cdots t^{\epsilon_n} u_{n+1}$ with $\delta_i \neq \epsilon_i$ for some i then, as in Case 1, we can find $N_1 \in \mathscr{N}$ such that, in $\overline{G} = G\pi_{N_1}$, $\|\bar{y}\| = \|y\|$, $\|\bar{x}\| = \|x\|$, and $\bar{y} \notin \langle \bar{h}^s \rangle \bar{x} \langle \bar{h}^s \rangle$. So we consider $y = v_1 t^{\epsilon_1} v_2 \cdots t^{\epsilon_n} v_{n+1}$ and $x = u_1 t^{\epsilon_1} u_2 \cdots t^{\epsilon_n} u_{n+1}$ where $y \notin \langle h^s \rangle x \langle h^s \rangle$. As before, we let $x_{n-1} = u_1 t^{\epsilon_1} u_2 \cdots t^{\epsilon_{n-1}} u_n$ and $y_{n-1} = v_1 t^{\epsilon_1} v_2 \cdots t^{\epsilon_{n-1}} v_n$. By Lemma 2.5, there exists $N_1 \in \mathscr{N}$ such that $\|\tilde{x}\| = \|\tilde{y}\| = n$ in $\tilde{G} = G\pi_{N_1}$.

Subcase 1. Suppose $\epsilon_n = 1$. If $y_{n-1} \notin \langle h^s \rangle x_{n-1} \langle h \rangle$ or $v_{n+1} \notin \langle k \rangle u_{n+1} \langle h^s \rangle$ then by induction, as in Subcase 1 of Case 1, we can find $N \in \mathscr{N}$ such that, in $\overline{G} = G\pi_N$, $\|\bar{x}\| = \|x\|$, $\|\bar{y}\| = \|y\|$ and $\bar{y}_{n-1} \notin \langle \bar{h}^s \rangle \bar{x}_{n-1} \langle \bar{h} \rangle$ or $\bar{v}_{n+1} \notin \langle \bar{k} \rangle \bar{u}_{n+1} \langle \bar{h}^s \rangle$. Then, in $\overline{G} = G\pi_N$, $\bar{y} \notin \langle \bar{h}^s \rangle \bar{x} \langle \bar{h}^s \rangle$ as in Case 1. Hence we can assume $y_{n-1} = h^{si_1} x_{n-1} h^{i_2}$ and $v_{n+1} = k^{j_1} u_{n+1} h^{sj_2}$ for some i_1, i_2, j_1, j_2. This implies

$$y = y_{n-1} t v_{n+1} = h^{si_1} x_{n-1} h^{i_2} t k^{j_1} u_{n+1} h^{sj_2} \notin \langle h^s \rangle x_{n-1} t u_{n+1} \langle h^s \rangle. \quad (3.7)$$

As in case 1, we have the following cases:

1-a. Suppose $\langle x_{n-1}^{-1} h x_{n-1} \rangle \cap \langle h \rangle = 1$ and $\langle u_{n+1}^{-1} k u_{n+1} \rangle \cap \langle h \rangle = 1$.
1-b. Suppose $\langle x_{n-1}^{-1} h x_{n-1} \rangle \cap \langle h \rangle = 1$ and $\langle u_{n+1}^{-1} k u_{n+1} \rangle \cap \langle h \rangle \neq 1$.
1-c. Suppose $\langle x_{n-1}^{-1} h x_{n-1} \rangle \cap \langle h \rangle \neq 1$ and $\langle u_{n+1}^{-1} k u_{n+1} \rangle \cap \langle h \rangle = 1$.
1-d. Suppose $\langle x_{n-1}^{-1} h x_{n-1} \rangle \cap \langle h \rangle \neq 1$ and $\langle u_{n+1}^{-1} k u_{n+1} \rangle \cap \langle h \rangle \neq 1$.

In the same manner we can find $N \in \mathscr{N}$ such that $\bar{y} \notin \langle \bar{h}^s \rangle \bar{x} \langle \bar{h}^s \rangle$ in $\overline{G} = G\pi_N$ for each of the cases.

Subcase 2. Suppose $\epsilon_n = -1$. If $y_{n-1} \notin \langle h^s \rangle x_{n-1} \langle k \rangle$ or $v_{n+1} \notin \langle h \rangle u_{n+1} \langle h^s \rangle$ then, as in Subcase 2 of Case 1, we can find $N \in \mathscr{N}$ such that $\bar{y} \notin \langle \bar{h}^s \rangle \bar{x} \langle \bar{h}^s \rangle$ in $\overline{G} = G\pi_N$. Thus we can assume $y_{n-1} = h^{si_1} x_{n-1} k^{i_2}$ and $v_{n+1} = h^{j_1} u_{n+1} h^{sj_2}$ for some i_1, i_2, j_1, j_2. This implies

$$y = y_{n-1} t^{-1} v_{n+1} = h^{si_1} x_{n-1} k^{i_2} t^{-1} h^{j_1} u_{n+1} h^{sj_2} \notin \langle h^s \rangle x_{n-1} t^{-1} u_{n+1} \langle h^s \rangle. \quad (3.8)$$

Again we have the following cases:

2-a. Suppose $\langle x_{n-1}^{-1} h x_{n-1} \rangle \cap \langle k \rangle = 1$ and $\langle u_{n+1}^{-1} h u_{n+1} \rangle \cap \langle h \rangle = 1$.
2-b. Suppose $\langle x_{n-1}^{-1} h x_{n-1} \rangle \cap \langle k \rangle = 1$ and $\langle u_{n+1}^{-1} h u_{n+1} \rangle \cap \langle h \rangle \neq 1$.
2-c. Suppose $\langle x_{n-1}^{-1} h x_{n-1} \rangle \cap \langle k \rangle \neq 1$ and $\langle u_{n+1}^{-1} h u_{n+1} \rangle \cap \langle h \rangle = 1$.
2-d. Suppose $\langle x_{n-1}^{-1} h x_{n-1} \rangle \cap \langle k \rangle \neq 1$ and $\langle u_{n+1}^{-1} h u_{n+1} \rangle \cap \langle h \rangle \neq 1$.

In each of the cases, we can find $N \in \mathcal{N}$ such that $\bar{y} \notin \langle \bar{h}^s \rangle \bar{x} \langle \bar{h}^s \rangle$ in $\overline{G} = G\pi_N$. Hence, as in Case 1, by induction G is $\langle h^s \rangle x \langle h^s \rangle$-separable for any $x \in G$.

Similarly, G is $\langle k^s \rangle x \langle k^s \rangle$-separable for any $x \in G$. Therefore, G is double coset separable at $\{h, k\}$. ∎

DEFINITION 3.5. Let A be a group and let $h, k \in A$ be of infinite order. Then A is said to be *s-quasi-regular* at $\{h, k\}$ if $\langle h \rangle \cap \langle k \rangle = 1$ and if, for each integer $\epsilon > 0$, there exist an integer $\lambda_\epsilon > 0$ and $N_\epsilon \triangleleft_f A$ such that $N_\epsilon \cap \langle h \rangle = \langle h^{\epsilon\lambda_\epsilon} \rangle$, $N_\epsilon \cap \langle k \rangle = \langle k^{\epsilon\lambda_\epsilon} \rangle$, and $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = 1$, in $\overline{A} = A/N_\epsilon$.

Clearly if A is s-quasi-regular at $\{h, k\}$, then A is quasi-regular at $\{h, k\}$. Hence, as in Lemma 2.5, we have the following:

LEMMA 3.6. *Let $G = \langle A, t \colon t^{-1}ht = k \rangle$. Suppose A is $\langle h \rangle$-separable, $\langle k \rangle$-separable, and s-quasi-regular at $\{h, k\}$. Then, for each $M \triangleleft_f A$, for each $s > 0$ and for each reduced $x \in G$, there exist $N \triangleleft_f A$ and $\lambda > 0$ such that $N \subset M$, $N \cap \langle h \rangle = \langle h^{s\lambda} \rangle$, $N \cap \langle k \rangle = \langle k^{s\lambda} \rangle$, $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = 1$, and $\|\bar{x}\| = \|x\|$ in $\overline{G} = G\pi_N$.*

THEOREM 3.7. *Let A be s-quasi-regular and double coset separable at $\{h, k\}$. Let $x \in G = \langle A, t \colon t^{-1}ht = k \rangle$ be cyclically reduced. If $\|x\| \geq 1$ then, for each $y \in G$ such that $x \nsim_G y$, there exists $N \triangleleft_f G$ such that $x \nsim_{\overline{G}} \bar{y}$, where $\overline{G} = G/N$.*

Proof. WLOG, we may assume that y is also cyclically reduced in G.

Case 1 ($\|x\| \neq \|y\|$). Since x is cyclically reduced, by Lemma 2.5 we can find $N \in \mathcal{N}$ such that, in $\overline{G} = G\pi_N$, $\|\bar{x}\| = \|x\|$, $\|\bar{y}\| = \|y\|$, and $\bar{x}, \bar{y}$ are cyclically reduced. This implies $\|\bar{x}\| \neq \|\bar{y}\|$. Hence, by Theorem 2.1, $\bar{x} \nsim_{\overline{G}} \bar{y}$.

Case 2 ($\|x\| = \|y\| \geq 1$). Let $x = t^{\epsilon_1}u_1t^{\epsilon_2} \cdots u_{n-1}t^{\epsilon_n}u_n$ and $y = t^{\delta_1}v_1t^{\delta_2} \cdots v_{n-1}t^{\delta_n}v_n$ be cyclically reduced, where $u_i, v_i \in A$. Since $x \nsim_G y$, by Theorem 2.1, $x \nsim_{\langle h \rangle \cup \langle k \rangle} y_i^*$ for any cyclic permutation $y_i^* = t^{\delta_i}v_i \cdots t^{\delta_n}v_nt^{\delta_1}v_1 \cdots t^{\delta_{i-1}}v_{i-1}$ of y. For each i, we shall find $N_i \in \mathcal{N}$ such that, in $G\pi_{N_i} = \widehat{G}$, $\|x\pi N_i\| = \|x\| = \|y_i^*\pi_{N_i}\| = \|y\|$ and $x\pi_{N_i}, y_i^*\pi_{N_i}$ are cyclically reduced with $x\pi_{N_i} \nsim_{\langle \hat{h} \rangle \cup \langle \hat{k} \rangle} y_i^*\pi_{N_i}$. Let $N = N_1 \cap \cdots \cap N_n$. Then, in $\overline{G} = G\pi_N$, we have $\|\bar{x}\| = \|x\| = \|\bar{y}\| = \|y\|$ with $\bar{x}, \bar{y}$ cyclically reduced and $\bar{x} \nsim_{\langle \bar{h} \rangle \cup \langle \bar{k} \rangle} \bar{y}_i^*$ for all i. Hence, by Theorem 2.2, $\bar{x} \nsim_{\overline{G}} \bar{y}$ as required.

To find such N_i's, it suffices to consider only the case $i = 1$, others being similar. As in Case 1, by Lemma 3.6 we can find $M_1 \in \mathcal{N}$ such that $\langle \hat{h} \rangle \cap \langle \bar{k} \rangle = 1$, $\|\hat{x}\| = \|x\| = \|\hat{y}\| = \|y\|$ with $\hat{x}, \hat{y}$ cyclically reduced in $\widehat{G} = G\pi_{M_1}$.

If $\epsilon_i \neq \delta_i$ for some i, then we choose $N_1 = M_1$. This implies $\bar{x} \nsim_{\langle \bar{h} \rangle \cup \langle \bar{k} \rangle} \bar{y}$ in $\overline{G} = G\pi_{N_1}$. Hence we can assume $\epsilon_i = \delta_i$ for all i and $\epsilon_1 = 1$, say.

Subcase 1. Suppose $y \notin \langle h \rangle x \langle h \rangle$. By Lemma 3.4, there exists $M_2 \triangleleft_f G$ such that $y \notin M_2 \langle h \rangle x \langle h \rangle$. Let $N_1 \in \mathscr{N}$ such that $N_1 \subset M_1 \cap M_2$. Then $\bar{x} \nsim_{\langle \bar{h} \rangle} \bar{y}$ in $\overline{G} = G\pi_{N_1}$.

Subcase 2. Suppose $y = h^{\alpha} x h^{\beta}$ but $h^{\alpha} \neq h^{-\beta}$. In this case, we need to find $N_1 \in \mathscr{N}$ such that $\bar{x} \nsim_{\langle \bar{h} \rangle} \bar{y}$ in $\overline{G} = G\pi_{N_1}$ in each of the following cases:

(a) Suppose $\langle x^{-1} h x \rangle \cap \langle h \rangle = 1$. By Lemma 3.4, G is $\langle h \rangle x \langle h \rangle$-separable. Moreover, by Lemma 3.3, for $\epsilon > |\alpha + \beta|$, there exists $M_2 \triangleleft_f G$ such that if $M_2(x^{-1} h^i x) = M_2 h^j$ then $\epsilon | i, j$. Furthermore, by Lemma 3.6, there exists $N_1 \in \mathscr{N}$ such that $N_1 \subset M_1 \cap M_2$, $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = 1$, and $\|\bar{x}\| = \|x\| = \|\bar{y}\| = \|y\|$ in $\overline{G} = G\pi_{N_1}$. If $\bar{y} \sim_{\langle \bar{h} \rangle} \bar{x}$ then $\bar{y} = \bar{h}^{-i} \bar{x} \bar{h}^{i} = \bar{h}^{\alpha} \bar{x} \bar{h}^{\beta}$. This implies $\bar{x}^{-1} \bar{h}^{-i-\alpha} \bar{x} = \bar{h}^{\beta - i}$, whence $\epsilon | -i - \alpha, \beta - i$. Therefore $\epsilon | \alpha + \beta$, contradicting the choice of ϵ. Hence $\bar{x} \nsim_{\langle \bar{h} \rangle} \bar{y}$ in $\overline{G} = G\pi_{N_1}$.

(b) Suppose there exists an integer $s > 0$ such that $x^{-1} h^s x = h^{-s}$. WLOG, we can assume s to be the smallest such integer. If $h^{\alpha+\beta} = h^{2sr}$ then $y = h^{\alpha} x h^{\beta} = h^{-\beta+2sr} x h^{\beta} = h^{-\beta+sr} x h^{\beta - sr}$, whence $y \sim_{\langle h \rangle} x$. Hence $h^{\alpha+\beta} \notin \langle h^{2s} \rangle$. Since G is $\langle h^{\epsilon} \rangle$-separable for any ϵ by Lemma 3.4, there exists $M_2 \triangleleft_f G$ such that $h^{\alpha+\beta} \notin M_2 \langle h^{2s} \rangle$ and $x^{-1} h^i x \notin M_2 \langle h \rangle$ for $1 \leq i < s$. By Lemma 3.6, we can choose $N_1 \in \mathscr{N}$ such that $N_1 \subset M_1 \cap M_2$, $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = 1$, $\bar{h}^{\alpha+\beta} \notin \langle \bar{h}^{2s} \rangle$, and $\bar{x}^{-1} \bar{h}^i \bar{x} \notin \langle \bar{h} \rangle$ for $1 \leq i < s$ in $\overline{G} = G\pi_{N_1}$. If $\bar{y} = \bar{h}^{-i} \bar{x} \bar{h}^{i}$ then, as before, we have $\bar{x}^{-1} \bar{h}^{-i-\alpha} \bar{x} = \bar{h}^{\beta-i}$. This implies $s | -i - \alpha$. Let $-i - \alpha = s\mu$ for some μ, whence $\bar{h}^{\beta - i} = \bar{x}^{-1} \bar{h}^{s\mu} \bar{x} = \bar{h}^{-s\mu}$. Thus $\bar{h}^{\alpha+\beta} = \bar{h}^{i+\alpha} \bar{h}^{\beta-i} = \bar{h}^{-2s\mu} \in \langle \bar{h}^{2s} \rangle$, contradicting $\bar{h}^{\alpha+\beta} \notin \langle \bar{h}^{2s} \rangle$. Hence $\bar{x} \nsim_{\langle \bar{h} \rangle} \bar{y}$ in $\overline{G} = G\pi_{N_1}$.

(c) Suppose there exists an integer $s > 0$ such that $x^{-1} h^s x = h^s$. WLOG, we can assume s to be the smallest such integer. Since, by Lemma 3.4, G is $\langle h^{\epsilon} \rangle$-separable for any ϵ, we can find $M_2 \triangleleft_f G$ such that $h^{\alpha+\beta} \notin M_2$ and $x^{-1} h^i x \notin M_2 \langle h \rangle$ for $1 \leq i < s$. Now, by Lemma 3.6, we can choose $N_1 \in \mathscr{N}$ such that $N_1 \subset M_1 \cap M_2$ and $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = 1$, so that in $\overline{G} = G\pi_{N_1}$, if $\bar{y} = \bar{h}^{-i} \bar{x} \bar{h}^{i}$ then $\bar{x}^{-1} \bar{h}^{-i-\alpha} \bar{x} = \bar{h}^{\beta-i}$ and $s | -i - \alpha$. Let $-i - \alpha = s\mu$ for some μ. Then $\bar{h}^{\beta-i} = \bar{x}^{-1} \bar{h}^{s\mu} \bar{x} = \bar{h}^{s\mu}$. This implies $\bar{h}^{\alpha+\beta} = \bar{h}^{i+\alpha} \bar{h}^{\beta-i} = 1$, contradicting $h^{\alpha+\beta} \notin M_2$. Hence $\bar{x} \nsim_{\langle \bar{h} \rangle} \bar{y}$ in $\overline{G} = G\pi_{N_1}$.

Thus, in each case, we have found N_1 such that, in $\overline{G} = G\pi_{N_1}$, $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = 1$ and $\bar{x} \nsim_{\langle \bar{h} \rangle} \bar{y}$. Since $\epsilon_1 = 1$ and $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = 1$, $\bar{x} \nsim_{\langle \bar{k} \rangle} \bar{y}$. Hence $\bar{x} \nsim_{\langle \bar{h} \rangle \cup \langle \bar{k} \rangle} \bar{y}$ in $G\pi_{N_1}$, as required. ∎

4. A CRITERION

In this section we shall prove a criterion for the conjugacy separability of HNN extensions.

DEFINITION 4.1. Let A be a group and let $h, k \in A$ be of infinite order. Then A is said to be *c-quasi-regular* at $\{h, k\}$ if $\langle h \rangle \cap \langle k \rangle = 1$ and there exists an integer α such that, for a fixed integer $m > 0$ and for each integer $\epsilon > 0$, there exist an integer $\lambda_\epsilon > 0$ and $N_\epsilon \triangleleft_f A$ such that: (1) $N_\epsilon \cap \langle h \rangle = \langle h^{\epsilon\lambda_\epsilon} \rangle$ and $N_\epsilon \cap \langle k \rangle = \langle k^{\epsilon\lambda_\epsilon} \rangle$, (2) in $\overline{A} = A/N_\epsilon$, $\langle \overline{h} \rangle \cap \langle \overline{k} \rangle = 1$, (3) if $\overline{h}^{\alpha m} \sim_{\overline{A}} \overline{h}^j$ then $\overline{h}^j = \overline{h}^{\pm \alpha m}$, (4) if $\overline{k}^{\alpha m} \sim_{\overline{A}} \overline{k}^j$ then $\overline{k}^j = \overline{k}^{\pm \alpha m}$, and (5) if $\overline{h}^i \sim_{\overline{A}} \overline{k}^j$ then $\overline{h}^i = \overline{k}^j = 1$.

For example, if A is polycyclic-by-finite and if $h, k \in Z(A)$ are of infinite order then A is c-quasi-regular at $\{h, k\}$ whenever $\langle h \rangle \cap \langle k \rangle = 1$. Clearly if A is c-quasi-regular at $\{h, k\}$, then A is s-quasi-regular at $\{h, k\}$ and, hence, quasi-regular at $\{h, k\}$.

Remark 4.2. If A is c-quasi-regular at $\{h, k\}$ and $h^i \sim_G k^j$, then $i = j = 0$. For, by the above definition, for each $\epsilon > 0$, there exist λ_ϵ and $N \triangleleft_f A$ such that $N \cap \langle h \rangle = \langle h^{\epsilon\lambda_\epsilon} \rangle$ and $N \cap \langle k \rangle = \langle k^{\epsilon\lambda_\epsilon} \rangle$. Then, in $\overline{A} = A\pi_N$, $\overline{h}^i \sim_{\overline{A}} \overline{k}^j$ implies $\overline{h}^i = \overline{k}^j = 1$. Thus i, j must be divisible by every $\epsilon > 0$. Hence $i = 0 = j$.

LEMMA 4.3. *If A is c-quasi-regular at $\{h, k\}$ then, for each $M \triangleleft_f A$, there exists an integer α such that, for a fixed integer $m > 0$ and for each integer $s > 0$, there exist λ_s and $N_s \triangleleft_f A$ such that $N_s \subset M$, $N_s \cap \langle h \rangle = \langle h^{s\lambda_s} \rangle$, and $N_s \cap \langle k \rangle = \langle k^{s\lambda_s} \rangle$ such that, in $\overline{A} = A/N_s$, $\langle \overline{h} \rangle \cap \langle \overline{k} \rangle = 1$, if $\overline{h}^{\alpha m} \sim_{\overline{A}} \overline{h}^j$ then $\overline{h}^j = \overline{h}^{\pm \alpha m}$, if $\overline{k}^{\alpha m} \sim_{\overline{A}} \overline{k}^j$ then $\overline{k}^j = \overline{k}^{\pm \alpha m}$, and if $\overline{h}^i \sim_{\overline{A}} \overline{k}^j$ then $\overline{h}^i = \overline{k}^j = 1$.*

Proof. Let $M \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $M \cap \langle k \rangle = \langle k^{s_2} \rangle$. Let $\epsilon = s_1 s_2 s$. By Definition 4.1, there exist $\lambda > 0$ and $N_1 \triangleleft_f A$ such that $N_1 \cap \langle h \rangle = \langle h^{\epsilon\lambda} \rangle$, $N_1 \cap \langle k \rangle = \langle k^{\epsilon\lambda} \rangle$, and in $\tilde{A} = A/N_1$, $\langle \tilde{h} \rangle \cap \langle \tilde{k} \rangle = 1$, if $\tilde{h}^{\alpha m} \sim_{\tilde{A}} \tilde{h}^j$ then $\tilde{h}^j = \tilde{h}^{\pm \alpha m}$, if $\tilde{k}^{\alpha m} \sim_{\tilde{A}} \tilde{k}^j$ then $\tilde{k}^j = \tilde{k}^{\pm \alpha m}$, and if $\tilde{h}^i \sim_{\tilde{A}} \tilde{k}^j$ then $\tilde{h}^i = \tilde{k}^j = 1$. Let $N_s = N_1 \cap M$ and $\lambda_s = s_1 s_2 \lambda$. Then $N_s \subset M$ and $N_s \cap \langle h \rangle = \langle h^{s_1 s_2 s \lambda} \rangle = \langle h^{s\lambda_s} \rangle$, $N_s \cap \langle k \rangle = \langle k^{s_1 s_2 s \lambda} \rangle = \langle k^{s\lambda_s} \rangle$. Since $N_s \cap \langle h \rangle = N_1 \cap \langle h \rangle$ and $N_s \cap \langle k \rangle = N_1 \cap \langle k \rangle$, we have, in $\overline{A} = A/N_s$, $\langle \overline{h} \rangle \cap \langle \overline{k} \rangle = 1$, if $\overline{h}^{\alpha m} \sim_{\overline{A}} \overline{h}^j$ then $\overline{h}^j = \overline{h}^{\pm \alpha m}$, if $\overline{k}^{\alpha m} \sim_{\overline{A}} \overline{k}^j$ then $\overline{k}^j = \overline{k}^{\pm \alpha m}$, and if $\overline{h}^i \sim_{\overline{A}} \overline{k}^j$ then $\overline{h}^i = \overline{k}^j = 1$. ∎

DEFINITION 4.4. A group G is said to be *cyclic conjugacy separable* for $\langle h \rangle$ if $\{x\}^G \cap \langle h \rangle = \varnothing$ for $x \in G$; then there exists $N \triangleleft_f G$ such that, in $\overline{G} = G/N$, $\{\overline{x}\}^{\overline{G}} \cap \langle \overline{h} \rangle = \varnothing$.

It was proved in [7] that free-by-finite groups are cyclic conjugacy separable for each $\langle h \rangle$. We are now ready to prove a criterion for the conjugacy separability of HNN extensions.

THEOREM 4.5. *Let A be π_c, conjugacy separable, c-quasi-regular, and double coset separable at $\{h, k\}$. If A is cyclic conjugacy separable for $\langle h \rangle$ and $\langle k \rangle$, then $G = \langle A, t\colon t^{-1}ht = k \rangle$ is conjugacy separable.*

Proof. Let $x, y \in G$ be such that $x \nsim_G y$ and that x, y are of minimal lengths in their respective conjugacy classes. By Theorem 2.7, G is π_c, whence residually finite. Therefore we may assume $x \neq 1 \neq y$. By Theorem 3.7, we need only prove the case $\|x\| = \|y\| = 0$. Also by Theorem 2.2, $\overline{G} = G\pi_N$ is conjugacy separable for each $N \in \mathscr{N}$. Therefore, throughout the proof, we shall find a suitable $N \in \mathscr{N}$ such that, in $\overline{G} = G\pi_N$, $\bar{x} \nsim_{\overline{G}} \bar{y}$. Since we need only consider the case $\|x\| = \|y\| = 0$, that is $x, y \in A$, we have the following three cases.

Case 1. $x, y \in \langle h \rangle$ (similarly $x, y \in \langle k \rangle$).

(a) Suppose $x = h^m$ and $y = h^n$, where $m \neq \pm n$. By the quasi-regularity, for $\epsilon = mn$, there exist λ and $N \in \mathscr{N}$ such that $|\bar{h}| = |\bar{k}| = \epsilon\lambda$. Then $|\bar{x}| = |\bar{h}^m|$ and $|\bar{y}| = |\bar{h}^n|$ are different; hence $\bar{x} \nsim_{\overline{G}} \bar{y}$ in $\overline{G} = G\pi_N$.

(b) Suppose $x = h^m$ and $y = h^{-m}$. This implies $h^m \nsim_A h^{-m}$ and $k^m \nsim_A k^{-m}$. Since A is conjugacy separable, there exists $N_1 \lhd_f A$ such that $N_1 h^m \nsim_{A/N_1} N_1 h^{-m}$ and $N_1 k^m \nsim_{A/N_1} N_1 k^{-m}$. Let $N_1 \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $N_1 \cap \langle k \rangle = \langle k^{s_2} \rangle$. By Lemma 4.3, there exists α such that, for $\epsilon = 2 s_1 s_2 \alpha m$, there exist $N \lhd_f A$ and λ such that $N \subset N_1$ and, in $\overline{A} = A/N$, $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = 1$ and $|\bar{h}| = \epsilon\lambda = |\bar{k}|$ satisfying the conditions (3)–(5) of Definition 4.1. Now, if $\bar{x} \sim_{\overline{G}} \bar{y}$ then, by Theorem 2.1, there exist $z_1, z_2, \ldots, z_r$ of elements in $\langle \bar{h} \rangle \cup \langle \bar{k} \rangle$ such that

$$\bar{h}^m \sim_{\overline{A}} z_1 \sim_{\overline{A}, \tau^*} z_2 \sim_{A, \tau^*} \cdots \sim_{\overline{A}, \tau^*} z_r \sim_{\overline{A}} \bar{h}^{-m}.$$

By the choice of N, $z_1 = \bar{h}^i \in \langle \bar{h} \rangle$. This implies $\bar{h}^{\alpha m} \sim_{\overline{A}} \bar{h}^{i\alpha}$. Thus $\bar{h}^{i\alpha} = \bar{h}^{\pm \alpha m}$. If $\bar{h}^{i\alpha} = \bar{h}^{-\alpha m}$ then $\epsilon\lambda = 2 s_1 s_2 \alpha m \lambda = |\bar{h}|$ divides $\alpha(i + m)$. Let $i + m = 2 s_1 s_2 \lambda \mu$ for some μ. Since $\bar{h}^m \sim_{\overline{A}} \bar{h}^i = z_1$, in $\tilde{A} = A/N_1$, $\tilde{h}^m \sim_{\tilde{A}} \tilde{h}^i = \tilde{h}^{-m + 2 s_1 s_2 m \lambda \mu} = \tilde{h}^{-m}$, contradicting $\tilde{h}^m \nsim_{\tilde{A}} \tilde{h}^{-m}$. Therefore $z_1^\alpha = \bar{h}^{i\alpha} = \bar{h}^{\alpha m}$. Now $z_1 \sim_{\overline{A}, \tau^*} z_2$. This implies $\bar{h}^{\alpha m} = z_1^\alpha \sim_{\overline{A}, \tau^*} z_2^\alpha$. Therefore, either $z_2^\alpha = \bar{h}^{\alpha m}$ (as above), or $z_2^\alpha = \tau^{-1} \bar{h}^{\alpha m} \tau = \bar{k}^{\alpha m}$. Repeating this process, we have either $z_r^\alpha = \bar{h}^{\alpha m}$ or $z_r^\alpha = \tau^{-1} \bar{h}^{\alpha m} \tau = \bar{k}^{\alpha m}$. Clearly $z_r^\alpha \neq \bar{k}^{\alpha m}$. On the other hand, $z_r \sim_{\overline{A}} \bar{h}^{-m}$ implies $z_r^\alpha = \bar{h}^{\alpha m} \sim_{\overline{A}} \bar{h}^{\alpha m}$. Then, as above, $\bar{h}^{-\alpha m} = \bar{h}^{\alpha m}$, contradicting $|\bar{h}| = 2 s_1 s_2 \alpha m \lambda > 2\alpha m$. Hence $\bar{x} \nsim_{\overline{G}} \bar{y}$.

Case 2. Suppose $x \in \langle h \rangle$ and $y \in \langle k \rangle$ (similarly $x \in \langle k \rangle$ and $y \in \langle h \rangle$).

Let $x = h^m$ and $y = k^n$. Since $k^n \sim_G h^n$, the case follows from Case 1.

Case 3. Suppose $x \in A$ and $\{x\}^A \cap \langle h \rangle = \emptyset$ (or $\{x\}^A \cap \langle k \rangle = \emptyset$). By cyclic conjugacy separability of $\langle h \rangle$, there exists $N_1 \lhd_f A$ such that in $\tilde{A} = A/N_1$, $\{\tilde{x}\}^{\tilde{A}} \cap \langle \tilde{h} \rangle = \emptyset$. Since A is conjugacy separable, there exists $N_2 \lhd_f A$ such that $N_2 x \nsim_{A/N_2} N_2 y$. By Lemma 4.3, we can choose $N \in \mathscr{N}$

such that $N \subset N_1 \cap N_2$. Let $\overline{G} = G\pi_N$. Since $\{\bar{x}\}^{\overline{A}} \cap \langle \overline{h} \rangle = \emptyset$ and $\bar{x} \nsim_{\overline{A}} \bar{y}$, by Theorem 2.1, $\bar{x} \nsim_{\overline{G}} \bar{y}$. ∎

The following is a generalization of Theorem 3.3 in [8].

COROLLARY 4.6. *If A is polycyclic-by-finite and $h, k \in Z(A)$ of infinite order such that $\langle h \rangle \cap \langle k \rangle = 1$ then $G = \langle A, t\colon t^{-1}ht = k \rangle$ is conjugacy separable.*

Proof. Since polycyclic groups are double coset separable [11], this implies A is π_c and double coset separable at $\{h, k\}$. It was also well known that polycyclic-by-finite groups are conjugacy separable [4].

We need to show that A is c-quasi-regular. For a given ϵ, consider $\tilde{A} = A/\langle h^\epsilon \rangle \langle k^\epsilon \rangle$. Since $\tilde{A}$ is again polycyclic-by-finite, $\tilde{A}$ is residually finite. Thus there exists $\tilde{N} \lhd_f \tilde{A}$ such that $\tilde{N}\tilde{h}^i \neq 1 \neq \tilde{N}\tilde{k}^j$ and $\tilde{N}\tilde{h}^i \neq \tilde{N}\tilde{k}^j$ for $1 \leq i, j < \epsilon$. Let N be the preimage of $\tilde{N}$ in A. Then $N \lhd_f A$. Moreover $N \cap \langle h \rangle = \langle h^\epsilon \rangle$, $N \cap \langle k \rangle = \langle k^\epsilon \rangle$ and, in $\overline{A} = A/N$, $\langle \overline{h} \rangle \cap \langle \overline{k} \rangle = 1$. Furthermore, since $h, k \in Z(A)$, if $\overline{h}^m \sim_{\overline{A}} \overline{h}^i$ then $\overline{h}^i = \overline{h}^m$, if $\overline{k}^m \sim_{\overline{A}} \overline{k}^i$ then $\overline{k}^i = \overline{k}^m$, if $\overline{h}^i \sim_{\overline{A}} \overline{k}^j$ then $\overline{h}^i = \overline{k}^j = 1$. Hence A is c-quasi-regular at $\{h, k\}$.

Now $h, k \in Z(A)$ and A is π_c. This implies A is cyclic conjugacy separable for $\langle h \rangle$ and $\langle k \rangle$. Therefore, by Theorem 4.5, G is conjugacy separable. ∎

5. ON HNN EXTENSIONS OF FREE PRODUCTS OF FINITE CYCLES

In this section we shall prove the conjugacy separability of certain HNN extensions of free products of cyclic groups. Let

$$G = \langle t, a_1, \ldots, a_m, b_1, \ldots, b_n\colon a_i^{\alpha_i}, b_j^{\beta_j}, t^{-1}ht = k \rangle,$$

where $h \in \langle a_1, \ldots, a_m \rangle$ and $k \in \langle b_1, \ldots, b_n \rangle$ are of infinite order. For convenience, we let $C = \langle a_1, \ldots, a_m \rangle$, $D = \langle b_1, \ldots, b_n \rangle$, and $A = C * D$.

LEMMA 5.1 [10, Lemma 2.8]. *Let G be a finite extension of a residually finitely generated torsion-free nilpotent group B. Let $h \in G$ be of infinite order. If $\langle h \rangle \cap B = \langle h^n \rangle$ then, for each integer t, there exists $\lambda_1 > 0$ such that for any $\alpha > 0$ there exists $N_\alpha \lhd_f G$ so that $N_\alpha \cap \langle h \rangle = \langle h^{\alpha n \lambda_1} \rangle$ and that $\overline{h}^{nt} \sim_{\overline{G}} \overline{h}^j$ implies $\overline{h}^j = \overline{h}^{\pm nt}$, where $\overline{G} = G/N_\alpha$.*

LEMMA 5.2. *The group $A = C * D$ above is c-quasi-regular at $\{h, k\}$.*

Proof. Since C, D are free-by-finite, let C_1, D_1 be free subgroups of finite index in C, D, respectively. Let $C_1 \cap \langle h \rangle = \langle h^{n_1} \rangle$ and $D_1 \cap \langle k \rangle = \langle k^{n_2} \rangle$. Let m be a given integer.

Let $t = n_2 m$ in Lemma 5.1; there exists λ_1 such that, for any $s > 0$, there exists $M_s \triangleleft_f C$ such that $M_s \cap \langle h \rangle = \langle h^{sn_1\lambda_1} \rangle$ and, in $\hat{C} = C/M_s$, if $\hat{h}^{n_1 n_2 m} \sim_{\hat{C}} \hat{h}^i$ then $\hat{h}^i = \hat{h}^{\pm n_1 n_2 m}$. Similarly, let $t = n_1 m$ in Lemma 5.1. Then there exists λ_2 such that, for any $s > 0$, there exists $L_s \triangleleft_f D$ such that $L_s \cap \langle k \rangle = \langle k^{sn_2\lambda_2} \rangle$ and, in $\hat{D} = D/L_s$, if $\hat{k}^{n_1 n_2 m} \sim_{\hat{D}} \hat{k}^i$ then $\hat{k}^i = \hat{k}^{\pm n_1 n_2 m}$.

For any ϵ, considering $s = n_2 \lambda_2 \epsilon$. Then there exists $M_1 \triangleleft_f C$ such that $M_1 \cap \langle h \rangle = \langle h^{n_1 n_2 \lambda_1 \lambda_2 \epsilon} \rangle$ and, in $\tilde{C} = C/M_1$, if $\tilde{h}^{n_1 n_2 m} \sim_{\tilde{C}} \tilde{h}^i$ then $\tilde{h}^i = \tilde{h}^{\pm n_1 n_2 m}$. Similarly, for $s = n_1 \lambda_1 \epsilon$, there exists $L_1 \triangleleft_f D$ such that $L_1 \cap \langle k \rangle = \langle k^{n_1 n_2 \lambda_1 \lambda_2 \epsilon} \rangle$ and, in $\tilde{D} = D/L_1$, if $\tilde{k}^{n_1 n_2 m} \sim_{\tilde{D}} \tilde{k}^i$ then $\tilde{k}^i = \tilde{k}^{\pm n_1 n_2 m}$. Let $\tilde{A} = \tilde{C} * \tilde{D}$. Then $\tilde{h}^i \nsim_{\tilde{A}} \tilde{k}^j$ for any $\tilde{h}^i \neq 1 \neq \tilde{k}^j$. Since $\tilde{A}$ is free-by-finite, it is conjugacy separable [2]. Thus there exists $\tilde{N} \triangleleft_f \tilde{A}$ such that $\tilde{h}^i, \tilde{k}^j \notin \tilde{N}$ and $\tilde{N}\tilde{h}^i \nsim_{\tilde{A}/\tilde{N}} \tilde{N}\tilde{k}^j$ for all $\tilde{h}^i \neq 1 \neq \tilde{k}^j$, $\tilde{N}\tilde{h}^i \nsim_{\tilde{A}/\tilde{N}} \tilde{N}\tilde{h}^j$ for all $\tilde{h}^i \nsim_{\tilde{A}} \tilde{h}^j$, $\tilde{N}\tilde{k}^i \nsim_{\tilde{A}/\tilde{N}} \tilde{N}\tilde{k}^j$ for all $\tilde{k}^i \nsim_{\tilde{A}} \tilde{k}^j$. Let N be the preimage of $\tilde{N}$ in A. Then $N \cap \langle h \rangle = \langle h^{n_1 n_2 \lambda_1 \lambda_c 2\epsilon} \rangle$, $N \cap \langle k \rangle = \langle k^{n_1 n_2 \lambda_1 \lambda_2 \epsilon} \rangle$, and, in $\bar{A} = A/N$, $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = 1$. Moreover, if $\bar{h}^{n_1 n_2 m} \sim_{\bar{A}} \bar{h}^j$ then $\bar{h}^j = \bar{h}^{\pm n_1 n_2 m}$, if $\bar{k}^{n_1 n_2 m} \sim_{\bar{A}} \bar{k}^j$ then $\bar{k}^j = \bar{k}^{\pm n_1 n_2 m}$, and if $\bar{h}^i \sim_{\bar{A}} \bar{k}^j$ then $\bar{h}^i = \bar{k}^j = 1$. Hence A is c-quasi-regular at $\{h, k\}$. ∎

Since A is c-quasi-regular at $\{h, k\}$, A is quasi-regular at $\{h, k\}$. Hence, by Theorem 2.7, we have the following:

THEOREM 5.3. *The group* $G = \langle t, a_1, \dots, a_m, b_1, \dots, b_n \colon a_i^{\alpha_i}, b_j^{\beta_j}, t^{-1}ht = k \rangle$, *where* $h \in \langle a_1, \dots, a_m \rangle$ *and* $k \in \langle b_1, \dots, b_n \rangle$, *is* π_c, *whence residually finite.*

THEOREM 5.4. *The group* $G = \langle t, a_1, \dots, a_m, b_1, \dots, b_n \colon a_i^{\alpha_i}, b_j^{\beta_j}, t^{-1}ht = k \rangle$, *where* $h \in \langle a_1, \dots, a_m \rangle$ *and* $k \in \langle b_1, \dots, b_n \rangle$, *is conjugacy separable.*

Proof. By Lemma 5.2, $A = \langle a_1, \dots, a_m, b_1, \dots, b_n \rangle$ is c-quasi-regular at $\{h, k\}$. Since A is free-by-finite and since free groups are double coset separable [15], A is π_c and double coset separable at $\{h, k\}$. Moreover, free-by-finite groups are conjugacy separable [2] and cyclic conjugacy separable [7]. The theorem follows from Theorem 4.5. ∎

6. ON HNN EXTENSIONS OF BAUMSLAG–SOLITAR GROUPS

In this section we consider the HNN extension

$$G = \langle t, a, b \colon t^{-1}a^{\nu}t = b^{\xi}, b^{-1}a^{\lambda}b = a^{\mu} \rangle, \tag{6.1}$$

of the Baumslag–Solitar group $A = \langle a, b \colon b^{-1}a^{\lambda}b = a^{\mu} \rangle$ with associated subgroups $\langle a^{\nu} \rangle$ and $\langle b^{\xi} \rangle$. Hopficity and residual finiteness of this type of groups were studied by Raptis, Talelli, and Varsos [16] which motivates this section. Using their result on residual finiteness, we prove the cyclic subgroup separability and conjugacy separability of G.

THEOREM 6.1 [16]. *The group G, as given by* (6.1), *is residually finite if and only if* $|\lambda| = |\mu|$.

THEOREM 6.2 [12, Theorem 4.6]. *Let $G = A *_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclically reduced, and that $x \sim_G y$.*

(1) *If $|x| = 0$, then $\|y\| \le 1$ and, if $y \in A$, then there is a sequence $h_1, h_2, \dots, h_r$ of elements in H such that $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim h_r = x$.*

(2) *If $\|x\| = 1$, then $\|y\| = 1$ and either $x, y \in A$ and $x \sim_A y$, or $x, y \in B$ and $x \sim_B y$.*

(3) *If $\|x\| \ge 2$, then $\|x\| = \|y\|$ and $y \sim_H x^*$ where x^* is a cyclic permutation of x.*

LEMMA 6.3. *The group $A = \langle a, b\colon b^{-1}a^{\lambda}b = a^{\mu}\rangle$ is c-quasi-regular at $\{a^{\nu}, b^{\xi}\}$ if $|\lambda| = |\mu|$.*

Proof.

Case 1. $A = \langle a, b\colon b^{-1}a^{\lambda}b = a^{\lambda}\rangle$.

Clearly $\langle h\rangle \cap \langle k\rangle = 1$, where $h = a^{\nu}$ and $k = b^{\xi}$. Let ϵ be a given integer. Since $A = \langle a\rangle *_{a^{\lambda}=x} \langle x, b\colon b^{-1}xb = x\rangle$, we consider the natural homomorphism

$$\chi_{\epsilon}\colon A \to \langle a\rangle/\langle a^{\lambda\nu\epsilon}\rangle \underset{\hat{a}^{\lambda}=\hat{x}}{*} (\langle x\rangle/\langle x^{\nu\epsilon}\rangle \times \langle b\rangle/\langle b^{\lambda\xi\epsilon}\rangle),$$

where $\hat{A} = A\chi_{\epsilon}$. Clearly $|\hat{h}| = \lambda\epsilon = |\hat{k}|$ and $\langle\hat{h}\rangle \cap \langle\hat{k}\rangle = 1$. Since $\hat{x} \in Z(\hat{A})$, it follows from Theorem 6.2 that if $\hat{h}^m \sim_{\hat{A}} \hat{h}^i$ then $\hat{h}^m = \hat{h}^i$. Similarly, if $\hat{h}^i \sim_{\hat{A}} \hat{k}^j$ then $\hat{h}^i = 1 = \hat{k}^j$.

Now suppose $\hat{k}^m \sim_{\hat{A}} \hat{k}^i$. Since $x \in Z(A)$, this implies that if $\hat{k}^m \sim_{\hat{A}} \hat{x}^s$ for some s, then $\hat{k}^m = \hat{x}^s \in \langle\hat{h}\rangle \cap \langle\hat{k}\rangle = 1$. Hence $\hat{k}^m = 1 = \hat{k}^i$. On the other hand, if $\hat{k}^m \not\sim_{\hat{A}} \hat{x}^s$ for any s then, by Theorem 6.2, $\hat{k}^m$ and $\hat{k}^i$ are conjugate in $\langle\hat{x}\rangle \times \langle\hat{b}\rangle$. Thus $\hat{k}^m = \hat{k}^i$.

We have shown that, in $\hat{A} = A\chi_{\epsilon}$, $|\hat{h}| = \lambda\epsilon = |\hat{k}|$ and $\langle\hat{h}\rangle \cap \langle\hat{k}\rangle = 1$. Moreover, if $\hat{h}^m \sim_{\hat{A}} \hat{h}^i$ then $\hat{h}^m = \hat{h}^i$, if $\hat{h}^i \sim_{\hat{A}} \hat{k}^j$ then $\hat{h}^i = 1 = \hat{k}^j$, and if $\hat{k}^m \sim_{\hat{A}} \hat{k}^i$ then $\hat{k}^m = \hat{k}^i$. Since $\hat{A}$ is free-by-finite, it is conjugacy separable. Thus, as in the proof of Lemma 5.2, we can find $N \triangleleft_f A$ such that $N \cap \langle h\rangle = \langle h^{\lambda\epsilon}\rangle$, $N \cap \langle k\rangle = \langle k^{\lambda\epsilon}\rangle$, and, in $\overline{A} = A/N$, conditions (2)–(5) of Definition 4.1 are satisfied. Hence A is c-quasi-regular at $\{h, k\} = \{a^{\nu}, b^{\xi}\}$.

Case 2. $A = \langle a, b\colon b^{-1}a^{\lambda}b = a^{-\lambda}\rangle$.

Clearly $\langle h\rangle \cap \langle k\rangle = 1$, where $h = a^{\nu}$ and $k = b^{\xi}$. Let ϵ be a given integer. Since $A = \langle a\rangle *_{a^{\lambda}=x} \langle x, b\colon b^{-1}xb = x^{-1}\rangle$, we consider the natural homomorphism

$$\phi_{\epsilon}\colon A \to \langle a\rangle/\langle a^{2\lambda\nu\epsilon}\rangle \underset{\hat{a}^{\lambda}=\hat{x}}{*} (\langle x, b\rangle/\langle x^{2\nu\epsilon}\rangle\langle b^{2\lambda\xi\epsilon}\rangle),$$

where $\hat{A} = A\phi_{\epsilon}$. Clearly $|\hat{h}| = 2\lambda\epsilon = |\hat{k}|$ and $\langle\hat{h}\rangle \cap \langle\hat{k}\rangle = 1$, since $\langle\hat{x}\rangle \cap \langle\hat{b}\rangle = 1$.

Suppose $\hat{h}^{2m} \sim_{\hat{A}} \hat{h}^{i}$. We note that $\hat{g}^{-1}\hat{x}^{s}\hat{g} = \hat{x}^{\pm s}$ for any s and any $g \in A$. If $\hat{h}^{2m} \sim_{\hat{A}} \hat{x}^{s}$ for some s, then $\hat{h}^{2m} = \hat{x}^{\pm s}$ and $\hat{h}^{i} = \hat{x}^{\pm s}$, thus $\hat{h}^{i} = \hat{h}^{\pm 2m}$. If $\hat{h}^{2m} \nsim_{\hat{A}} \hat{x}^{s}$ for any s then, by Theorem 6.2, $\hat{h}^{2m} \sim_{\langle\hat{a}\rangle} \hat{h}^{i}$. This implies $\hat{h}^{i} = \hat{h}^{2m}$.

Now suppose $\hat{k}^{2m} \sim_{\hat{A}} \hat{k}^{i}$. We note that $\hat{k}^{2m} \in Z(\langle\hat{x}, \hat{b}\rangle)$. If $\hat{k}^{2m} \sim_{\hat{A}} \hat{x}^{s}$ for some s then, by Theorem 6.2, $\hat{k}^{2m} = \hat{x}^{\pm s} \in \langle\hat{k}\rangle \cap \langle\hat{x}\rangle = 1$. Hence $\hat{k}^{2m} = 1 = \hat{k}^{i}$. On the other hand, if $\hat{k}^{2m} \nsim_{\hat{A}} \hat{x}^{s}$ for any s then, by Theorem 6.2, $\hat{k}^{2m}$ and $\hat{k}^{i}$ are conjugate in $\langle\hat{x}, \hat{b}\rangle$. This implies $\hat{k}^{2m} = \hat{k}^{i}$.

Suppose $\hat{h}^{i} \sim_{\hat{A}} \hat{k}^{j}$; then, by Theorem 6.2, $\hat{h}^{i} \sim_{\hat{A}} \hat{x}^{s}$ for some s. Therefore, as above, $\hat{h}^{i} = \hat{x}^{\pm s}$. This implies $\hat{k}^{j} \sim_{\hat{A}} \hat{x}^{\pm s}$. This means $\hat{k}^{j} = \hat{x}^{\pm s} \in \langle\hat{k}\rangle \cap \langle\hat{x}\rangle = 1$. Hence $\hat{h} = 1 = \hat{k}^{j}$.

Thus, as in Case 1, we can find $N \lhd_f A$ such that conditions (1)–(5) of Definition 4.1 are satisfied. Hence A is c-quasi-regular at $\{h, k\} = \{a^{\nu}, b^{\xi}\}$. ∎

THEOREM 6.4. *The group* $G = \langle t, a, b\colon t^{-1}a^{\nu}t = b^{\xi}, b^{-1}a^{\lambda}b = a^{\mu}\rangle$ *is* π_c *if and only if* $|\lambda| = |\mu|$.

Proof. Suppose $|\lambda| = |\mu|$. By [17], $A = \langle a, b\colon b^{-1}a^{\lambda}b = a^{\mu}\rangle$ is π_c if and only if $|\lambda| = |\mu|$. Therefore, by Theorem 2.7 and Lemma 6.3, G is π_c. Conversely, if G is π_c then A is π_c. Hence, $|\lambda| = |\mu|$ by [17]. ∎

LEMMA 6.5. *The group* $A = \langle a, b\colon b^{-1}a^{\lambda}b = a^{\mu}\rangle$ *is cyclic conjugacy separable for* $\langle a^{\nu}\rangle$ *and* $\langle b^{\xi}\rangle$ *if* $|\lambda| = |\mu|$.

Proof.

Case 1. $A = \langle a, b\colon b^{-1}a^{\lambda}b = a^{\lambda}\rangle$. Let $A = \langle a\rangle *_{a^{\lambda}=x} \langle x, b\colon b^{-1}xb = x\rangle$ and let $g \in A$ be of minimal length in its conjugate class in A.

(a) A is cyclic conjugacy separable for $\langle h\rangle = \langle a^{\nu}\rangle$.

Let $\{g\}^{A} \cap \langle h\rangle = \emptyset$. Suppose $\|g\| \geq 2$ or $g \in \langle x, b\rangle \setminus \langle a\rangle$. Since $a^{\lambda} = x \in Z(A)$, we can find a homomorphism χ_{ϵ}, as in Lemma 6.3, Case 1, such that, in $\hat{A} = A\chi_{\epsilon}$, $\|\hat{g}\| = \|g\| \geq 2$ or $\hat{g} \in \langle\hat{x}, \hat{b}\rangle \setminus \langle\hat{a}\rangle$. Then, by Theorem 6.2, $\hat{g} \nsim_{\hat{A}} \hat{h}^{i}$ for any i. Since $\hat{A}$ is conjugacy separable and $|\hat{h}| < \infty$, there exists $\hat{N} \lhd_f \hat{A}$ such that $\hat{N}\hat{g} \nsim_{\hat{A}/\hat{N}} \hat{N}\hat{h}^i$ for all i. Let N be the preimage of $\hat{N}$ in A. Then, in $\bar{A} = A/N, \{\bar{g}\}^{\bar{A}} \cap \langle \bar{h} \rangle = \emptyset$.

Suppose $g \in \langle a \rangle$. Since $\{g\}^A \cap \langle h \rangle = \emptyset$, $g = a^s \notin \langle a^\nu \rangle$. Then, in $\hat{A} = A\chi_\epsilon$ as above, $\hat{g} \notin \langle \hat{h} \rangle$. Thus, by Theorem 6.2, $\hat{g} \nsim_{\hat{A}} \hat{h}^i$ for all i. Then, as in (a), we can find $N \lhd_f A$ such that, in $\bar{A} = A/N$, $\{\bar{g}\}^{\bar{A}} \cap \langle \bar{h} \rangle = \emptyset$. This proves that A is cyclic conjugacy separable for $\langle h \rangle$.

(b) A is cyclic conjugacy separable for $\langle k \rangle = \langle b^\xi \rangle$.

Let $\{g\}^A \cap \langle k \rangle = \emptyset$. Suppose $\|g\| \geq 2$ or $g \in \langle a \rangle \setminus \langle a^\lambda \rangle$. As before, we can find a homomorphic χ_ϵ above such that, in $\hat{A} = A\chi_\epsilon, \|\hat{g}\| = \|g\| \geq 2$ or $\hat{g} \in \langle \hat{a} \rangle \setminus \langle \hat{a}^\lambda \rangle$. Then, by Theorem 6.2, $\hat{g} \nsim_{\hat{A}} \hat{b}^{\xi i}$ for all i. As in (a) we can find $N \lhd_f A$ such that, in $\bar{A} = A/N$, $\{\bar{g}\}^{\bar{A}} \cap \langle \bar{k} \rangle = \emptyset$.

So, suppose $g \in \langle a^\lambda, b \rangle$. Since $\{g\}^A \cap \langle k \rangle = \emptyset, g \notin \langle b^\xi \rangle$. Again, we can choose $\hat{A} = A\chi_\epsilon$ such that $\hat{g} \notin \langle \hat{b}^\xi \rangle$. Thus, by Theorem 6.2, $\hat{g} \nsim_{\hat{A}} \hat{b}^{\xi i}$ for any i. Hence, as in (a), we can find $N \lhd_f A$ such that, $\bar{A} = A/N$, $\{\bar{g}\}^{\bar{A}} \cap \langle \bar{k} \rangle = \emptyset$. This proves that A is cyclic conjugacy separable for $\langle k \rangle$.

Case 2. $A = \langle a, b \colon b^{-1}a^\lambda b = a^{-\lambda} \rangle$. Let $A = \langle a \rangle *_{a^\lambda = x} \langle x, b \colon b^{-1}xb = x^{-1} \rangle$ and let $g \in A$ be of minimal length in its conjugate class in A.

(a) A is cyclic conjugacy separable for $\langle h \rangle = \langle a^\nu \rangle$.

Let $\{g\}^A \cap \langle h \rangle = \emptyset$. Suppose $\|g\| \geq 2$ or $g \in \langle a^\lambda, b \rangle \setminus \langle a^\lambda \rangle$. As in Lemma 6.3, Case 2, we can find a homomorphism ϕ_ϵ such that, in $\hat{A} = A\phi_\epsilon, \|\hat{g}\| = \|g\| \geq 2$ or $\hat{g} \in \langle \hat{a}^\lambda, \hat{b} \rangle \setminus \langle \hat{a}^\lambda \rangle$. Then, by Theorem 6.2, $\hat{g} \nsim_{\hat{A}} \hat{h}^i$ for all i. If $g \in \langle a \rangle$, then $g = a^s \notin \langle h \rangle = \langle a^\nu \rangle$. If $a^s \notin \langle a^\lambda \rangle$, we can choose $\hat{A} = A\phi_\epsilon$ such that $\hat{g} \notin \langle \hat{a}^\lambda \rangle \cap \langle \hat{a}^\nu \rangle$. Then, by Theorem 6.2, $\hat{g} \nsim_{\hat{A}} \hat{h}^i$ for all i. If $g = a^s \in \langle a^\lambda \rangle$ we can choose $\hat{A} = A\phi_\epsilon$ such that $\hat{g} \notin \langle \hat{a}^\nu \rangle$. Then, again by Theorem 6.2, $\hat{g} \nsim_{\hat{A}} \hat{h}^i$ for all i. Hence, as in Case 1, A is cyclic conjugacy separable for $\langle h \rangle$.

(b) A is cyclic conjugacy separable for $\langle k \rangle = \langle b^\xi \rangle$.

Let $\{g\}^A \cap \langle k \rangle = \emptyset$. Suppose $\|g\| \geq 2$ or $g \in \langle a \rangle \setminus \langle a^\lambda \rangle$. As in (a), we can find a homomorphism ϕ_ϵ above such that, in $\hat{A} = A\phi_\epsilon, \|\hat{g}\| = \|g\| \geq 2$ or $\hat{g} \in \langle \hat{a} \rangle \setminus \langle \hat{a}^\lambda \rangle$. Then, by Theorem 6.2, $\hat{g} \nsim_{\hat{A}} \hat{k}^i$ for any i.

Let $B = \langle a^\lambda, b \rangle$. If $g \in B$, then $\{g\}^B \cap \langle k \rangle = \emptyset$. Let $g = b^s x^r$ where $x = a^\lambda$. We note that $b^s x^r \sim_B b^{\xi i} = k^i$ if and only if $b^s = b^{\xi i}$ and that r is even if s is odd, and $r = 0$ if s is even. Thus, if $b^s \notin \langle b^\xi \rangle$ then, in $\hat{B} = B\phi_\epsilon$, we have $\hat{b}^s \hat{x}^r \nsim_{\hat{B}} \hat{b}^{\xi i}$ for any i. So we can assume that $b^s = b^{\xi j}$ for some j. This means $x^r \neq 1$. If s is even, then we choose $\hat{B} = B\phi_\epsilon$ such that $\hat{x}^r \neq 1$. This implies $\hat{b}^s \hat{x}^r \nsim_{\hat{B}} \hat{b}^{\xi i}$ for any i. If s is odd, then r is not even. Hence we choose $\hat{B} = B\phi_\epsilon$ such that $\hat{x}^r \notin \langle \hat{x}^2 \rangle$. Then $\hat{b}^s \hat{x}^r \nsim_{\hat{B}} \hat{b}^{\xi i}$ for any i. In this way we can find $\hat{A} = A\phi_\epsilon$ such that $\{\hat{g}\}^{\hat{B}} \cap \langle \hat{k} \rangle = \emptyset$. Since any nontrivial element $\hat{k}^i$ has the minimal length 1 in its conjugate class in $\hat{A}$, by Theorem 6.2, $\hat{g} \nsim_{\hat{A}} \hat{k}^i$ for any i. Therefore, as in Case 1, A is cyclic conjugacy separable for $\langle k \rangle$. ∎

LEMMA 6.6. *The group* $A = \langle a, b\colon b^{-1}a^{\lambda}b = a^{\mu}\rangle$ *is double coset separable at* $\{a^{\nu}, b^{\xi}\}$ *if* $|\lambda| = |\mu|$.

Proof.

Case 1. $A = \langle a, b\colon b^{-1}a^{\lambda}b = a^{\lambda}\rangle$.
Let $A = \langle a\rangle *_{a^{\lambda}=x} B$, where $B = \langle x, b\colon b^{-1}xb = x\rangle$.

(a) A is $\langle h^s\rangle u\langle k^s\rangle$-separable for $h = a^{\nu}$, $k = b^{\xi}$.

We first show that A is $\langle a^{\lambda\mu}\rangle\langle b^{\delta}\rangle$-separable for any μ, δ. Let $v \in A$ and $v \notin \langle a^{\lambda\mu}\rangle\langle b^{\delta}\rangle$. If $v \notin B$ then we choose $\hat{A} = A\chi_{\epsilon}$, as in Lemma 6.5, Case 1, such that $\hat{v} \notin \hat{B}$ by considering the length of v in $A = \langle a\rangle *_{a^{\lambda}=x} B$. Then $\hat{v} \notin \langle \hat{a}^{\lambda\mu}\rangle\langle \hat{b}^{\delta}\rangle \subset \hat{B}$. If $v \in B$, we let $\epsilon = \mu\delta$ in $\hat{A} = A\chi_{\epsilon}$. Then $\hat{v} \notin \langle \hat{a}^{\lambda\mu}\rangle\langle \hat{b}^{\delta}\rangle$. Since $\hat{A}$ is free-by-finite, whence residually finite, there exists $\hat{N} \lhd_f \hat{A}$ such that $\hat{v} \notin \hat{N}\langle \hat{a}^{\lambda\mu}\rangle\langle \hat{b}^{\delta}\rangle$. Let N be the preimage of $\hat{N}$ in A. Then $N \lhd_f G$ and $v \notin N\langle a^{\lambda\mu}\rangle\langle b^{\delta}\rangle$. Thus A is $\langle a^{\lambda\mu}\rangle\langle b^{\delta}\rangle$-separable for any μ, δ.

To show that A is $\langle a^{\nu s}\rangle u\langle b^{\xi s}\rangle$-separable for any s, let $v \notin \langle a^{\nu s}\rangle u\langle b^{\xi s}\rangle$, that is, $u^{-1}v \notin \langle u^{-1}a^{\nu s}u\rangle\langle b^{\xi s}\rangle$. Let $d = (\nu s, \lambda)$. Since $a^{\lambda} \in Z(A)$, $\langle u^{-1}a^{\nu s}u\rangle = \bigcup_{i=0}^{(\lambda/d)-1} u^{-1}a^{\nu s i}u\langle a^{(\nu s/d)\lambda}\rangle$. Thus, for each $0 \le i < \frac{\lambda}{d}$, $u^{-1}a^{-\nu s i}u \cdot u^{-1}v \notin \langle a^{(\nu s/d)\lambda}\rangle\langle b^{\xi s}\rangle$. By the above note, there exists $N \lhd_f A$ such that $u^{-1}a^{-\nu s i}u \cdot u^{-1}v \notin N\langle a^{(\nu s/d)\lambda}\rangle\langle b^{\xi s}\rangle$ for all $0 \le i < \frac{\lambda}{d}$. It follows that $u^{-1}v \notin N\langle u^{-1}a^{\nu s}u\rangle\langle b^{\xi s}\rangle$; hence $v \notin N\langle a^{\nu s}\rangle u\langle b^{\xi s}\rangle$.

(b) A is $\langle h^s\rangle u\langle h^s\rangle$-separable for $h = a^{\nu}$. This follows easily by using similar argument as in (a).

(c) A is $\langle k^s\rangle u\langle k^s\rangle$-separable for $k = b^{\xi}$.

By Theorem 6.4, A is π_c. Therefore it is sufficient to show that A is $\langle k^s\rangle u\langle k^s\rangle$-separable for $u = a^{r_1}b^{s_1} \cdots b^{s_{n-1}}a^{r_n}$ where $n \ge 1$, $0 < r_1, \ldots, r_{n-1} < \lambda$ and $r_n \ne 0$. Let $v \notin \langle k^s\rangle u\langle k^s\rangle$.

Suppose $\|v\| > \|u\| + 2$ or $v = a^{e_1}b^{c_1} \cdots b^{c_n}a^{e_{n+1}}$ with $\|v\| = \|u\| + 2$ or $v = b^{c_1}a^{e_1} \cdots a^{e_{n-1}}b^{c_n}$ with $\|v\| = \|u\|$ or $\|v\| < \|u\|$. In this case, we can choose $\hat{A} = A\chi_{\epsilon}$ such that $\|\hat{v}\| = \|v\|$ and $\|\hat{u}\| = \|u\|$. It follows that $\hat{v} \notin \langle \hat{k}^s\rangle\hat{u}\langle \hat{k}^s\rangle$.

So, suppose $v = b^{c_1}a^{e_1} \cdots a^{e_n}b^{c_{n+1}}$. Since $\langle a^{\lambda}, b\rangle = \langle a^{\lambda}\rangle \times \langle b\rangle$, we note that $v = b^{c_1}a^{e_1} \cdots a^{e_n}b^{e_{n+1}} \in \langle k^s\rangle u\langle k^s\rangle$, where $k = b^{\xi}$, if and only if $b^{c_1} = b^{\xi s\alpha}$, $a^{e_1} = a^{r_1}a^{\lambda\delta_1}$, $b^{c_2} = a^{-\lambda\delta_1}b^{s_1}a^{\lambda\delta_1}, \ldots, b^{c_n} = a^{-\lambda\delta_{n-1}}b^{s_{n-1}}a^{\lambda\delta_{n-1}}$, $a^{e_n} = a^{-\lambda\delta_{n-1}}a^{r_n}$, and $b^{c_{n+1}} = b^{\xi s\beta}$ for some α, β, δ_i. Thus $b^{c_1} \in \langle b^{\xi s}\rangle$, $a^{e_i - r_i} \in \langle a^{\lambda}\rangle$, $b^{c_{i+1}} = b^{s_i}$, $b^{c_{n+1}} \in \langle b^{\xi s}\rangle$, and $a^{e_1 + \cdots + e_n} = a^{r_1 + \cdots + r_n}$. Hence $v \notin \langle b^{\xi s}\rangle u\langle b^{\xi s}\rangle$ if and only if $b^{c_1} \notin \langle b^{\xi s}\rangle$, or $a^{e_i - r_i} \notin \langle a^{\lambda}\rangle$, or $b^{c_{i+1}} \ne b^{s_i}$, or $b^{c_{n+1}} \notin \langle b^{\xi s}\rangle$, or $a^{e_1 + \cdots + e_n} \ne a^{r_1 + \cdots + r_n}$. As usual, we can find $\hat{A} = A\chi_{\epsilon}$ such that $\|\hat{v}\| = \|v\|$ and $\|\hat{u}\| = \|u\|$, and $\hat{b}^{c_1} \notin \langle \hat{b}^{\xi s}\rangle$, or $\hat{a}^{e_i - r_i} \notin \langle \hat{a}^{\lambda}\rangle$, or $\hat{b}^{c_{i+1}} \ne \hat{b}^{s_i}$, or $\hat{b}^{c_{n+1}} \notin \langle \hat{b}^{\xi s}\rangle$, or $\hat{a}^{e_1 + \cdots + e_n} \ne \hat{a}^{r_1} + \cdots + r_n$. This implies

$\hat{v} \notin \langle \hat{b}^{\xi s} \rangle \hat{u} \langle \hat{b}^{\xi s} \rangle$. Thus, as in (a), we can find $N \lhd_f A$ such that $v \notin N \langle b^{\xi s} \rangle u \langle b^{\xi s} \rangle$.

Case 2. $A = \langle a, b \colon b^{-1} a^{\lambda} b = a^{-\lambda} \rangle$.

Let $A = \langle a \rangle *_{a^{\lambda} = x} B$, where $B = \langle x, b \colon b^{-1} x b = x^{-1} \rangle$. Since $\langle h^s \rangle u \langle h^s \rangle$-separability and $\langle h^s \rangle u \langle k^s \rangle$-separability are very similar to Case 1, we shall only show that A is $\langle k^s \rangle u \langle k^s \rangle$-separable for $k = b^{\xi}$.

Let $v \in A$ such that $v \notin \langle k^s \rangle u \langle k^s \rangle = \langle b^{\xi s} \rangle u \langle b^{\xi s} \rangle$. As in (c) of Case 1, we need only consider $u = a^{r_1} b^{s_1} \cdots b^{s_{n-1}} a^{r_n}$ and $v = b^{c_1} a^{e_1} \cdots a^{e_n} b^{c_{n+1}}$ where $n \geq 1$. Since $\langle a^{\lambda} \rangle \cap \langle b \rangle = 1$, we note that $v \in \langle b^{\xi s} \rangle u \langle k^{\xi s} \rangle$ iff $b^{c_1} = b^{\xi s \alpha}$, $a^{e_1} = a^{r_1} a^{\lambda \delta_1}$, $b^{c_2} = a^{-\lambda \delta_1} b^{s_1} a^{\lambda \mu_1}$, $a^{e_2} = a^{-\lambda \mu_1} a^{r_2} a^{\lambda \delta_2}, \ldots, b^{c_n} = a^{-\lambda \delta_{n-1}} b^{s_{n-1}} a^{\lambda \mu_{n-1}}$, $a^{e_n} = a^{-\lambda \mu_{n-1}} a^{r_n}$, and $b^{c_{n+1}} = b^{\xi s \beta}$ for some $\alpha, \beta, \delta_i, \mu_i$, where $b^{c_{i+1}} = b^{s_i}$. If s_i is even, then $\lambda \delta_i = \lambda \mu_i$, and if s_i is odd, then $\lambda \delta_i = -\lambda \mu_i$. Thus $v \notin \langle b^{\xi s} \rangle u \langle b^{\xi s} \rangle$ if and only if $b^{c_1} \notin \langle b^{\xi s} \rangle$, or $a^{e_i - r_i} \notin \langle a^{\lambda} \rangle$, or $b^{c_{i+1}} \neq b^{s_i}$, or $b^{c_{n+1}} \notin \langle b^{\xi s} \rangle$, or $b^{c_1} a^{e_1} \cdots b^{c_n} = b^{\xi s \alpha} a^{r_1} b^{s_1} \cdots b^{s_{n-1}} a^{\lambda \mu_{n-1}}$ and $a^{e_n} = a^{r_n} a^{\lambda \mu_n}$ but $a^{\lambda \mu_{n-1}} \neq a^{-\lambda \mu_n}$. Then, as usual, we can find $\hat{A} = A \phi_{\epsilon}$ such that $\|\hat{v}\| = \|v\|$ and $\|\hat{u}\| = \|u\|$, and $\hat{b}^{c_1} \notin \langle \hat{b}^{\xi s} \rangle$, or $\hat{a}^{e_i - r_i} \notin \langle \hat{a}^{\lambda} \rangle$, or $\hat{b}^{c_{i+1}} \neq \hat{b}^{s_i}$, or $\hat{b}^{c_{n+1}} \notin \langle \hat{b}^{\xi s} \rangle$, or $\hat{a}^{\lambda \mu_{n-1}} \neq \hat{a}^{-\lambda \mu_n}$ if $b^{c_1} a^{e_1} \cdots b^{c_n} = b^{\xi s \alpha} a^{r_1} b^{s_1} \cdots b^{s_{n-1}} a^{\lambda \mu_{n-1}}$ and $a^{e_n} = a^{r_n} a^{\lambda \mu_n}$. This implies $\hat{v} \notin \langle \hat{b}^{\xi s} \rangle \hat{u} \langle \hat{b}^{\xi s} \rangle$. As in Case 1, we can find $N \lhd_f A$ such that $v \notin N \langle b^{\xi s} \rangle u \langle b^{\xi s} \rangle$. ∎

THEOREM 6.7. *The group* $G = \langle t, a, b \colon t^{-1} a^{\nu} t = b^{\xi},\ b^{-1} a^{\lambda} b = a^{\mu} \rangle$ *is conjugacy separable if and only if* $|\lambda| = |\mu|$.

Proof. If $|\lambda| = |\mu|$, then $A = \langle a, b \colon b^{-1} a^{\lambda} b = a^{\mu} \rangle$ is conjugacy separable by [5]. Applying Lemmas 6.3–6.6 and Theorem 4.5, G is conjugacy separable. Conversely, if G is conjugacy separable then, G is residually finite. Hence, by Theorem 6.1, $|\lambda| = |\mu|$. ∎

Brunner [1] has studied the epimorphisms of the class of groups

$$G(l, m; k) = \langle a, t \colon t^{-1} a^{-k} t a^{l} t^{-1} a^{k} t = a^{m} \rangle,$$

where l, m, k are integers and $|l| > m$ and obtained various results concerning the Hopficity and the automorphism groups of certain groups in $G(l, m; k)$. Clearly $G(l, m; k)$ is a special case of G above when $\xi = 1$. Thus we have the following:

THEOREM 6.8. *The following statements on* $G(l, m; k)$ *are equivalent*:

(1) $G(l, m; k)$ *is conjugacy separable*.

(2) $|l| = |m|$.

(3) $G(l, m; k)$ *is cyclic subgroup separable*.

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