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Separability Properties of Certain Tree Products of Groups

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We introduce the concept of finite compatibility to prove certain double coset separability of tree products of central subgroup separable groups. We also prove a criterion for the conjugacy separability of generalized free products of two conjugacy separable groups amalgamating a central subgroup in one of the factors. Using this we prove that tree products of finitely many central subgroup separable and conjugacy separable groups are conjugacy separable. We apply the results to polycyclic-by-finite groups and groups of linkages of torus knots. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

Let S be a subset of a group G. Then G is said to be *S-separable* if, for each $g \in G \backslash S$, there exists a normal subgroup N of finite index in G such

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that $g \notin NS$. Equivalently, S is a closed subset in the profinite topology of G. In particular, if $S = \{1\}$ then G is *residually finite*. Residual and separability properties are of interest to both group theorists and topologists. They are related to the solvability of the word problem, conjugacy problem, generalized word problem (Mal'cev [16] and Mostowski [17]). Topologically separability properties are related to questions of embeddability of equivariant subspaces in their regular covering space (Scott [21] and Niblo [18]).

In general it is difficult to show whether a given subset of a group is separable. Free and surface groups are known to be subgroup separable and conjugacy separable (Hall [9], Scott [21], and Stebe [23]). Fine and Rosenberger [7] showed that Fuchsian groups are conjugacy separable. Niblo [18] showed that these groups are also double coset separable. It is also known that polycyclic-by-finite groups are subgroup separable and conjugacy separable (Malcev [16] and Remeslennikov [19] or Formanek [8]). In [14], Lennox and Wilson showed that if G is polycyclic and H, K are subgroups of G then G is HK-separable. It follows easily that polycyclic-by-finite groups are double coset separable (cf. Tang [24]). It seems natural to ask under what conditions separability properties of groups are preserved under generalized free products and HNN-extensions. In [2], Allenby and Tang showed that generalized free products of free-by-finite groups with cyclic amalgamation are subgroup separable. Thus groups of F-type (Fine and Rosenberger [6]) are subgroup separable. In [11, 12], Kim and Tang gave criteria for generalized free products and HNN-extensions with cyclic amalgamated subgroups or associated subgroups to be conjugacy separable. It follows that tree products of free and surface groups with cyclic amalgamated subgroups are conjugacy separable. In [20], Ribes *et al.* showed that tree products of polycyclic-by-finite groups with cyclic amalgamated subgroups are conjugacy separable. This answered a problem posed by Tang [13]. Generalizing to tree products with noncyclic amalgamated subgroups seems difficult. Recently, Wong and C. K. Tang [26] proved that tree products of conjugacy separable groups amalgamating central subgroups are conjugacy separable if the amalgamated subgroups intersect trivially. This is a generalization of a result of Baumslag [3]. Wong and Tang's proof is quite long. In this paper we prove a criterion for generalized free products of certain conjugacy separable groups to be conjugacy separable. Applying this result we simplified and generalized Wong and Tang's result by removing the restriction that the amalgamated subgroups intersect trivially. In this process we prove the double coset separability for central subgroups in the vertices of tree products of central subgroup separable groups amalgamating central subgroups.

In Section 2, we list the terms and notations which we shall be using in this paper. We also list some results that will be used frequently. In Section 3, we show that if G is a tree product of groups whose edge groups are separable and finitely compatible in their respective vertex groups then G is H-separable and H-compatible, where H is an H-separable and finitely compatible subgroup of a vertex group. We also introduce the concept of subgroup conjugacy separability. We prove that tree products of finitely many central subgroup separable groups amalgamating central subgroups are subgroup conjugacy separable for certain subgroups. In Section 4 we show that if G is a tree product of finitely many central subgroup separable groups amalgamating finitely generated central subgroups then, relative to central subgroups H, K in some vertex groups, G is HK-separable and double coset separable. Although the full extent of the results was not needed for our main result, we include these results because of their independent interest. In Section 5, we prove a criterion for the conjugacy separability of generalized free products of two conjugacy separable groups amalgamating a central subgroup in one of the factors. From this we derive our main result that tree products of central subgroup separable and conjugacy separable groups, amalgamating finitely generated central subgroups, are conjugacy separable. Hence tree products of finitely many polycyclic-by-finite groups, amalgamating central subgroups, are conjugacy separable. As another application we show that the groups of linkage of torus knots are conjugacy separable.

2. TERMS AND NOTATIONS

Throughout this paper we use standard terms and notations. For convenience we list the following:

The letter G always denotes a group.

If $x \in G$, $\{x\}^G$ denotes the set of all conjugates of x in G.

$x \sim_G y$ means x, y are conjugate in G.

$N \triangleleft_f G$ means N is a normal subgroup of finite index in G.

If $x \in G = A *_H B$ then $\|x\|$ denotes the free product length of x in G.

We use $\mathcal{RF}$ to denote the class of *residually finite* groups. By abuse of notation, we also use $\mathcal{RF}$ to mean *residually finite*.

Let S be a subset of a group G. Then G is said to be *S-separable* if for every $g \in G \backslash S$, there exists $N \triangleleft_f G$ such that $g \notin NS$. Equivalently, S is a *closed subset* in the profinite topology on G. Then a group G is *subgroup separable*, if G is H-separable for all finitely generated subgroups H of G.

Let $x, y \in G$ such that $x \not\sim_G y$. If there exists $N \triangleleft_f G$ such that, in $\overline{G} = G/N$, $\bar{x} \not\sim_{\overline{G}} \bar{y}$ then x, y are said to be *conjugacy distinguishable* in G. If each pair of nonconjugate elements $x, y \in G$ are conjugacy distinguishable then G is said to be *conjugacy separable*.

Let Γ be a tree. To each vertex v of Γ assign a group G_v called a vertex group. Similarly to each edge e of Γ assign a group G_e. Let u, v be vertices at the ends of e. Let α_e and β_e be monomorphisms of G_e to G_u and G_v, respectively. Then $G_e\alpha_e$ is called an edge group of the vertex group G_u and $G_e\beta_e$ an edge group of G_v. The *tree product* of Γ is defined to be the group generated by all the generators and relations of the vertex groups of Γ together with the extra relations obtained by identifying $g_e\alpha_e$ and $g_e\beta_e$ for each $g_e \in G_e$ and each e in Γ.

The following well-known results will be used extensively in this paper:

THEOREM 2.1 [15, Theorem 4.6]. *Let $G = A *_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclically reduced, and that $x \sim_G y$.*

(1) *If $\|x\| = 0$, then $\|y\| \leq 1$ and, if $y \in A$, then there is a sequence $h_1, h_2, \ldots, h_r$ of elements in H such that $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim_B h_r = x$.*

(2) *If $\|x\| = 1$, then $\|y\| = 1$ and, either $x, y \in A$ and $x \sim_A y$, or $x, y \in B$ and $x \sim_B y$.*

(3) *If $\|x\| \geq 2$, then $\|x\| = \|y\|$ and $y \sim_H x^*$, where x^* is a cyclic permutation of x.*

THEOREM 2.2 [5, Theorem 4]. *If A and B are conjugacy separable and H is finite, then $A *_H B$ is conjugacy separable.*

3. SOME SEPARABILITY PROPERTIES

In this section we shall introduce the concepts of finite compatibility and subgroup conjugacy separability. We first prove some properties of these concepts and apply them to tree products of central subgroup separable groups amalgamating central edge groups.

DEFINITION 3.1. Let G be a group and let H be a subgroup of G. We say H is *finitely compatible* in G or *G is H-finite* if, for every $D \lhd_f H$, there exists $N_D \lhd_f G$ such that $N_D \cap H = D$.

Finitely compatible groups are useful in the study of residual properties of generalized free products of groups. The concept was first discussed by Wehrfritz [25] without giving it a name. Using a similar argument as that in the criterion [25] or Theorem 2.1 [1], it is not difficult to prove the following result:

THEOREM 3.2. *Let $G = A *_H B$, where H is a $\mathcal{RF}$ subgroup of the groups A and B. If A, B are H-finite and H-separable then G is $\mathcal{RF}$.*

If a subgroup H of A is $\mathcal{RF}$ and if A is H-finite and H-separable then A is $\mathcal{RF}$. Hence, in the above theorem, A and B are $\mathcal{RF}$. The theorem can be compared with Proposition 2 and Corollary 2.41 in [3].

LEMMA 3.3. *Let $G = A *_C B$, where A, B are C-finite. Let H be a subgroup of A such that A is H-finite. Then G is H-finite.*

Proof. Let $D \lhd_f H$. Since A is H-finite, there exists $L \lhd_f A$ such that $L \cap H = D$. Clearly $L \cap C \lhd_f C$. Since B is C-finite, there exists $M \lhd_f B$ such that $M \cap C = L \cap C$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{B}$, where $\overline{A} = A/L$, $\overline{B} = B/M$, and $\overline{C} = CL/L \cong CM/M$. Now $\overline{G}$ is $\mathcal{RF}$ and $\overline{H}$ is finite. This implies that there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{N} \cap \overline{H} = 1$. Let $\theta: G \to \overline{G}$ be the natural homomorphism of G to $\overline{G}$. Let $N = \theta^{-1}(\overline{N})$, the preimage image of $\overline{N}$ in G. Then $N \lhd_f G$ and $N \cap H = D$. Hence G is H-finite. ∎

The following result was proved in [10]:

LEMMA 3.4 [10, Theorem 2.3]. *Let $G = A *_C B$, where A, B are C-finite and C-separable. Let H be a subgroup of A such that A is H-separable. Then G is H-separable.*

Remark 3.5. If G is a tree product of n vertices ($n \geq 2$), then we can always consider $G = A *_C T$, where A is the vertex group of an extremal vertex of the tree and T is the tree product on the remaining $n-1$ vertices and C is the edge group joining A to its adjacent vertex group, say B.

Since the next result is immediate from Lemmas 3.3 and 3.4, we omit the proof.

THEOREM 3.6. *Let G be a tree product of n vertices such that the edge groups in each vertex group are finitely compatible in the vertex group. Let H be a subgroup of a vertex group V.*

(1) *If H is finitely compatible in V, then H is finitely compatible in G.*

(2) *If edge groups are separable in their respective vertex groups and if V is H-separable, then G is H-separable.*

DEFINITION 3.7. A group G is *central subgroup separable* if G is H-separable for any finitely generated subgroup H of $Z(G)$.

In particular, every subgroup separable group is central subgroup separable.

LEMMA 3.8. *Let A be a central subgroup separable group and let $H, C \subset Z(A)$ be finitely generated. For each $U \lhd_f C$, there exists $N \lhd_f A$ such that $N \cap C = U$ and $NH \cap NC = N(H \cap C)$. In particular, A is C-finite.*

Proof. Let $c_0, c_1, \ldots, c_n$ be coset representatives of U in C, where $c_0 = 1$. Since A is central subgroup separable and $U \subset Z(A)$ is finitely generated, there exists $N_1 \lhd_f A$ such that $c_i \notin N_1 U$ for $i \geq 1$. Similarly, there exists $N_2 \lhd_f A$ such that $c_i \notin N_2 HU$ for all $c_i \notin HU$. Let $N = N_1 U \cap N_2 U$. Then $N \lhd_f A$ and clearly $N \cap C = U$. We shall show that $NH \cap NC \subset N(H \cap C)$. Let $n_1 h = n_2 c$, where $n_1, n_2 \in N$, $h \in H$, and $c \in C$. Let $c = c_i u$ for $u \in U$. Then $c_i = n_2^{-1} n_1 h u^{-1} \in NHU \subset N_2 HU$. This follows from the choice of N_2 that $c_i \in HU$. Let $c_i = h_1 u_1$ for $h_1 \in H$ and $u_1 \in U$. Since $U \subset C$, $h_1 \in H \cap C$. Also, since $U \subset N$, $u_1 u \in N$. Hence $n_2 c = n_2 c_i u = n_2 h_1 u_1 u = n_2 u_1 u h_1 \in N(H \cap C)$. This proves $NH \cap NC \subset N(H \cap C)$. Hence $NH \cap NC = N(H \cap C)$. ∎

We thank the referee for informing us that C-finiteness of Lemma 3.8 can be also obtained by Proposition 2.2 in [18].

Since infinitely generated abelian groups, for example, the Prüfer group $\mathbb{Z}_{p^\infty}$, may not be $\mathcal{RF}$, we now restrict ourselves to tree products of central subgroup separable groups whose edge groups are finitely generated and contained in the centers of the vertex groups; briefly, we say *the tree products of central subgroup separable groups amalgamating central edge groups* and we assume the edge groups are finitely generated.

Since the central subgroup separable group A is C-finite (Lemma 3.8) for finitely generated $C \subset Z(A)$, using Theorem 3.6, we have

COROLLARY 3.9. *Let G be a tree product of finitely many central subgroup separable groups amalgamating central edge groups. Let H be a finitely generated central subgroup of a vertex group. Then G is H-finite and H-separable. In particular, G is $\mathcal{RF}$.*

LEMMA 3.10. *Let G be a tree product of n central subgroup separable groups amalgamating central edge groups. Let H, K be finitely generated central subgroups of some vertex groups. Then, for each $U \triangleleft_f H$, there exists $N \triangleleft_f G$ such that $N \cap H = U$ and $NH \cap NK = N(H \cap K)$. Moreover, if $M \triangleleft_f G$ such that $M \subset N$ and $M \cap H = U$, then $MH \cap MK = M(H \cap K)$.*

Proof. We shall prove the lemma by induction on n. By Lemma 3.8, the lemma is true for $n = 1$. As in Remark 3.5, let $G = A *_C T$.

Case 1. $H \subset T$ and $K \subset A$ (similarly, $H \subset A$ and $K \subset T$). Since H, $C \cap K$ are central subgroups in vertex groups of T, by induction we can find $M \triangleleft_f T$ such that $M \cap H = U$ and $MH \cap M(C \cap K) = M(H \cap (C \cap K)) = M(H \cap K)$. By Lemma 3.8, there exists $L \triangleleft_f A$ such that $L \cap C = M \cap C$ and $LK \cap LC = L(K \cap C)$. Let $\overline{A} = A/L$, $\overline{T} = T/M$, and $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$. Then we have $\overline{H} \cap (\overline{C \cap K}) = \overline{H \cap K}$ and $\overline{K} \cap \overline{C} = \overline{K \cap C}$. Since $\overline{G}$ is $\mathcal{RF}$ and $\overline{HK}$ is finite, there exists $\overline{N} \triangleleft_f \overline{G}$ such that $\overline{N} \cap \overline{HK} = 1$. Let N be the preimage of $\overline{N}$ in G. Then $N \triangleleft_f G$. Clearly $N \cap H = U$. We shall show that $NH \cap NK = N(H \cap K)$. Suppose $n_1 h = n_2 k \in NH \cap NK$, where $n_1, n_2 \in N$, $h \in H$, and $k \in K$. Then $\overline{hk}^{-1} = \bar{n}_1^{-1} \bar{n}_2 \in \overline{N} \cap \overline{HK} = 1$. Hence $\bar{h} = \bar{k} \in \overline{H} \cap \overline{K} \subset \overline{C}$ in $\overline{G}$. Thus $\bar{k} \in \overline{K} \cap \overline{C} = \overline{K \cap C}$ and $\bar{h} = \bar{k} \in \overline{H} \cap \overline{(K \cap C)} = \overline{H \cap K}$. This implies $h \in M(H \cap K) \subset N(H \cap K)$. Hence $n_1 h \in N(H \cap K)$. Therefore $NH \cap NK \subset N(H \cap K)$; hence $NH \cap NK = N(H \cap K)$.

Case 2. $H, K \subset T$ (similarly, $H, K \subset A$). By induction, there exists $M \lhd_f T$ such that $M \cap H = U$ and $MH \cap MK = M(H \cap K)$. Since A is C-finite by Lemma 3.8, there exists $L \lhd_f A$ such that $L \cap C = M \cap C$. Let $\overline{A} = A/L$, $\overline{T} = T/M$ and $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$. Then $\overline{H} \cap \overline{K} = \overline{H \cap K}$ in $\overline{T}$. Since $\overline{G}$ is $\mathcal{RF}$ and $\overline{HK}$ is finite, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{N} \cap \overline{HK} = 1$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_f G$, $N \cap H = U$, and $NH \cap NK = N(H \cap K)$ as before.

To prove the last part of the lemma, let $M \lhd_f G$ such that $M \subset N$ and $M \cap H = U$. If $m_1 h = m_2 k$, where $m_1, m_2 \in M$, $h \in H$, and $k \in K$, then $h = m_1^{-1} m_2 k \in NH \cap NK = N(H \cap K)$. Hence $h = ns$ for some $n \in N$ and $s \in H \cap K$. Thus $n = hs^{-1} \in N \cap H = M \cap H \subset M$. Therefore $m_1 h = m_1 ns \in M(H \cap K)$. This proves $MH \cap MK \subset M(H \cap K)$. Hence $MH \cap MK = M(H \cap K)$. ∎

COROLLARY 3.11. *Let G be a tree product of central subgroup separable groups amalgamating central edge groups. Let H, K, J be finitely generated central subgroups of some vertex groups. Then, for each $U \lhd_f H$, there exists $N \lhd_f G$ such that $N \cap H = U$, $NH \cap NK = N(H \cap K)$, and $NH \cap NJ = N(H \cap J)$. Moreover, if $M \lhd_f G$ such that $M \subset N$ and $M \cap H = U$, then $MH \cap MK = M(H \cap K)$ and $MH \cap MJ = M(H \cap J)$.*

Proof. By Lemma 3.10, there exists $N_1 \lhd_f G$ such that $N_1 \cap H = U$, $N_1 H \cap N_1 K = N_1(H \cap K)$. Also there exists $N_2 \lhd_f G$ such that $N_2 \cap H = U$, $N_2 H \cap N_2 J = N_2(H \cap J)$. Let $N = N_1 \cap N_2$. Then $N \lhd_f G$ and $N \cap H = U$. Since $N \subset N_1$ and $N \cap H = U$, by the second part of Lemma 3.10 we have $NH \cap NK = N(H \cap K)$. Similarly, $NH \cap NJ = N(H \cap J)$. The last part of the corollary is similar to the proof of the last part of Lemma 3.10. ∎

To prove a criterion for the conjugacy separability of generalized free products of conjugacy separable groups we introduce the concept of subgroup conjugacy separable.

DEFINITION 3.12. Let G be a group and let H be a subgroup of G. We say G is *H-conjugacy separable* if, for each $x \in G$ such that $\{x\}^G \cap H = \varnothing$, there exists $N \lhd_f G$ such that, in $\overline{G} = G/N$, $\{\bar{x}\}^{\overline{G}} \cap \overline{H} = \varnothing$.

We note that if $H \subset Z(G)$ then G is H-conjugacy separable if and only if G is H-separable. Thus, if A is central subgroup separable and if $H \subset Z(A)$ is finitely generated, then A is H-conjugacy separable. More generally, we have the following:

THEOREM 3.13. *Let G be a tree product of n central subgroup separable groups amalgamating central edge groups. Let H be a finitely generated central subgroup of a vertex group of G. Then G is H-conjugacy separable.*

Proof. We shall prove the lemma by induction on n. Clearly the lemma is true for $n = 1$. By Remark 3.5, let $G = A *_C T$. Let $x \in G$ such that $\{x\}^G \cap H = \emptyset$. We can assume x to be of minimal length in its conjugate class $\{x\}^G$. We shall find $L \triangleleft_f A$ and $M \triangleleft_f T$ with $L \cap C = M \cap C$ such that $\{\bar{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$ in $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$ and $\overline{T} = T/M$. Then $\overline{G}$ is conjugacy separable by Theorem 2.2 and $\overline{H}$ is finite. Thus there exists $\overline{N} \triangleleft_f \overline{G}$ such that $\overline{N}\bar{x} \not\sim_{\overline{G}/\overline{N}} \overline{N}\bar{h}$ for every $\bar{h} \in \overline{H}$. Let N be the preimage of $\overline{N}$ of G. Then $N \triangleleft_f G$ and we have $\{\hat{x}\}^{\widehat{G}} \cap \widehat{H} = \emptyset$ in $\widehat{G} = G/N$, as required. We shall find $\overline{G}$ as follows.

Case 1. $H \subset A$.

(a) $x \in C$. Since $\{x\}^G \cap H = \emptyset$, we have $\{x\}^T \cap (H \cap C) = \emptyset$. By induction, there exists $M \triangleleft_f T$ such that $\{\bar{x}\}^{\overline{T}} \cap \overline{H \cap C} = \emptyset$, where $\overline{T} = T/M$. By Lemma 3.8, there exists $L \triangleleft_f A$ such that $L \cap C = M \cap C$ and $LH \cap LC = L(H \cap C)$. Let $\overline{A} = A/L$ and $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$. Then we have $\overline{H} \cap \overline{C} = \overline{H \cap C}$ and $\{\bar{x}\}^{\overline{T}} \cap \overline{H \cap C} = \emptyset$. We shall show $\{\bar{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$. If $\bar{x} \sim_{\overline{G}} \bar{h}$ for some $\bar{h} \in \overline{H}$ then, by Theorem 2.1, there exists $\bar{c}_1, \ldots, \bar{c}_r \in \overline{C}$ such that $\bar{h} \sim_{\overline{A}} \bar{c}_1 \sim_{\overline{T}} \bar{c}_2 \sim_{\overline{A}} \cdots \sim_{\overline{T}} \bar{c}_r = \bar{x}$. Since $\bar{c}_i \in Z(\overline{A})$, it is clear that $\bar{h} = \bar{c}_1$ and that $\bar{c}_i \sim_{\overline{A}} \bar{c}_{i+1}$ implies that $\bar{c}_i = \bar{c}_{i+1}$. Thus $\bar{h} = \bar{c}_1 \sim_{\overline{T}} \bar{x}$. Since $\bar{h} = \bar{c}_1 \in \overline{H} \cap \overline{C} = \overline{H \cap C}$, this contradicts the choice of M. Thus $\{\bar{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$.

(b) $x \in A \backslash C$. Since $\{x\}^G \cap H = \emptyset$, clearly $x \notin H$. Let $L \triangleleft_f A$ such that $x \notin LC$ and $x \notin LH$. Since T is C-finite by Corollary 3.9, there exists $M \triangleleft_f T$ with $M \cap C = L \cap C$. Consider $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$ and $\overline{T} = T/M$. Since $\bar{x} \notin \overline{C} \subset Z(\overline{A})$, $\bar{x}$ is of the minimal length 1 in its conjugacy class in $\overline{G}$. Thus, if $\bar{x} \sim_{\overline{G}} \bar{h}$ for some $\bar{h} \in \overline{H}$ then, by Theorem 2.1, $\bar{x} \sim_{\overline{A}} \bar{h}$. Since $H \subset Z(A)$, this implies that $\bar{x} \in \overline{H}$. Clearly this is impossible by the choice of L. Thus $\{\bar{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$.

(c) $x \in T \backslash C$. Since x is of the minimal length 1 in its conjugacy class in G, $\{x\}^T \cap C = \emptyset$. By induction, there exists $M \triangleleft_f T$ such that $\{\bar{x}\}^{\overline{T}} \cap \overline{C} = \emptyset$, where $\overline{T} = T/M$. Let $L \triangleleft_f A$ such that $L \cap C = M \cap C$. Consider $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$. We have $\{\bar{x}\}^{\overline{G}} \cap \overline{C} = \emptyset$ by Theorem 2.1. Hence $\bar{x}$ is of the minimal length 1 in its conjugacy class in $\overline{G}$. Since $\overline{H} \subset \overline{A}$ and $\overline{x} \in \overline{T}$, this implies that $\{\bar{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$ by Theorem 2.1.

(d) $\|x\| > 1$. Since x is of minimal length in its conjugate class, x is cyclically reduced in G. Let $x = a_1 b_1 \cdots a_s b_s$, where $a_i \in A \backslash C$ and $b_i \in T \backslash C$. Since G is C-separable by Corollary 3.9, there exists $N \triangleleft_f G$ such that $a_i \notin NC$ and $b_i \notin NC$ for all i. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/(N \cap A)$ and $\overline{T} = T/(N \cap T)$. Then $\bar{a}_i \in \overline{A} \backslash \overline{C}$ and $\bar{b}_i \in \overline{T} \backslash \overline{C}$ for all i. Hence, $\|\bar{x}\| = \|x\| > 1$ and $\bar{x}$ is cyclically reduced in $\overline{G}$. Thus $\bar{x}$ is of minimal length in its conjugacy class. This implies that $\{\bar{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$.

Case 2. $H \subset T$.

(a) $x \in C$. Since $\{x\}^G \cap H = \emptyset$, $\{x\}^T \cap H = \emptyset$. By induction, there exists $M \lhd_f T$ such that $\{\bar{x}\}^{\overline{T}} \cap \overline{H} = \emptyset$, where $\overline{T} = T/M$. Let $L \lhd_f A$ such that $L \cap C = M \cap C$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$. If $\bar{x} \sim_{\overline{G}} \bar{h}$ for some $\bar{h} \in \overline{H}$ then, as in (a) of Case 1, we have $\bar{x} \sim_{\overline{T}} \bar{h}$, which contradicts the choice of M. Thus $\{\bar{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$.

(b) $x \in T \backslash C$. Since x has the minimal length 1, $\{x\}^T \cap C = \emptyset$. By induction, there exists $M \lhd_f T$ such that, in $\overline{T} = T/M$, $\{\bar{x}\}^{\overline{T}} \cap \overline{H} = \emptyset$ and $\{\bar{x}\}^{\overline{T}} \cap \overline{C} = \emptyset$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$ be as in (a). Then $\bar{x}$ has the minimal length 1 in its conjugacy class in $\overline{G}$ and $\{\bar{x}\}^{\overline{T}} \cap \overline{H} = \emptyset$. If $\bar{x} \sim_{\overline{G}} \bar{h}$ for some $\bar{h} \in \overline{H}$ then, by Theorem 2.1, we have $\bar{x} \sim_{\overline{T}} \bar{h}$, which contradicts the choice of M. Hence $\{\bar{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$.

(c) $x \in A \backslash C$. Let $L \lhd_f A$ such that $x \notin LC$. Consider $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$. As before, $\bar{x}$ has the minimal length 1 in its conjugacy class in $\overline{G}$. Since $\overline{H} \subset \overline{T}$, $\{\bar{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$ by Theorem 2.1.

(d) $\|x\| > 1$. The proof is similar to that of Case 1(d). ∎

The following lemma will be useful in the study of conjugacy classes of the tree product of any groups amalgamating central edge groups.

LEMMA 3.14. *Let G be a tree product of any groups A_i $(1 \le i \le n)$ amalgamating central edge groups. Let $x \in Z(A_i)$ and $y \in Z(A_j)$. If $x \sim_G y$ then $x = y$.*

Proof. We shall prove the lemma by induction on n. Clearly the lemma is true for $n = 1$. By Remark 3.5, let $G = A *_C T$. Since $x, y \in A \cup T$, x and y are cyclically reduced.

Case 1. $x \in C$. By Theorem 2.1, $x \sim_G y$ implies that there exists $c_1, \ldots, c_r \in C$ such that $x = c_1 \sim_A c_2 \sim_T \cdots \sim_{T(A)} c_r \sim_{A(T)} y$, where $c_r \sim_{A(T)} y$ means either $c_r \sim_A y$ if $y \in A$ or $c_r \sim_T y$ if $y \in T$. By induction, $c_i \sim_T c_{i+1}$ implies $c_i = c_{i+1}$. Also $c_i \sim_A c_{i+1}$ implies $c_i = c_{i+1}$. Hence $x \sim_G y$ implies either $x = c_r \sim_A y$ if $y \in A$ or $x = c_r \sim_T y$ if $y \in T$. Therefore, by induction, $x = y$.

Case 2. $x \in A \backslash C$. Clearly, $\{x\}^A \cap C = \emptyset$. Thus x is of the minimal length 1 in its conjugacy class in G. Since y is cyclically reduced, by Theorem 2.1, $x \sim_G y$ implies that $y \in A$ and $x \sim_A y$. Hence $x = y$.

Case 3. $x \in T \backslash C$. By induction, $\{x\}^T \cap C = \emptyset$. Thus x is of the minimal length 1 in its conjugacy class in G. By Theorem 2.1, $x \sim_G y$ implies that $y \in T$ and $x \sim_T y$. Thus, by induction, $x = y$. ∎

4. DOUBLE COSET SEPARABILITY

In this section we shall study the double coset separability of tree products of central subgroup separable groups amalgamating finitely generated central edge groups.

LEMMA 4.1. *Let A be central subgroup separable and let $H, K \subset Z(A)$ be finitely generated. Then A is HxK-separable for any $x \in A$.*

Proof. Let $g \notin HxK$. Then $x^{-1}g \notin HK$ and $HK \subset Z(A)$. By central subgroup separability, there exists $N \lhd_f A$ such that $x^{-1}g \notin NHK$; i.e., $g \notin NHxK$. ∎

LEMMA 4.2. *Let G be a tree product of m central subgroup separable groups amalgamating central edge groups. Let H, K be finitely generated central subgroups of some vertex groups. Then G is HK-separable.*

Proof. We shall prove the lemma by induction on m. If $m = 1$, then G is central subgroup separable and the lemma is true by Lemma 4.1. To apply induction, we let $G = A *_C T$ as in Remark 3.5. Let $g \in G$ such that $g \notin HK$.

Case 1. $H, K \subset T$ (similarly $H, K \subset A$). Suppose $g \in T$. Then, by induction, there exists $M \lhd_f T$ such that $\bar{g} \notin \overline{HK}$ in $\overline{T} = T/M$. Since A is C-finite by Corollary 3.9, there exists $L \lhd_f A$ such that $L \cap C = M \cap C$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$. Since $\overline{G}$ is $\mathcal{RF}$ and $\overline{HK}$ is finite, there exists $\overline{N} \lhd_f \overline{G}$ such that $\bar{g} \notin \overline{N}(\overline{HK})$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_f G$ such that $g \notin NHK$.

Suppose $g \notin T$. Let $g = a_1 t_1 \cdots a_n t_n$, say, $a_i \in A \backslash C$ and $t_i \in T \backslash C$. Since G is C-separable by Corollary 3.9, there exists $N \lhd_f G$ such that $a_i \notin NC$ and $t_i \notin NC$ for all i. Let $\overline{A} = A/(N \cap A)$, $\overline{T} = T/(N \cap T)$, and $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$. Then $\|\bar{g}\| = \|g\|$ and $\bar{g} \notin \overline{T}$. Since $\overline{HK} \subset \overline{T}$, this implies that $\bar{g} \notin \overline{HK}$. Thus as before we can find $N \lhd_f G$ such that $g \notin NHK$.

Case 2. $H \subset A$ and $K \subset T$ (similarly $H \subset T$ and $K \subset A$).

(a) $g \in C$. Clearly $g \notin (C \cap H)(C \cap K)$. Since C is finitely generated abelian, there exists $U \lhd_f C$ such that $g \notin U(C \cap H)(C \cap K)$. By Corollary 3.11, there exists $N \lhd_f G$ such that $N \cap C = U$, $NC \cap NH = N(C \cap H)$, and $NC \cap NK = N(C \cap K)$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/(N \cap A)$ and $\overline{T} = T/(N \cap T)$. Then $\overline{H} \cap \overline{C} = \overline{H \cap C}$ and $\overline{K} \cap \overline{C} = \overline{K \cap C}$. If $\bar{g} \in \overline{HK}$ then $\bar{g} = \overline{hk}$ for some $\bar{h} \in \overline{H}$ and $\bar{k} \in \overline{K}$. Since $g \in C$, $H \subset A$, and $K \subset T$, $\bar{h} \in \overline{H} \cap \overline{C}$, and $\bar{k} \in \overline{K} \cap \overline{C}$. Thus $\bar{g} = \bar{h}\bar{k} \in (\overline{H \cap C})(\overline{K \cap C})$, contradicting the choice of U. Hence $\bar{g} \notin \overline{HK}$ in $\overline{G}$. As usual, we can find $N \lhd_f G$ such that $g \notin NHK$.

(b) $g \in T\backslash C$ (or $g \in A\backslash C$). If $g \notin CK$ then, by induction, there exists $M \lhd_f T$ such that $\bar{g} \notin \overline{CK}$ in $\overline{T} = T/M$. By Corollary 3.9, there exists $L \lhd_f A$ such that $L \cap C = M \cap C$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$. Then $\bar{g} \notin \overline{C}\overline{K}$; hence $\bar{g} \notin \overline{HK}$. Thus as usual we can find $N \lhd_f G$ such that $g \notin NHK$. If $g \in CK$ then $g = ck$, where $c \in C$ and $k \in K$. Since $g \notin HK$, $c \notin HK$. Thus by (a), we can find $N \lhd_f G$ such that $c \notin NHK$. This implies that $g \notin NHK$.

(c) $g = at$, where $a \in A\backslash C$ and $t \in T\backslash C$. If $a \notin HC$ (or $t \notin CK$) then there exists $L \lhd_f A$ and $M \lhd_f T$ such that $a \notin LHC$, $t \notin MC$, and $L \cap C = M \cap C$. In $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$ and $\overline{T} = T/M$. Clearly $\bar{g} = \bar{a}\bar{t} \notin \overline{HK}$. Thus as usual we can find $N \lhd_f G$ such that $g \notin NHK$. Suppose $a = hc_1$ and $t = c_2k$ for some $h \in H$, $k \in K$ and $c_1, c_2 \in C$. Then $g \notin HK$ implies that $c_1c_2 \notin HK$. As in (a) above, there exists $N \lhd_f G$ such that $c_1c_2 \notin NHK$. Hence $g \notin NHK$.

(d) $g = ta$, where $t \in T\backslash C$ and $a \in A\backslash C$ or $\|g\| \geq 3$. As in Case 1, we can find $N \lhd_f G$ such that, in $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/(N \cap A)$ and $\overline{T} = T/(N \cap T)$, we have $\|\bar{g}\| = \|g\|$. This implies that $\bar{g} \notin \overline{HK}$. Thus, as before, we can find $N \lhd_f G$ such that $g \notin NHK$. This completes the proof. ∎

Throughout this section, we shall use induction on the number m of vertex groups of the tree product to show that the tree product $G = A *_C T$ of m central subgroup separable groups amalgamating central edge groups is HxK-separable for $x \in G$ and for finitely generated central subgroups H, K of some vertex groups of G (Theorem 4.9).

LEMMA 4.3. *Let $G = A *_C T$ be a tree product of m ($m \geq 2$) central subgroup separable groups amalgamating central edge groups. Suppose T is UtV-separable for any $t \in T$ and any finitely generated central subgroups U, V of some vertex groups of T. Let H, K be finitely generated central subgroups of some vertex groups of G. Then G is HxK-separable for $x \in A \cup T$.*

Proof. We shall only prove the case where $x \in T$. The proof for the case, $x \in A$, is similar. Let $g \in G$ such that $g \notin HxK$, where $x \in T$. Throughout the proof we shall find $L \lhd_f A$ and $M \lhd_f T$ such that $L \cap C = M \cap C$ and $\bar{g} \notin \overline{H}\bar{x}\overline{K}$ in $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$ and $\overline{T} = T/M$. Since $\overline{H}, \overline{K}$ are finite and since $\overline{G}$ is $\mathcal{RF}$, we can find $N \lhd_f G$ such that $g \notin NHxK$. It follows that G is HxK-separable for $x \in T$.

Case 1. $H, K \subset T$. If $g \in T$ then, by assumption, there exists $M \lhd_f T$ such that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$, where $\overline{T} = T/M$. Since A is C-finite, there exists $L \lhd_f A$ such that $L \cap C = M \cap C$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$ and $\overline{T} = T/M$. Then $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

If $g \in A \backslash C$, then there exists $L \triangleleft_f A$ such that $g \notin LC$. By Corollary 3.9, T is C-finite. Therefore there exists $M \triangleleft_f T$ such that $M \cap C = L \cap C$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$ and $\overline{T} = T/M$. Then clearly $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

If $\|g\| \geq 2$ then, by the usual procedure, we can find $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$ such that, in $\overline{G}$, $\|\bar{g}\| = \|g\|$. This implies that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

Case 2. $H \subset T$ and $K \subset A$ (similarly $H \subset A$ and $K \subset T$).

(a) $g \in T$. If $g \notin HxC$ then, by Case 1, there exists $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$ such that $\bar{g} \notin \overline{H}\bar{x}\overline{C}$. Since $\overline{K} \subset \overline{A}$, it follows that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. Thus let $g = hxc$, where $h \in H$ and $c \in C$. Then $g \notin HxK$ implies that $xc \notin Hx(K \cap C)$. By assumption, there exists $M \triangleleft_f T$ such that $xc \notin MHx(K \cap C)$. Let $M \cap C = U$. Then, by Lemma 3.8, there exists $L \triangleleft_f A$ such that $L \cap C = U$ and $LK \cap LC = L(K \cap C)$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$ and $\overline{T} = T/M$. If $\bar{g} \in \overline{H}\bar{x}\overline{K}$, then $\bar{g} = \bar{h}\overline{xc} = \bar{h}_1\bar{x}\bar{k}_1$ for some $\bar{h}_1 \in \overline{H}$ and $\bar{k}_1 \in \overline{K}$. Since $\bar{c}, \bar{h}, \bar{h}_1, \bar{x} \in \overline{T}$, we have $\bar{k}_1 \in \overline{C}$. This implies that $\bar{k}_1 \in \overline{K} \cap \overline{C} = \overline{K \cap C}$. Thus $\overline{xc} = \bar{h}^{-1}\bar{h}_1\bar{x}\bar{k}_1 \in \overline{H}\bar{x}(\overline{K \cap C})$, contradicting the choice of M. Hence $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

(b) $g \in A \backslash C$. If $g \notin CK$, by Lemma 4.2, there exists $L \triangleleft_f A$ such that $g \notin LCK$. Let $M \triangleleft_f T$ such that $M \cap C = L \cap C$. Then, $\bar{g} \notin \overline{CK}$ in $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$; hence $\bar{g} \notin \overline{H}\bar{x}\overline{K}$ in $\overline{G}$. So we assume $g = ck$ for some $c \in C$ and $k \in K$. Since $g \notin HxK$, $c \notin HxK$. Then by (a), there exists $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$ such that $\bar{c} \notin \overline{H}\bar{x}\overline{K}$. This implies that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

(c) $g = ta$, where $t \in T \backslash C$ and $a \in A \backslash C$. If $t \notin HxC$ (or similarly $a \notin CK$), by assumption, there exists $M_1 \triangleleft_f T$ such that $t \notin M_1HxC$. Let $L_1 \triangleleft_f A$ such that $L_1 \cap C = M_1 \cap C$. Since G is C-separable by Corollary 3.9, there exists $N \triangleleft_f G$ such that $t \notin NC$ and $a \notin NC$. Let $L = L_1 \cap N$ and $M = M_1 \cap N$. Consider $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$ and $\overline{T} = T/M$. Then $\bar{t} \in \overline{T} \backslash \overline{C}$, $\bar{a} \in \overline{A} \backslash \overline{C}$ and $\bar{t} \notin \overline{H}\bar{x}\overline{C}$. If $\bar{g} = \bar{h}\bar{x}\bar{k}$ for some $\bar{h} \in \overline{H}$ and $\bar{k} \in \overline{K}$, then $\bar{t}\bar{a} = \bar{h}\bar{x}\bar{k}$. This implies that $\bar{t} = \bar{h}\bar{x}\bar{k}\bar{a}^{-1}$. Since $\bar{k}\bar{a}^{-1} \in \overline{A}$ and $\bar{t}, \bar{h}, \bar{x} \in \overline{T}$, we have $\bar{k}\bar{a}^{-1} \in \overline{C}$, which implies that $\bar{t} \in \overline{H}\bar{x}\overline{C}$, contradicting the choice of $M \subset M_1$. Hence $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. Thus we can assume $t = hxc$ and $a = c'k$, where $h \in H$, $k \in K$, and $c, c' \in C$. This implies that $xcc' \notin HxK$. Thus, by (a) above, there exists $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$ such that $\overline{xcc'} \notin \overline{H}\bar{x}\overline{K}$. Hence $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

(d) $g = at$, where $a \in A \backslash C$ and $t \in T \backslash C$ or $\|g\| \geq 3$. By C-separability of G, we can choose $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$ such that $\|\bar{g}\| = \|g\|$. Then it is clear that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

Case 3. $H, K \subset A$. If $x \in C$, then $x^{-1}g \notin HK$. By Lemma 4.2, there exists $N \triangleleft_f G$ such that $x^{-1}g \notin NHK$. Hence $g \notin NHxK$. Thus we can assume $x \in T \backslash C$.

(a) $g \in T$. Clearly $g \notin (H\cap C)x(K\cap C)$ in T. Thus, by assumption, there exists $M \lhd_f T$ such that $g \notin M(H\cap C)x(K\cap C)$. By Corollary 3.11, there exists $N \lhd_f G$ such that $N\cap C = M\cap C$, $NH\cap NC = N(H\cap C)$, and $NK\cap NC = N(K\cap C)$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/(N\cap A)$ and $\overline{T} = T/(N\cap M)$. Then, as usual, we have $\bar{g} \notin (\overline{H\cap C})\bar{x}(\overline{K\cap C})$ $\overline{H}\cap\overline{C} = \overline{H\cap C}$ and $\overline{K}\cap\overline{C} = \overline{K\cap C}$. If $\bar{g} \in \overline{H}\bar{x}\overline{K}$ then $\bar{g} = \bar{h}\bar{x}\bar{k}$ for some $\bar{h} \in \overline{H}$ and some $\bar{k} \in \overline{K}$. Since $H, K \subset A$ and $g, x \in T$, we must have $\bar{h} \in \overline{C}$ and $\bar{k} \in \overline{C}$. Hence $\bar{h} \in \overline{H}\cap\overline{C} = \overline{H\cap C}$ and $\bar{k} \in \overline{K}\cap\overline{C} = \overline{K\cap C}$. Thus $\bar{g} = \bar{h}\bar{x}\bar{k} \in (\overline{H\cap C})\bar{x}(\overline{K\cap C})$, which contradicts the choice of M. Hence $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

(b) $g \in A\backslash C$. Since $x \in T\backslash C$, by Corollary 3.9 there exists $N \lhd_f G$ such that $x \notin NC$. Clearly $(N\cap A)\cap C = (N\cap T)\cap C$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/(N\cap A)$ and $\overline{T} = T/(N\cap T)$. Then $\bar{x} \in \overline{T}\backslash\overline{C}$. Since $H, K \subset A$ and $g \in A$, $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

(c) $g = a_1 t$, $g = t a_1$, or $g = a_1 t a_2$, where $a_i \in A\backslash C$ and $t \in T\backslash C$. We shall just consider the case where $g = a_1 t a_2$, others being similar. If $a_1 \notin HC$ or $a_2 \notin CK$ then, as in the previous cases, we can find $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$ such that $\|\bar{g}\| = \|g\|$ and $\bar{a}_1 \notin \overline{HC}$ or $\bar{a}_2 \notin \overline{CK}$. This implies that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. Hence we can assume $a_1 = hc$ and $a_2 = c'k$, where $h \in H$, $k \in K$, and $c, c' \in C$. Thus $g = a_1 t a_2 \notin HxK$ implies that $ctc' \notin HxK$. Since $ctc' \in T$, applying part (a), we can find $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$ such that $\overline{ctc'} \notin \overline{H}\bar{x}\overline{K}$. It follows that $\bar{g} = \bar{a}_1\bar{t}\bar{a}_2 \notin \overline{H}\bar{x}\overline{K}$.

(d) $g = t_1 a t_2$, where $t_i \in T\backslash C$ and $a \in A\backslash C$ or $\|g\| \geq 4$. Applying the usual length preserving method we can find $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$ such that $\|\bar{g}\| = \|g\|$. Then $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. This completes the proof. ∎

LEMMA 4.4. *Let C be a finitely generated abelian group and let $D, S_1, \dots, S_n$ be subgroups of C. Let $S = S_1 \cap \cdots \cap S_n$, $c \in C$ and $c \notin DS$. Then there exists $N \lhd_f C$ such that $S \subset N$, $c \notin ND$, and $N = NS_1 \cap \cdots \cap NS_n$.*

Proof. Consider the natural homomorphism $\phi_1\colon C \to \overline{C}_1 \times \cdots \times \overline{C}_n = \overline{C}$, where $\overline{C}_i = C/S_i$. Then $\bar{c} \notin \overline{D}$, since $\ker\phi_1 = S_1 \cap \cdots \cap S_n = S$. Since $\overline{C}$ is also finitely generated abelian, there exists $\overline{L} \lhd_f \overline{C}$ such that $\bar{c} \notin \overline{LD}$. Let $\overline{U}_i = \overline{L} \cap \overline{C}_i$ and $\overline{\overline{C}} = (\overline{C}_1/\overline{U}_1) \times \cdots \times (\overline{C}_n/\overline{U}_n)$. Then $\overline{\overline{C}}$ is finite and $\bar{\bar{c}} \notin \overline{\overline{D}}$. Let N be the kernel of the natural homomorphism $\phi\colon C \to (\overline{C}_1/\overline{U}_1) \times \cdots \times (\overline{C}_n/\overline{U}_n) = \overline{\overline{C}}$. Then $N \lhd_f C$, $S \subset N$, and $c \notin ND$. We need to show $N = NS_1 \cap \cdots \cap NS_n$. Suppose $x \in NS_i$. Then the ith coordinate of $\phi(x)$ is trivial. Hence, if $x \in NS_1 \cap \cdots \cap NS_n$, then $\phi(x)$ is trivial. This implies $x \in \ker\phi = N$. Thus $NS_1 \cap \cdots \cap NS_n \subset N$. Since $N \subset NS_1 \cap \cdots \cap NS_n$, we have $N = NS_1 \cap \cdots \cap NS_n$. ∎

LEMMA 4.5. *Let $G = A *_C T$ be a tree product of central subgroup separable groups amalgamating central edge groups. Suppose T is UtV-separable* *for any $t \in T$ and any finitely generated central subgroups U, V of some vertex groups of T. Let H, K be finitely generated central subgroups of some vertex groups of G and let $t_i \in T \backslash C$ $(1 \leq i \leq n)$. Let $S_1 = Z_{H \cap C}(t_1)$, $S_\alpha = Z_{K \cap C}(t_\alpha)$ $(2 \leq \alpha \leq n)$, and $S = \bigcap_{\alpha=2}^n S_\alpha$. Suppose $c \in C \backslash S_1 S$. Then there exists $L \lhd_f C$ such that $S \subset L$, $c \notin (LS_1 \cap H)(L \cap K)$, and $L = \bigcap_{\alpha=2}^n LS_\alpha$. Moreover, there exists $M \lhd_f T$ such that*

(1) *$t_i \notin MC$ for $1 \leq i \leq n$ and $c \notin M(LS_1 \cap H)(L \cap K)$, and*

(2) *$Mt_1^{-1}(H \cap C)t_1 \cap M(H \cap C) \subset M(LS_1 \cap H)$, $\bigcap_{\alpha=2}^n Mt_\alpha^{-1}(K \cap C)t_\alpha \cap M(K \cap C) \subset M(L \cap K)$, and*

(3) *$MH \cap MC = M(H \cap C)$ and $MK \cap MC = M(K \cap C)$.*

Proof. The edge group C of G is finitely generated. By Lemma 4.4, there exists $L \lhd_f C$ such that $S \subset L$, $c \notin LS_1$ and $L = \bigcap_{\alpha=2}^n LS_\alpha$. Since $(LS_1 \cap H)(L \cap K) \subset LS_1 L = LS_1$, $c \notin (LS_1 \cap H)(L \cap K)$. Let $s_0, s_1, \ldots, s_{m_1}$ be coset representatives of $LS_1 \cap H$ in $C \cap H$ and $x_0, x_1, \ldots, x_{m_2}$ be coset representatives of $L \cap K$ in $C \cap K$, where $s_0 = 1 = x_0$.

(I) We note that $s_i t_1 s_j^{-1} \notin (LS_1 \cap H)t_1(LS_1 \cap H)$ for all i, j except $i = j = 0$. For, if $s_i t_1 s_j^{-1} = h_1 t_1 h_2$ for $h_1, h_2 \in LS_1 \cap H$, then $t_1^{-1} h_1^{-1} s_i t_1 = h_2 s_j$. Hence, by Lemma 3.14, $h_1^{-1} s_i = h_2 s_j$. Since s_i, s_j are coset representatives of $LS_1 \cap H$ in $C \cap H$, we have $s_i = s_j$. Hence, $h_1^{-1} = h_2$. Thus $s_i t_1 s_i^{-1} = h_1 t_1 h_1^{-1}$. This implies that $h_1^{-1} s_i \in Z_{H \cap C}(t_1) = S_1 \subset LS_1$. Since $h_1 \in LS_1 \cap H$, we have $s_i \in LS_1 \cap H$. Hence $s_i = 1 = s_j$; that is, $i = j = 0$. Therefore $s_i t_1 s_j^{-1} \notin (LS_1 \cap H)t_1(LS_1 \cap H)$ for all i, j except $i = j = 0$.

(II) For each α $(2 \leq \alpha \leq n)$, if there exists i_α, depending on α, such that $x_{i_\alpha} t_\alpha x_j^{-1} \in (L \cap K)t_\alpha(L \cap K)$, then $x_j = 1$. To prove this, let $x_{i_\alpha} t_\alpha x_j^{-1} = k_1 t_\alpha k_2$, where $k_1, k_2 \in L \cap K$. As in (I), we can show $x_{i_\alpha} = x_j$, $k_1^{-1} = k_2$ and $k_1^{-1} x_j \in Z_{K \cap C}(t_\alpha) = S_\alpha$. Since $k_1 \in L \cap K \subset L$, we have $x_j \in LS_\alpha$. Thus, if there exists i_α such that $x_{i_\alpha} t_\alpha x_j^{-1} \in (L \cap K)t_\alpha(L \cap K)$ for each α, then $x_j \in LS_\alpha$ for all α. Hence $x_j \in \bigcap_{\alpha=2}^n LS_\alpha = L$. Since x_j is a coset representative of $L \cap K$ in $C \cap K$, $x_j = 1$.

Since T is UtV-separable for any $t \in T$ and any finitely generated central subgroups U, V in the vertex groups of T, there exists $M_1 \lhd_f T$ such that $t_i \notin M_1 C$ for $1 \leq i \leq n$, $c \notin M_1(LS_1 \cap H)(L \cap K)$, $s_i t_1 s_j^{-1} \notin M_1(LS_1 \cap H)t_1(LS_1 \cap H)$ for all i, j except $i = j = 0$, and $x_i t_\alpha x_j^{-1} \notin M_1(L \cap K)t_\alpha(L \cap K)$ for all possible i, j, α such that $x_i t_\alpha x_j^{-1} \notin (L \cap K)t_\alpha(L \cap K)$. Let $M_1 \cap C = U$. Then, by Corollary 3.11, there exists $N \lhd_f G$ such that $N \cap C = U$, $NC \cap NH = N(C \cap H)$, and $NC \cap NK = N(C \cap K)$. Let $M = M_1 \cap N$. Then, we have $MC \cap MH = M(C \cap H)$ and $MC \cap MK = M(C \cap K)$, whence (3) holds. Clearly (1) holds by the choice of M_1.

To prove (2), let $c_1, c_2 \in H \cap C$ such that $m_1 t_1^{-1} c_1 t_1 = m_2 c_2$, where $m_1, m_2 \in M$. Since $M \subset M_1$, in $\overline{T} = T/M_1$, $\overline{t_1^{-1} c_1 t_1} = \bar{c}_2$. Let $c_1 = s_i h_1$ and $c_2 = s_j h_2$ for $h_1, h_2 \in (LS_1 \cap H)$. Then $\overline{s_i t_1 s_j^{-1}} = \bar{h}_1^{-1} \bar{t}_1 \bar{h}_2$. Hence $s_i t_1 s_j^{-1} \in M_1(LS_1 \cap H) t_1 (LS_1 \cap H)$. By the choice of M_1, we have $s_i = s_j = 1$. Thus $c_2 = h_2 \in (LS_1 \cap H)$. Therefore $m_2 c_2 \in M(LS_1 \cap H)$. This proves the first part of (2).

To prove the second part of (2), let $mc \in \cap_{\alpha=2}^{n} M t_\alpha^{-1} \overline{(K \cap C) t_\alpha}$, where $m \in M$ and $c \in (K \cap C)$. In $\overline{T} = T/M_1$, we have $\bar{c} \in \overline{t_\alpha^{-1}(K \cap C) t_\alpha}$ for all $\alpha = 2, \dots, n$. Let $c = x_j k_1$, where $k_1 \in L \cap K$. Let $\bar{c} = \bar{t}_\alpha^{-1} \bar{x}_{j_\alpha} \bar{k}_2 \bar{t}_\alpha$ for some x_{j_α} and some $k_2 \in L \cap K$. Then $\bar{x}_{j_\alpha} \bar{t}_\alpha \bar{x}_j^{-1} = \bar{k}_2^{-1} \bar{t}_\alpha \bar{k}_1$. Thus $x_{j_\alpha} t_\alpha x_j^{-1} \in M_1(L \cap K) t_\alpha (L \cap K)$. By the choice of M_1, $x_{j_\alpha} t_\alpha x_j^{-1} \in (L \cap K) t_\alpha (L \cap K)$. This implies that $x_j = 1$ from (II) above. Thus $mc = m x_j k_1 = m k_1 \in M(L \cap K)$. This proves that $\cap_{\alpha=2}^{n} M t_\alpha^{-1} (K \cap C) t_\alpha \cap M(K \cap C) \subset M(L \cap K)$. ∎

LEMMA 4.6. *Let G be a tree product of m vertex groups A_i amalgamating central edge groups. Suppose U, V are finitely generated central subgroups of some vertex groups of G. Then $G/\langle UV \rangle^G$ is also a tree product of vertex groups $\overline{A_i} = A_i/(UV \cap A_i)$ amalgamating finitely generated central edge groups. Moreover, if H is a subgroup of the vertex A_H then $\langle UV \rangle^G \cap H = UV \cap H \subset Z(A_H)$.*

Note. If $H = A_i$, then $UV \cap A_i \subset Z(A_i)$.

Proof. We shall prove the lemma by induction on m. Clearly the lemma holds for $m = 1$. Suppose the lemma holds for any tree product of at most $m - 1$ vertex groups amalgamating central edge groups. Let $G = A *_C T$ as in Remark 3.5.

Case 1. $U, V \subset A$. By induction, we have $\langle UV \cap C \rangle^T \cap C = (UV \cap C) \cap C = (UV \cap A) \cap C$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/(UV \cap A)$ and $\overline{T} = T/\langle UV \cap C \rangle^T$. Now consider the natural homomorphism $\phi_1 \colon G \to \overline{G}$. Then $\ker \phi_1 = \langle UV \cap A, \langle UV \cap C \rangle^T \rangle^G = \langle UV \rangle^G$. By induction, $\overline{T}$ is a tree product of $A_j/(UV \cap C \cap A_j)$, where the A_j's are the vertex groups of T. Since $A_j \subset T$ and $UV \subset A$, we have $UV \cap A_j \subset C$. Hence $\overline{T}$ is a tree product of $A_j/(UV \cap A_j)$. Thus $\overline{G} \cong G/\langle UV \rangle^G$ is a tree product of $A_i/(UV \cap A_i)$ amalgamating central edge groups.

If $H \subset A$ then $\langle UV \rangle^G \cap H = \langle UV \rangle^G \cap A \cap H = (UV \cap A) \cap H = UV \cap H \subset Z(A)$. If $H \subset T$ then

$$\langle UV \rangle^G \cap H = \langle UV \rangle^G \cap T \cap H = \langle UV \cap C \rangle^T \cap H. \tag{1}$$

Now, by induction, $\langle UV \cap C \rangle^T \cap H = (UV \cap C) \cap H \subset Z(A_H)$. Since $UV \subset A$ and $H \subset T$, we have $(UV \cap C) \cap H = UV \cap H$. Thus, by (1), $\langle UV \rangle^G \cap H = UV \cap H \subset Z(A_H)$.

Case 2. $U \subset T$ and $V \subset A$. By induction, we have $\langle U(V \cap C)\rangle^T \cap C = U(V \cap C) \cap C = (U \cap C)(V \cap C)$. Also $(UV \cap A) \cap C = UV \cap C = (U \cap C)(V \cap C)$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/(UV \cap A)$ and $\overline{T} = T/\langle U(V \cap C)\rangle^T$. Now consider the natural homomorphism ϕ_2: $G \to \overline{G}$. Then $\ker \phi_2 = \langle UV \cap A, \langle U(V \cap C)\rangle^T \rangle^G = \langle UV \rangle^G$. By induction, $\overline{T}$ is a tree product of $A_j/(U(V \cap C) \cap A_j)$, where A_j's are vertex groups of T. Since $U, A_j \subset T$ and $V \subset A$, we have $U(V \cap C) \cap A_j = UV \cap A_j$. Hence $\overline{T}$ is a tree product of $A_j/(UV \cap A_j)$. Thus $\overline{G} \cong G/\langle UV \rangle^G$ is a tree product of $A_i/(UV \cap A_i)$ amalgamating central edge groups. As in Case 1, we have $\langle UV \rangle^G \cap H = UV \cap H \subset Z(A_H)$.

Case 3. $U, V \subset T$. By induction, we have $\langle UV \rangle^T \cap C = UV \cap C = (UV \cap A) \cap C$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/(UV \cap A)$ and $\overline{T} = T/\langle UV \rangle^T$. Then, as before, we can show that $\overline{G} \cong G/\langle UV \rangle^G$ is a tree product of $A_i/(UV \cap A_i)$ amalgamating central edge groups and $\langle UV \rangle^G \cap H = UV \cap H \subset Z(A_H)$. ∎

COROLLARY 4.7. *Let G be a tree product of m central subgroup separable groups A_i amalgamating central edge groups. Let U, V, H be finitely generated central subgroups in some vertex groups of G. Then $\overline{G} = G/\langle UV \rangle^G$ is $\overline{H}$-separable. In particular, $\overline{G}$ is $\mathcal{RF}$.*

Proof. By Lemma 4.6, $\overline{G}$ is a tree product of $A_i/(UV \cap A_i)$ amalgamating finitely generated central edge groups. Let $W = UV \cap A_i$. By Theorem 3.6, it suffices to show that if K is a finitely generated central subgroup of A_i then $\overline{A}_i = A_i/W$ is $\overline{K}$-finite and $\overline{K}$-separable. Let $\overline{D} \lhd_f \overline{K}$. Then $DW \lhd_f KW$ and $KW \subset Z(A_i)$ is finitely generated. By Lemma 3.8, since A_i is central subgroup separable, A_i is KW-finite. Thus there exists $N \lhd_f A_i$ such that $N \cap KW = DW$. Then $\overline{N} \lhd_f \overline{A}_i$ and $\overline{N} \cap \overline{K} = \overline{D}$. Hence $\overline{A}_i = A_i/W$ is $\overline{K}$-finite.

To show that $\overline{A}_i$ is $\overline{K}$-separable, let $\overline{a} \in \overline{A}_i \backslash \overline{K}$. Then $a \notin KW$ and $KW \subset Z(A_i)$. Since A_i is central subgroup separable, there exists $N \lhd_f A_i$ such that $a \notin NKW$. Then $\overline{N} \lhd_f \overline{A}_i$ and $\bar{a} \in \overline{NK}$. ∎

LEMMA 4.8. *Let $G = A *_C T$ be a tree product of m central subgroup separable groups amalgamating central edge groups. Suppose T is UtV-separable for every $t \in T$ and every pair of central edge subgroups U, V of T. Let H, K be finitely generated central subgroups of some vertex groups of G and let $1 \neq c \in C$, $t_i \in T \backslash C$, and $a_i \in A \backslash C$. Then there exists $N \lhd_f G$ such that*

(1) *if $x = t_1 a_1 \cdots t_n a_n$ and $t_1 c a_1 \cdots t_n a_n \notin HxK$, then $t_1 c a_1 \cdots t_n a_n \notin NHxK$;*

(2) *if $x = t_1 a_1 \cdots a_{n-1} t_n$ and $t_1 c a_1 \cdots a_{n-1} t_n \notin HxK$, then $t_1 c a_1 \cdots a_{n-1} t_n \notin NHxK$;*

(3) *if* $x = a_1 t_1 \cdots a_n t_n$ *and* $a_1 c t_1 \cdots a_n t_n \notin HxK$, *then* $a_1 c t_1 \cdots a_n t_n \notin NHxK$;

(4) *if* $x = a_1 t_1 \cdots t_{n-1} a_n$ *and* $a_1 c t_1 \cdots t_{n-1} a_n \notin HxK$, *then* $a_1 c t_1 \cdots t_{n-1} a_n \notin NHxK$.

Proof. We shall only prove (1), since the others are similar. Let $u = a_1 t_2 \cdots t_n a_n$.

Case 1. $H, K \subset T$. Let S_1, S be as in Lemma 4.5. Since $t_1 c a_1 \cdots t_n a_n \notin HxK$, we have $c \notin S_1 S$. Let $L \lhd_f C$ and $M \lhd_f T$ be as in Lemma 4.5. Let $R = (M \cap H)(M \cap K)$ and let $\widetilde{T} = T/R^T$. By Lemma 4.6, $\widetilde{T}$ is the tree product of $\widetilde{A}_i = A_i/(R^T \cap A_i)$. Let $D = R^T \cap C$ and $\widetilde{A} = A/D$. Since, in $\widetilde{T}$, $CR^T/R^T \cong C/(C \cap R^T) = C/D$, we can form $\widetilde{G} = \widetilde{A} *_{\widetilde{C}} \widetilde{T}$. Clearly $\widetilde{G}$ is also a tree product with $\tilde{a}_i \notin \widetilde{C}$. Moreover by the choice of M in Lemma 4.5, $\tilde{t}_i \notin \widetilde{C}$. We shall show that $\tilde{t}_1 \tilde{c} \tilde{a}_1 \cdots \tilde{t}_n \tilde{a}_n \notin \widetilde{H} \tilde{x} \widetilde{K}$. Suppose $\tilde{t}_1 \tilde{c} \tilde{a}_1 \cdots \tilde{t}_n \tilde{a}_n = \tilde{h} \tilde{x} \tilde{k}$ for some $\tilde{h} \in \widetilde{H}$ and $\tilde{k} \in \widetilde{K}$. Then there exist $\tilde{z}_i \in \widetilde{C}$ such that $\tilde{t}_1 \tilde{c} = \tilde{h} \tilde{t}_1 \tilde{z}_1$, $\tilde{a}_1 = \tilde{z}_1^{-1} \tilde{a}_1 \tilde{z}_1$, $\tilde{t}_2 = \tilde{z}_1^{-1} \tilde{t}_2 \tilde{z}_2, \ldots, \tilde{t}_n = \tilde{z}_{n-1}^{-1} \tilde{t}_n \tilde{z}_n$, and $\tilde{a}_n = \tilde{z}_n^{-1} \tilde{a}_n \tilde{k}$. Since $\widetilde{G}$ is a tree product of groups amalgamating central edge groups, by Lemma 3.14, we have $\tilde{h}^{-1} = \tilde{z}_1 \tilde{c}^{-1}$ and $\tilde{t}_1^{-1} \tilde{h} \tilde{t}_1 = \tilde{h}$. Also $\tilde{z}_i = \tilde{z}_{i+1}$, $\tilde{z}_n = \tilde{k}$ and $\tilde{t}_i^{-1} \tilde{z} \tilde{t}_i = \tilde{z}$ for $2 \le i \le n$, where $\tilde{z} = \tilde{z}_i$. Thus, in $\overline{T} = T/M$, we have $\bar{c} = \bar{h} \bar{z}$, $\bar{t}_1^{-1} \bar{h} \bar{t}_1 = \bar{h}$, and $\bar{t}_i^{-1} \bar{z} \bar{t}_i = \bar{z}$ for $2 \le i \le n$. Since $\bar{h} = \overline{cz^{-1}} \in \overline{C}$, we have $\bar{h} \in \overline{H} \cap \overline{C} = \overline{H \cap C}$ by the choice of M in Lemma 4.5. Similarly $\bar{k} = \bar{z} \in \overline{K} \cap \overline{C} = \overline{K \cap C}$. Thus $\bar{t}_1^{-1} \bar{h} \bar{t}_1 = \bar{h} \in \bar{t}_1^{-1} (\overline{H \cap C}) \bar{t}_1 \cap (\overline{H \cap C}) \subset \overline{LS_1 \cap H}$ and $\bar{t}_i^{-1} \bar{z} \bar{t}_i = \bar{z} \in (\bigcap_{\alpha=2}^{n} \bar{t}_\alpha^{-1} (\overline{K \cap C}) \bar{t}_\alpha) \cap \overline{K \cap C} \subset \overline{L \cap K}$. Thus $\bar{c} = \overline{hk} \in (\overline{LS_1 \cap H})(\overline{L \cap K})$—that is, $c \in M(LS_1 \cap H)(L \cap K)$—contradicting the choice of M. Hence $\tilde{t}_1 \tilde{c} \tilde{a}_1 \cdots \tilde{t}_n \tilde{a}_n \notin \widetilde{H} \tilde{x} \widetilde{K}$. Since $\widetilde{H}, \widetilde{K}$ are finite and, by Corollary 4.7, $\widetilde{G}$ is residually finite, there exists $\widetilde{N} \lhd_f \widetilde{G}$ such that $\tilde{t}_1 \tilde{c} \tilde{a}_1 \cdots \tilde{t}_n \tilde{a}_n \notin \widetilde{N} \widetilde{H} \tilde{x} \widetilde{K}$. Let N be the preimage of $\widetilde{N}$ in G. Then $N \lhd_f G$ and $t_1 c a_1 \cdots t_n a_n \notin NHxK$.

Case 2. $H \subset A$ and $K \subset T$ (similarly, $H \subset T$ and $K \subset A$). Since $t_1 c a_1 \cdots t_n a_n \notin HxK$, $t_1 c u \notin (H \cap C)xK$, where $u = a_1 t_2 \cdots t_n a_n$. Since $H \cap C$ and K are central subgroups of vertex groups of T, by Case 1 above, there exists $N_1 \lhd_f G$ such that $t_1 c u \notin N_1 (H \cap C) x K$, $t_i \notin N_1 C$ and $a_i \notin N_1 C$ for all i. By Lemma 3.10, there exists $N_2 \lhd_f G$ such that $N_2 \cap C = N_1 \cap C$ and $N_2 C \cap N_2 H = N_2 (C \cap H)$. Let $P = N_1 \cap N_2$. Then $t_1 c u \notin P(H \cap C)xK$ and $\overline{H} \cap \overline{C} = \overline{H \cap C}$ in $\overline{A} = A/(P \cap C)$. Let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{T} = T/\langle P \cap C \rangle^T$. Then we have $\overline{t_1 c u} \notin \overline{(H \cap C)xK} = (\overline{H} \cap \overline{C}) \bar{x} \overline{K}$. We shall show that $\overline{t_1 c u} \notin \overline{HxK}$. Suppose $\overline{t_1 c u} = \bar{h} \bar{x} \bar{k}$ for some $\bar{h} \in \overline{H}$ and $\bar{k} \in \overline{K}$. Since $\bar{t}_i \notin \overline{C}$ and $\bar{a}_i \notin \overline{C}$, there exist $\bar{z}_i \in \overline{C}$ such that $\bar{t}_1 \bar{c} = \bar{h} \bar{t}_1 \bar{z}_1$, $\bar{a}_1 = \bar{z}_1^{-1} \bar{a}_1 \bar{z}_1$, $\bar{t}_2 = \bar{z}_1^{-1} \bar{t}_2 \bar{z}_2, \ldots, \bar{t}_n = \bar{z}_{n-1}^{-1} \bar{t}_n \bar{z}_n$, and $\bar{a}_n = \bar{z}_n^{-1} \bar{a}_n \bar{k}$. Now $\overline{G} = G/\langle P \cap C \rangle^G$

is a tree product of groups amalgamating central edge groups. Thus, by Lemma 3.14, $\bar{h}^{-1} = \bar{z}_1\bar{c}^{-1} \in \overline{H} \cap \overline{C}$, $\bar{z}_i = \bar{z}_{i+1}$ for $1 \le i \le n-1$, and $\bar{z}_n = \bar{k}$. Thus $\bar{z}_1\bar{u} = \overline{uz_1} = \bar{u}\bar{k}$. Hence $\bar{t}_1\overline{cu} = \bar{h}\bar{t}_1\bar{z}_1\bar{u} \in (\overline{H} \cap \overline{C})\bar{t}_1\bar{z}_1\bar{u}\overline{K} = (\overline{H} \cap \overline{C})\bar{t}_1\bar{u}\overline{K} = (\overline{H \cap C})\bar{x}\overline{K}$, contradicting the choice of N_1. Thus $\bar{t}_1\overline{cu} \notin \overline{H}\bar{x}\overline{K}$, equivalently, $\bar{x}^{-1}\bar{h}^{-1}\bar{t}_1\overline{cu} \notin \overline{K}$ for all $\bar{h} \in \overline{H}$. Since $\overline{H}$ is finite and $\overline{G}$ is $\overline{K}$-separable by Corollary 4.7, there exists $\overline{N} \triangleleft_f \overline{G}$ such that $\bar{x}^{-1}\bar{h}^{-1}\bar{t}_1\overline{cu} \notin \overline{NK}$ for all $\bar{h} \in \overline{H}$; that is, $\bar{t}_1\overline{cu} \notin \overline{NH}\bar{x}\overline{K}$. Let N be the preimage of $\overline{N}$ in G. Then $N \triangleleft_f G$ and $t_1ca_1 \cdots t_na_n \notin NHxK$.

Case 3. $H, K \subset A$. Clearly $t_1cu \notin (H \cap C)x(K \cap C)$, where $u = a_1t_2 \cdots t_na_n$. By Case 1, there exists $N_1 \triangleleft_f G$ such that and $t_1cu \notin N_1(H \cap C)x(K \cap C)$, $a_i \notin N_1C$, and $t_i \notin N_1C$ for all i. By Corollary 3.11, there exists $N_2 \triangleleft_f G$ such that $N_2 \cap C = N_1 \cap C$, $N_2C \cap N_2H = N_2(C \cap H)$, and $N_2C \cap N_2K = N_2(C \cap K)$. Let $P = N_1 \cap N_2$. Then $t_1cu \notin P(H \cap C)x(K \cap C)$ and, by Corollary 3.11, $PH \cap PC = P(H \cap C)$ and $PK \cap PC = P(K \cap C)$. Since, by Lemma 4.6, $\langle P \cap C\rangle^T \cap C = P \cap C$, let $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{T} = T/\langle P \cap C\rangle^T$ and $\overline{A} = A/(P \cap A)$. Then $\overline{G}$ is a tree product of groups amalgamating central edge groups with $\overline{H} \cap \overline{C} = \overline{H \cap C}$ and $\overline{K} \cap \overline{C} = \overline{K \cap C}$. Thus $\overline{t_1cu} \notin \overline{(H \cap C)x(K \cap C)} = (\overline{H} \cap \overline{C})\bar{x}(\overline{K} \cap \overline{C})$. As in Case 2, we can show that $\bar{t}_1\overline{cu} \notin \overline{H}\bar{x}\overline{K}$. Since $\overline{H}, \overline{K}$ are finite and $\overline{G}$ is residually finite by Corollary 4.7, there exists $\overline{N} \triangleleft_f \overline{G}$ such that $\bar{t}_1\overline{cu} \notin \overline{NH}\bar{x}\overline{K}$. Let N be the preimage of $\overline{N}$ in G. Then $N \triangleleft_f G$ and $t_1ca_1 \cdots t_na_n \notin NHxK$. ∎

THEOREM 4.9. *Let G be a tree product of m central subgroup separable groups amalgamating central edge groups. Let H, K be finitely generated central subgroups of some vertex groups. Then G is HxK-separable for $x \in G$.*

Proof. We prove this by induction on m. By Lemma 4.1, the theorem is true for $m = 1$. Assume that any tree product T of at most $m-1$ central subgroup separable groups, amalgamating central edge groups, is UtV-separable for any $t \in T$ and central subgroups U, V in the vertex groups of T. Let $G = A *_C T$ as before. By Lemma 4.3, the theorem is true for $\|x\| \le 1$. Thus, by induction on $\|x\|$, we can assume that G is HyK-separable for all $y \in G$, whenever $\|y\| < \|x\|$. Let $g \in G$ such that $g \notin HxK$. We shall find $L \triangleleft_f A$ and $M \triangleleft_f T$ with $L \cap C = M \cap C$ such that in $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$, where $\overline{A} = A/L$ and $\overline{T} = T/M$, $\|\bar{x}\| = \|x\|$, $\|\bar{g}\| = \|g\|$, and $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. Since $\overline{G}$ is $\mathscr{RF}$ and $\overline{H}\bar{x}\overline{K}$ is finite, we can find $N \triangleleft_f G$ such that $g \notin NHxK$. We shall only consider the case where $x = t_1a_1 \cdots t_na_n$, where $t_i \in T \backslash C$ and $a_i \in A \backslash C$, other cases being similar. Since $g \notin HxK$ is equivalent to $x \notin HgK$, we note that, if $\|g\| < \|x\|$ then, by induction, there exists $\overline{G}$ such that $\bar{x} \notin \overline{H}\bar{g}\overline{K}$, whence $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. Moreover, if $\|g\| > \|x\| + 2$ then we can find $\overline{G}$ such that $\|\bar{x}\| = \|x\|$ and $\|\bar{g}\| = \|g\|$. This implies that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. Hence we need only consider $g \in G$ such that $\|x\| \le \|g\| \le \|x\| + 2$.

Case 1. $H, K \subset T$.

(a) $\|g\| = \|x\|$. Suppose $g = d_1 s_1 \cdots d_n s_n$, where $d_i \in A \backslash C$ and $s_i \in T \backslash C$. If $t_1 \notin HC$ then, by Lemma 4.2, there exists $M \lhd_f T$ such that $t_1 \notin MHC$. Moreover, M can be chosen such that $s_i, t_i \notin MC$. Let $L \lhd_f A$ such that $a_i, d_i \notin LC$, and $L \cap C = M \cap C$. Thus, in $\overline{G} = A/L *_{\overline{C}} T/M$, we have $\|\bar{g}\| = \|g\|$, $\|\bar{x}\| = \|x\|$, and $\bar{t}_1 \notin \overline{HC}$. This implies that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. So let $t_1 = hc$, where $h \in H$ and $c \in C$. This implies that $g \notin Hca_1 t_2 \cdots t_n a_n K$. Since $ca_1 \in A$, we have $\|ca_1 t_2 \cdots t_n a_n\| < \|x\|$. By induction, there exists $\overline{G}$ such that $\bar{g} \notin \overline{H}\overline{ca_1 \cdots t_n a_n}\overline{K}$, $\|\bar{g}\| = \|g\|$, and $\|\bar{x}\| = \|x\|$, whence $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

Suppose $g = s_1 d_1 \cdots s_n d_n$. Let $u = d_1 \cdots s_n d_n$. If $s_1 \notin Ht_1 C$ or $u \notin Ca_1 t_2 \cdots t_n a_n K$ then, by induction, we can find $\overline{G}$ such that $\|\bar{g}\| = \|g\|$, $\|\bar{x}\| = \|x\|$, and $\bar{s}_1 \notin \overline{H}\bar{t}_1\overline{C}$ or $\bar{u} \notin \overline{Ca_1 t_2 \cdots t_n a_n}\overline{K}$. This implies that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. So let $s_1 = ht_1 c$, where $h \in H$ and $c \in C$, and $u = c' a_1 t_2 \cdots t_n a_n k$, where $c' \in C$ and $k \in K$. This implies that $t_1 cc' a_1 t_2 \cdots t_n a_n \notin HxK$. Thus, by Lemma 4.8, there exists $P \lhd_f G$ such that $t_1 cc' a_1 t_2 \cdots t_n a_n \notin PHxK$, and $t_i, s_i, a_i, d_i \notin PC$. Let $\overline{A} = A/(P \cap A)$, $\overline{T} = T/(P \cap T)$ and $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$. Then we have $\overline{t_1 cc' a_1 t_2 \cdots t_n a_n} \notin \overline{H}\bar{x}\overline{K}$, $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. Thus $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

(b) $\|g\| = \|x\| + 1$. Suppose $g = d_1 s_1 \cdots d_n s_n d_{n+1}$, where $d_i \in A \backslash C$ and $s_i \in T \backslash C$. Then we can find $\overline{G}$ such that $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. Since $\bar{d}_{n+1} \in \overline{A} \backslash \overline{C}$, $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. So let $g = s_1 d_1 \cdots s_n d_n s_{n+1}$. If $s_{n+1} \notin CK$, by Lemma 4.2, we can find $\overline{G}$ such that $\|\bar{g}\| = \|g\|$, $\|\bar{x}\| = \|x\|$ and $\bar{s}_{n+1} \notin \overline{CK}$. Thus $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. If $s_{n+1} = ck$ for some $c \in C$ and $k \in K$, then $gk^{-1} = s_1 d_1 \cdots s_n d_n c \notin HxK$. Since $d_n c \in A$, by the second part of (a), there exists $\overline{G}$ such that $\|\bar{g}\| = \|g\|$, $\|\bar{x}\| = \|x\|$, and $\overline{gk^{-1}} \notin \overline{H}\bar{x}\overline{K}$. Thus $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

(c) $\|g\| = \|x\| + 2$. We can choose $\overline{G}$ such that $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. Then $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

Case 2. $H \subset T$ and $K \subset A$.

(a) $\|g\| > \|x\|$. We can choose $\overline{G}$ such that $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. Then $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

(b) $\|g\| = \|x\|$. If $g = d_1 s_1 \cdots d_n s_n$, we can choose $\overline{G}$ such that $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. This implies that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. If $g = s_1 d_1 \cdots s_n d_n$. Then we can apply the method of the second part of Case 1(a).

Case 3. $H, K \subset A$.

(a) $\|g\| = \|x\|$. Suppose $g = d_1 s_1 \cdots d_n s_n$. If $a_n \notin CK$ then, by Lemma 4.2, we can choose $\overline{G}$ such that $\|\bar{g}\| = \|g\|$, $\|\bar{x}\| = \|x\|$, and $\bar{a}_n \notin \overline{CK}$. Then $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. So let $a_n = ck$, where $c \in C$ and $k \in K$. This implies that $g \notin Ht_1 a_1 \cdots t_n cK$. Since $t_n c \in T$, by induction, there exists $\overline{G}$ such that $\|\bar{g}\| = \|g\|$, $\|\bar{x}\| = \|x\|$, and $\bar{g} \notin \overline{H}\overline{t_1 a_1 \cdots t_n c}\overline{K}$. Hence $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

If $g = s_1 d_1 \cdots s_n d_n$, then we can apply the method of the second part of Case 1 (a).

(b) $\|g\| = \|x\| + 1$ or $\|g\| = \|x\| + 2$. Suppose $g = d_1 s_1 \cdots d_n s_n d_{n+1}$. If $d_1 s_1 \notin Ht_1 C$ or $u = d_2 s_2 \cdots d_n s_n d_{n+1} \notin Ca_1 t_2 \cdots t_n a_n K$ then, by induction, we can choose $\overline{G}$ so that $\|\bar{g}\| = \|g\|$, $\|\bar{x}\| = \|x\|$, and $\overline{d_1 s_1} \notin \overline{H}\bar{t}_1\overline{C}$ or $\bar{u} \notin \overline{Ca_1 t_2 \cdots t_n a_n}\overline{K}$. This implies that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. So let $d_1 s_1 = ht_1 c$ and $u = c' a_1 t_2 \cdots t_n a_n k$, where $h \in H$, $k \in K$, and $c, c' \in C$. This means $g = d_1 s_1 u = ht_1 cc' a_1 \cdots t_n a_n k \notin HxK$, whence $t_1 cc' a_1 \cdots t_n a_n \notin HxK$. Thus, by Lemma 4.8, we can find $P \triangleleft_f G$ such that $t_1 cc' a_1 \cdots t_n a_n \notin PHxK$ and $t_i, s_i, a_i, d_i \notin PC$. Let $\overline{A} = A/(P \cap A)$, $\overline{T} = T/(P \cap T)$ and $\overline{G} = \overline{A} *_{\overline{C}} \overline{T}$. Then we have $\|\bar{g}\| = \|g\|$, $\|\bar{x}\| = \|x\|$, and $\overline{t_1 cc' a_1 \cdots t_n a_n} \notin \overline{H}\bar{x}\overline{K}$. Hence $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

If $g = s_1 d_1 \cdots s_n d_n s_{n+1}$ or $\|g\| = \|x\| + 2$, we can choose $\overline{G}$ such that $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. Then $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

Case 4. $H \subset A$ and $K \subset T$.

(a) $\|g\| = \|x\|$. Suppose $g = d_1 s_1 \cdots d_n s_n$. Then we can choose $\overline{G}$ such that $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. This implies that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. If $g = s_1 d_1 \cdots s_n d_n$, then we can apply the method of Case 1(a) to get $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

(b) $\|g\| = \|x\| + 1$. If $g = s_1 d_1 \cdots s_n d_n s_{n+1}$, we can apply the method of Case 1(a) to get $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. If $g = d_1 s_1 \cdots d_n s_n d_{n+1}$, we can apply the method of Case 3(b) to get $\bar{g} \notin \overline{H}\bar{x}\overline{K}$.

(c) $\|g\| = \|x\| + 2$. If $g = s_1 d_1 \cdots d_n s_{n+1} d_{n+1}$, we can choose $\overline{G}$ such that $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. This implies that $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. If $g = d_1 s_1 \cdots s_n d_{n+1} s_{n+1}$, we can apply the method of Case 3(b) to get $\bar{g} \notin \overline{H}\bar{x}\overline{K}$. This completes the proof. ∎

Since polycyclic-by-finite groups are subgroup separable [16], we have

COROLLARY 4.10. *Let G be a tree product of m polycyclic-by-finite groups amalgamating central edge groups. Let H, K be central subgroups of some vertex groups. Then G is HxK-separable for* $x \in G$.

5. CONJUGACY SEPARABILITY

In this section we derive a criterion for the conjugacy separability of generalized free products of two conjugacy separable groups amalgamating a subgroup which is central in one of the factors. Applying this criterion we prove that tree products of finitely many polycyclic-by-finite groups amalgamating central edge groups are conjugacy separable.

THEOREM 5.1. *Let $G = A *_H B$ and $H \subset Z(A)$. Suppose A, B satisfy the following*:

C1. *Both A and B are conjugacy separable.*

C2. *A, B are H-finite.*

C3. *B is H-conjugacy separable.*

C4. *G is HxH-separable for $x \in G$.*

C5. *For every $M_1 \triangleleft_f B$, there exists $M \triangleleft_f B$ such that $M \subset M_1$ and, in $\overline{B} = B/M$, we have $\bar{h} \not\sim_{\overline{B}} \bar{k}$ for all $\bar{h} \neq \bar{k}$ in $\overline{H}$.*

Then G is conjugacy separable.

We note that A and B are H-separable by C4. Since $H \subset Z(A)$, A is H-conjugacy separable. Moreover, $\overline{A} = A/H$ is $\mathcal{RF}$, since A is H-separable.

Proof. Let $x, y \in G$ such that $x \not\sim_G y$, where x and y are chosen to be of minimal lengths in their respective conjugacy classes. Since A is conjugacy separable by C1, A is $\mathcal{RF}$. Hence H is $\mathcal{RF}$. Thus, by C2, C4, and Theorem 3.2, G is $\mathcal{RF}$. Hence we may assume $x \neq 1 \neq y$. In most of the following cases we shall find a group $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$ such that $\bar{x} \not\sim_{\overline{G}} \bar{y}$, where $\overline{A}, \overline{B}$ are finite. Then $\overline{G}$ is conjugacy separable by Theorem 2.2. Hence there exists $\overline{N} \triangleleft_f \overline{G}$ such that $\overline{N}\bar{x} \not\sim_{\overline{G}/\overline{N}} \overline{N}\bar{y}$. Let N be the preimage of $\overline{N}$ in G. Then $N \triangleleft_f G$ and $Nx \not\sim_{G/N} Ny$, as required.

Case 1. $x, y \in H$. This means $x \not\sim_B y$. Thus, by C1, there exists $M \triangleleft_f B$ such that, in $\overline{B} = B/M$, $\bar{x} \not\sim_{\overline{B}} \bar{y}$. By C2, there exists $L \triangleleft_f A$ such that $L \cap H = M \cap H$. Let $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{A} = A/L$. If $\bar{x} \sim_{\overline{G}} \bar{y}$ then, by Theorem 2.1, there exists a sequence of elements $\bar{h}_1, \ldots, \bar{h}_r \in \overline{H}$ such that $\bar{x} = \bar{h}_0 \sim_{\overline{A}} \bar{h}_1 \sim_{\overline{B}} \bar{h}_2 \sim_{\overline{A}} \cdots \sim_{\overline{B}} \bar{h}_r = \bar{y}$. Since $H \subset Z(A)$, $\bar{h}_i \sim_{\overline{A}} \bar{h}_{i+1}$ implies that $\bar{h}_i = \bar{h}_{i+1}$. Thus we have $\bar{x} \sim_{\overline{B}} \bar{y}$, contradicting the choice of M. Therefore $\bar{x} \not\sim_{\overline{G}} \bar{y}$.

Case 2. $x \in H$ and $y \in A \backslash H$. Consider the natural homomorphism π: $G \to \overline{A}$, where $\overline{A} = G/B^G$. Then $\bar{x} = 1$ and $\bar{y} \neq 1$. Since $\overline{A} \cong A/H$ is $\mathcal{RF}$, there exists $\overline{N} \triangleleft_f \overline{A}$ such that $\bar{y} \notin \overline{N}$. Let $N = \pi^{-1}(\overline{N})$. Then $N \triangleleft_f G$ such that $N = Nx \not\sim_{G/N} Ny$.

Case 3. $x \in H$ and $y \in B \backslash H$. Since $y \in B \backslash H$, y is of minimal length 1 in its conjugacy class. Hence we have $\{y\}^B \cap H = \varnothing$. Moreover, since B is H-conjugacy separable by C3, there exists $M \triangleleft_f B$ such that $\{\bar{y}\}^{\overline{B}} \cap \overline{H} = \varnothing$, where $\overline{B} = B/M$. By C2, there exists $L \triangleleft_f A$ such that $L \cap H = M \cap H$. Let $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{A} = A/L$. Then $\{\bar{y}\}^{\overline{B}} \cap \overline{H} = \varnothing$. This implies that $\bar{x} \not\sim_{\overline{G}} \bar{y}$, by Theorem 2.1.

Case 4. $\|x\| \neq \|y\|$ and $\|x\| \geq 2$ (or $\|y\| \geq 2$). Since A, B are H-separable and H-finite, we can choose $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$ such that $\|\bar{x}\| = \|x\|$ and $\|\bar{y}\| = \|y\|$, where $\bar{x}, \bar{y}$ are cyclically reduced. Hence $\bar{x}$ is of minimal length in its conjugacy class and $\|\bar{x}\| \neq \|\bar{y}\|$. This implies that $\bar{x} \not\sim_{\overline{G}} \bar{y}$ by Theorem 2.1.

Case 5. $\|x\| = 1 = \|y\|$.

(a) Suppose $x \in A\backslash H$ and $y \in B\backslash H$ (or $x \in B\backslash H$ and $y \in A\backslash H$). Since y is of minimal length 1 in its conjugacy class, we have $\{y\}^B \cap H = \emptyset$. By C3, B is H-conjugacy separable. Therefore, there exists $M_1 \lhd_f B$ such that $\{\tilde{y}\}^{\widetilde{B}} \cap \widetilde{H} = \emptyset$, where $\widetilde{B} = B/M_1$. Since A is H-separable, there exists $L_1 \lhd_f A$ such that $x \notin L_1 H$. By C2, there exist $L_2 \lhd_f A$ and $M_2 \lhd_f B$ such that $L_2 \cap H = L_1 \cap M_1 = M_2 \cap H$. Let $L = L_1 \cap L_2$ and $M = M_1 \cap M_2$. Consider $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{A} = A/L$ and $\overline{B} = B/M$. Then $\bar{x} \notin \overline{H} \subset Z(\overline{A})$ and $\{\bar{y}\}^{\overline{B}} \cap \overline{H} = \emptyset$. Hence $\bar{x}, \bar{y}$ are of minimal length 1 in their conjugacy classes in $\overline{G}$, where $\bar{x} \in \overline{A}\backslash\overline{H}$ and $\bar{y} \in \overline{B}\backslash\overline{H}$. Thus, by Theorem 2.1, $\bar{x} \not\sim_{\overline{G}} \bar{y}$.

(b) $x, y \in A\backslash H$. Since A is conjugacy separable and H-separable, there exists $L \lhd_f A$ such that $Lx \not\sim_{A/L} Ly$ and $x, y \notin LH$. By C2, there exists $M \lhd_f B$ such that $M \cap H = L \cap H$. Let $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{B} = B/M$. Since $\bar{x} \notin \overline{H} \subset Z(\overline{A})$, $\bar{x}$ is of minimal length 1 in its conjugacy class in $\overline{G}$. Since $\bar{x} \not\sim_{\overline{A}} \bar{y}$, by Theorem 2.1, we have $\bar{x} \not\sim_{\overline{G}} \bar{y}$.

(c) $x, y \in B\backslash H$. This implies $\{x\}^B \cap H = \emptyset = \{y\}^B \cap H$. Since B is conjugacy separable by C1 and H-conjugacy separable by C3, there exists $M \lhd_f B$ such that $\bar{x} \not\sim_{\overline{B}} \bar{y}$ and $\{\bar{x}\}^{\overline{B}} \cap \overline{H} = \emptyset = \{\bar{y}\}^{\overline{B}} \cap \overline{H}$, where $\overline{B} = B/M$. By C2, there exists $L \lhd_f A$ such that $L \cap H = M \cap H$. Let $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{A} = A/L$. Then $\bar{x}, \bar{y}$ are of minimal length 1 in their conjugacy classes in $\overline{G}$ and $\bar{x} \not\sim_{\overline{B}} \bar{y}$. Thus, by Theorem 2.1, $\bar{x} \not\sim_{\overline{G}} \bar{y}$.

Case 6. $\|x\| = \|y\| = r \geq 2$. Let $x = u_1 u_2 \cdots u_r$ and $y = v_1 v_2 \cdots v_r$ be cyclically reduced words in G. Since $x \not\sim_G y$, by Theorem 2.1, $x \not\sim_H y^*$ for any cyclic permutation y^* of y. Thus, for each i, the equation

$$I(i) : u_1 u_2 \cdots u_r = h^{-1} v_i \cdots v_r v_1 \cdots v_{i-1} h \tag{2}$$

has no solution for $h \in H$. Thus, for each i, we shall find $L_i' \lhd_f A$ and $M_i' \lhd_f B$ such that $L_i' \cap H = M_i' \cap H$ so that, in each $\overline{G} = A/L_i' *_{\overline{H}} B/M_i'$, $\|\bar{x}\| = \|x\|$ and $\|\bar{y}\| = \|y\|$ and the equation

$$\overline{I(i)} \ : \ \overline{u_1 u_2 \cdots u_r} = \bar{h}^{-1} \overline{v_i \cdots v_r v_1 \cdots v_{i-1}} \bar{h} \tag{3}$$

has no solution for $\bar{h} \in \overline{H}$. Let $L = \bigcap_{i=1}^r L_i'$, $M = \bigcap_{i=1}^r M_i'$, and $\widetilde{G} = A/L *_{\widetilde{H}} B/M$. Then $\tilde{x}$ and $\tilde{y}$ are cyclically reduced and $\tilde{x} \not\sim_{\widetilde{H}} \tilde{y}^*$ for all cyclic permutations $\tilde{y}^*$ of $\tilde{y}$. Thus, by Theorem 2.1, we have $\tilde{x} \not\sim_{\widetilde{G}} \tilde{y}$. We can consider $\widetilde{G}$ to be the $\overline{G}$, which we mentioned in the beginning of the proof.

To find such L_i', M_i', since all the cases are similar, we need only consider the case $i = 1$. Suppose $I(1)$ having no solution in H. Then $x = u_1 u_2 \cdots u_r \not\sim_H v_1 v_2 \cdots v_r = y$. If u_i and v_i are not in the same factor of $G = A *_H B$, we can choose $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$ such that $\|\bar{x}\| = \|x\|$ and $\|\bar{y}\| = \|y\|$. Then equation $\overline{I(1)}$ has no solution in $\overline{H}$, since $\overline{u_i}$ and $\overline{v_i}$ are not in the same factor of $\overline{G}$. Thus we need only consider the case $x = e_1 d_1 \cdots e_n d_n$ and $y = a_1 b_1 \cdots a_n b_n$, where $a_i, e_i \in A \backslash H$ and $b_i, d_i \in B \backslash H$, such that $x = h^{-1} y h$ has no solution h in H.

(a) Suppose $x \notin HyH$. By condition C4, there exists $N \lhd_f G$ such that $x \notin NHyH$, $a_i, b_i, e_i, d_i \notin NH$. Let $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{B} = B/(N \cap B)$ and $\overline{A} = A/(N \cap A)$. Then $\bar{x} \notin \overline{HyH}$ and $\|\bar{x}\| = \|\bar{y}\| = 2n$. Hence $\bar{x} \not\sim_{\overline{H}} \bar{y}$.

(b) Suppose $x = h_1 y h_2$ for $h_1, h_2 \in H$ and $h_1 h_2 \neq 1$. Since B is residually finite by C1 and B is H-separable by C4, there exists $M_1 \lhd_f B$ such that $h_1 h_2 \notin M_1$ and $b_i \notin M_1 H$. Also there exists $L_1 \lhd_f A$ such that $a_i \notin L_1 H$. By C2, there exists $L_2 \lhd_f A$ and $M_2 \lhd_f B$ such that $L_2 \cap H = L_1 \cap M_1 = M_2 \cap H$. By C5, there exists $M \lhd_f B$ such that $M \subset M_1 \cap M_2$ and $\bar{h} \not\sim_{\overline{B}} \bar{k}$ for all $\bar{h} \neq \bar{k}$ in $\overline{H}$, where $\overline{B} = B/M$. Let $L_3 \lhd_f A$ such that $L_3 \cap H = M \cap H$. Let $L = L_1 \cap L_2 \cap L_3$. Then $L \cap H = M \cap H$. Consider $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{A} = A/L$. We have $\|\bar{x}\| = \|x\| = \|\bar{y}\| = \|y\|$ and $\bar{h}_1 \bar{h}_2 \neq 1$. If $\bar{x} \sim_{\overline{H}} \bar{y}$, then $\bar{x} = \bar{h}^{-1} \bar{y} \bar{h}$ for some $\bar{h} \in \overline{H}$. Hence $\bar{y} = \bar{h} \bar{h}_1 \bar{y} \bar{h}_2 \bar{h}^{-1}$; that is, $\overline{a_1 b_1 \cdots a_n b_n} = \bar{h} \bar{h}_1 \overline{a_1 b_1 \cdots a_n b_n} \bar{h}_2 \bar{h}^{-1}$. This implies that there exists $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n, \bar{z}_1, \ldots, \bar{z}_{n-1} \in \overline{H}$ such that $\bar{a}_1 = \overline{h h_1} \bar{a}_1 \bar{c}_1$, $\bar{b}_1 = \bar{c}_1^{-1} \bar{b}_1 \bar{z}_1$, $\bar{a}_2 = \bar{z}_1^{-1} \bar{a}_2 \bar{c}_2, \ldots, \bar{a}_n = \bar{z}_{n-1}^{-1} \bar{a}_n \bar{c}_n$, and $\bar{b}_n = \bar{c}_n^{-1} \bar{b}_n \bar{h}_2 \bar{h}^{-1}$. Since $\overline{H} \subset Z(\overline{A})$ and $\bar{h} \not\sim_{\overline{B}} \bar{k}$ for all $\bar{h} \neq \bar{k}$ in $\overline{H}$, we have $(\overline{h h_1})^{-1} = \bar{c}_1$, $\bar{c}_1 = \bar{z}_1$, $\bar{z}_1 = \bar{c}_2, \ldots, \bar{z}_{n-1} = \bar{c}_n$, and $\bar{c}_n = \bar{h}_2 \bar{h}^{-1}$. This implies that $(\overline{h h_1})^{-1} = \bar{h}_2 \bar{h}^{-1}$, whence $\bar{h}_1 \bar{h}_2 = 1$, contradicting the choice of $M \subset M_1$. Hence $\bar{x} \not\sim_{\overline{H}} \bar{y}$.

Thus, in all cases, we have found $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$ such that $\bar{x}$ and $\bar{y}$ are cyclically reduced and $\bar{x} \not\sim_{\overline{H}} \bar{y}$. Then, as we mentioned at the beginning of Case 6, we can find $\widetilde{G} = \widetilde{A} *_{\widetilde{H}} \widetilde{B}$ such that $\|\tilde{x}\| = \|x\| = \|\tilde{y}\| = \|y\|$ and $\tilde{x} \not\sim_{\widetilde{H}} \tilde{y}^*$ for all cyclic permutations $\tilde{y}^*$ of $\tilde{y}$. Hence $\tilde{x} \not\sim_{\widetilde{G}} \tilde{y}$ as required. This completes the proof. ∎

We immediately have the following:

COROLLARY 5.2. *Let A, B be conjugacy separable and central subgroup separable and let $H \subset Z(A) \cap Z(B)$ be finitely generated. Then $G = A *_H B$ is conjugacy separable.*

Proof. Since A, B are central subgroup separable and $H \subset Z(A) \cap Z(B)$, it is clear that C2 (Lemma 3.8), C3, and C5 are satisfied. Since A, B

are H-finite and H-separable, G is H-separable by Lemma 3.4. Moreover, $H \subset Z(G)$ implies C4. Hence G is conjugacy separable by Theorem 5.1. ∎

The following shows that the tree products discussed in Section 4 satisfy C5 in Theorem 5.1.

LEMMA 5.3. *Let G be a tree product of m central subgroup separable groups amalgamating finitely generated central edge groups. Let H be a central subgroup of a vertex group. For every $N_1 \triangleleft_f G$, there exists $N \triangleleft_f G$ such that $N \subset N_1$ and, in $\overline{G} = G/N$, we have $\bar{h} \not\sim_{\overline{G}} \bar{k}$ for all $\bar{h} \neq \bar{k}$ in $\overline{H}$.*

Proof. We shall prove this by induction on m. Clearly the lemma is true for $m = 1$. Let $G = A *_C T$ be as in Remark 3.5. By induction, we can assume T has the property stated in the lemma.

Case 1. $H \subset A$. By induction, there exists $M \triangleleft_f T$ such that $M \subset N_1 \cap T$ and $\tilde{c}_1 \not\sim \tilde{c}_2$ in $\widetilde{T} = T/M$ for all $\tilde{c}_1 \neq \tilde{c}_2$ in $\widetilde{C}$. By Lemma 3.8, there exists $L \triangleleft_f A$ such that $L \cap C = M \cap C$. Let $\widetilde{G} = \widetilde{A} *_{\widetilde{C}} \widetilde{T}$, where $\widetilde{A} = A/(L \cap N_1)$. We first show that $\tilde{h} \not\sim_{\widetilde{G}} \tilde{k}$ for all $\tilde{h} \neq \tilde{k}$ in $\widetilde{H}$. Suppose $\tilde{h} \sim_{\widetilde{G}} \tilde{k}$. If $\tilde{h} \sim_{\widetilde{G}} \tilde{c}$ for some $c \in C$ then, by Theorem 2.1, there exist $\tilde{c}_1, \ldots, \tilde{c}_r \in \widetilde{C}$ such that $\tilde{h} \sim_{\widetilde{A}} \tilde{c}_1 \sim_{\widetilde{T}} \tilde{c}_2 \sim_{\widetilde{A}} \cdots \sim_{\widetilde{T}} \tilde{c}_r = \tilde{c}$. By the choice of M, $\tilde{c}_i \sim_{\widetilde{T}} \tilde{c}_{i+1}$ implies that $\tilde{c}_i = \tilde{c}_{i+1}$. Therefore $\tilde{h} \sim_{\widetilde{A}} \tilde{c}$, whence $\tilde{h} = \tilde{c}$, since $H \subset Z(A)$. Similarly, since $\tilde{k} \sim_{\widetilde{G}} \tilde{h} = \tilde{c}$, we have $\tilde{k} = \tilde{c}$. Hence $\tilde{h} = \tilde{k}$. If $\tilde{h} \not\sim_{\widetilde{G}} \tilde{c}$ for any $c \in C$, then $\tilde{h}$ has the minimal length 1 in its conjugacy class in $\widetilde{G}$. Thus, by Theorem 2.1, $\tilde{h} \sim_{\widetilde{A}} \tilde{k}$, whence $\tilde{h} = \tilde{k}$. This proves that $\tilde{h} \not\sim_{\widetilde{G}} \tilde{k}$ for all $\tilde{h} \neq \tilde{k}$ in $\widetilde{H}$. Since $\widetilde{H}$ is finite and $\widetilde{G}$ is conjugacy separable, there exists $\widetilde{N}_2 \triangleleft_f \widetilde{G}$ such that, in $\widetilde{G}/\widetilde{N}_2$, $\widetilde{N}_2\tilde{h} \not\sim \widetilde{N}_2\tilde{k}$ for all $\tilde{h} \neq \tilde{k}$ in $\widetilde{H}$. Let N_2 be the preimage of $\widetilde{N}_2$ in G and let $N = N_1 \cap N_2$. Then $N \triangleleft_f G$ and $N \subset N_1$. Moreover, if $\bar{h} \sim_{\overline{G}} \bar{k}$ for $h, k \in H$, where $\overline{G} = G/N$, then $\widetilde{N}_2\tilde{h} \sim \widetilde{N}_2\tilde{k}$ in $\widetilde{G}/\widetilde{N}_2$. Hence $\tilde{h} = \tilde{k}$ by the choice of $\widetilde{N}_2$. This implies that $hk^{-1} \in L \cap N_1 \subset N_1$. Clearly $\tilde{h}\tilde{k}^{-1} = 1 \in \widetilde{N}_2$; hence $hk^{-1} \in N_2$. Therefore, $hk^{-1} \in N_1 \cap N_2 = N$; hence $\bar{h} = \bar{k}$, as required.

Case 2. $H \subset T$. By induction, there exists $M \triangleleft_f T$ such that $M \subset N_1 \cap T$ and $\tilde{h} \not\sim \tilde{k}$ in $\widetilde{T} = T/M$ for all $\tilde{h} \neq \tilde{k}$ in $\widetilde{H}$. Let $L \triangleleft_f A$ such that $L \cap C = M \cap C$. Consider $\widetilde{G} = \widetilde{A} *_{\widetilde{C}} \widetilde{T}$, where $\widetilde{A} = A/(L \cap N_1)$. As in Case 1, we shall show that $\tilde{h} \not\sim_{\widetilde{G}} \tilde{k}$ for all $\tilde{h} \neq \tilde{k}$ in $\widetilde{H}$. Suppose $\tilde{h} \sim_{\widetilde{G}} \tilde{k}$. If $\tilde{h} \sim_{\widetilde{G}} \tilde{c}$ for some $c \in C$ then, by Theorem 2.1, there exist $\tilde{c}_1, \ldots, \tilde{c}_r \in \widetilde{C}$ such that $\tilde{h} \sim_{\widetilde{T}} \tilde{c}_1 \sim_{\widetilde{A}} \tilde{c}_2 \sim_{\widetilde{T}} \cdots \sim_{\widetilde{T}} \tilde{c}_r \sim_{\widetilde{A}} \tilde{c}$. Since $C \subset Z(A)$, $\tilde{c}_i \sim_{\widetilde{A}} \tilde{c}_{i+1}$ implies that $\tilde{c}_i = \tilde{c}_{i+1}$. Thus $\tilde{h} \sim_{\widetilde{T}} \tilde{c}$. Similarly, since $\tilde{k} \sim_{\widetilde{G}} \tilde{h} \sim_{\widetilde{G}} \tilde{c}$, we have $\tilde{k} \sim_{\widetilde{T}} \tilde{c}$. Hence $\tilde{h} \sim_{\widetilde{T}} \tilde{k}$. This implies that $\tilde{h} = \tilde{k}$ by the choice of M. If $\tilde{h} \not\sim_{\widetilde{G}} \tilde{c}$ for any $c \in C$, then $\tilde{h}$ has the minimal length 1 in its conjugacy class in $\widetilde{G}$. Thus, by Theorem 2.1, $\tilde{h} \sim_{\widetilde{T}} \tilde{k}$. Then, as before, $\tilde{h} = \tilde{k}$. Hence $\tilde{h} \not\sim_{\widetilde{G}} \tilde{k}$ for all $\tilde{h} \neq \tilde{k}$ in $\widetilde{H}$. Now, choose $N \lhd_f G$ as in Case 1. Then $N \subset N_1$ and, in $\overline{G} = G/N$, $\bar{h} \not\sim_{\overline{G}} \bar{k}$ for all $\bar{h} \neq \bar{k}$ in $\overline{H}$. ∎

THEOREM 5.4. *Let G be a tree product of finitely many central subgroup separable and conjugacy separable groups amalgamating finitely generated central edge groups. Then G is conjugacy separable.*

Proof. We shall prove this by induction on the number m of vertex groups of G. If $m = 2$ then G is conjugacy separable by Corollary 5.2. Let $G = A *_C T$ be as in Remark 3.5. By induction, we assume T is conjugacy separable. We shall apply Theorem 5.1. Clearly C1 holds. The conditions C2–C5 are true by Corollary 3.9, Theorems 3.13 and 4.9, and Lemma 5.3, respectively. Hence G is conjugacy separable by Theorem 5.1. ∎

As an immediate consequence we have the following result:

THEOREM 5.5. *Let G be a tree product of m polycyclic-by-finite groups amalgamating central edge groups. Then G is conjugacy separable. In particular, a tree product of m finitely generated abelian groups is conjugacy separable.*

In [26], Wong and Tang proved that tree products of subgroup and conjugacy separable groups amalgamating central subgroups, where the edge subgroups intersect trivially, are conjugacy separable. They applied it to tree products of polycyclic-by-finite or free-by-finite groups amalgamating central subgroups where the edge subgroups intersect trivially. We note that if A is a free-by-finite group, then either $Z(A)$ is finite or A is cyclic-by-finite. Since generalized free products of conjugacy separable groups amalgamating finite subgroups are conjugacy separable, Theorems 5.4 and 5.5 are generalizations of the main results (Theorem 2 and Corollary 1) in [26].

In [22], Stebe showed that Brauner's groups in [4] are cyclic subgroup separable (π_c). Using our criterion, we can show that some of them are conjugacy separable.

The group for the linkage of a torus knot with a cycle is presented as

$$\langle x, y : [x, y] \rangle \underset{x^n y^m = z^m}{*} \langle z \rangle.$$

And the group for the linkage of a torus knot with cycles within and outside the torus is presented as

$$\langle x, y : [x, y] \rangle \underset{x^n y^m = a^m b^n}{*} \langle a, b : [a, b] \rangle.$$

The groups of linkages of torus knots are presented as

$$\langle a \rangle \underset{a^{n_1} = x_1^{\delta_1}}{*} \langle x_1, y_1 : [x_1, y_1] \rangle \underset{x_1^{\alpha_1} y_1^{\beta_1} = x_2^{\delta_2}}{*} \langle x_2, y_2 : [x_2, y_2] \rangle \underset{x_2^{\alpha_2} y_2^{\beta_2} = x_3^{\delta_3}}{*}$$
$$\cdots \underset{x_{n-1}^{\alpha_{n-1}} y_{n-1}^{\beta_{n-1}} = x_n^{\delta_n}}{*} \langle x_n, y_n : [x_n, y_n] \rangle \underset{x_n^{\alpha_n} y_n^{\beta_n} = x_{n+1}^{\delta_{n+1}}}{*} \langle x_{n+1} \rangle.$$

Thus all of the above groups are tree products of free abelian groups where the amalgamated subgroups may intersect nontrivially. Hence they are conjugacy separable by Theorem 5.4.

COROLLARY 5.6. *The groups of linkages of torus knots are conjugacy separable*.

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