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Subgroup Separability of HNN-Extensions with Abelian Base Group

V. Metaftsis[1]

Department of Mathematics, University of the Aegean, Karlovassi, 83 200 Samos, Greece
E-mail: vmet@aegean.gr

and

E. Raptis

Department of Mathematics, University of Athens, Panepistimiopolis, 15 784 Athens, Greece
E-mail: eraptis@math.uoa.gr

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In the present paper we give a complete characterisation of subgroup separability of HNN-extensions with finitely generated abelian base group. We are also able to characterise subgroup separability for some families of free-by-cyclic groups and thus answer partially a question of Scott (1987, "Combinatorial Group Theory and Topology," *Annals of Mathematics Studies*, Vol. 111, Princeton Univ. Press, Princeton, NJ). © 2001 Academic Press

1. INTRODUCTION

A group G is called *residually finite* if the trivial subgroup is the intersection of the finite index subgroups of G.

The group G is said to be *subgroup separable* (or *LERF*) if for every finitely generated subgroup H of G, H is the intersection of finite index subgroups of G. Equivalently, every finitely generated subgroup of G is

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closed in the profinite topology, the topology whose closed basis consists of the cosets of finite index subgroups of the group.

There are other equivalent definitions of subgroup separability. Hence, G is subgroup separable if for every finitely generated subgroup H of G and every $g \in G$ with $g \notin H$ there is a normal subgroup N of finite index in G such that $g \notin NH$. This is equivalent to saying that $gN \notin HN/N$ or that $\bigcap_{N \in \mathcal{N}} NH = H$ where $\mathcal{N}$ is the set of all normal subgroups of finite index in G.

Malćev [16] showed that subgroup separable groups have decidable membership problem with respect to their finitely generated subgroups. Also, as shown by Thurston [20], subgroup separability allows certain immersions to lift to an embedding in a finite cover. Nevertheless, very few families of groups are known to be subgroup separable. Hall [14] showed subgroup separability for free groups. Malćev also showed that polycyclic-by-finite groups are subgroup separable and Scott [18, 19] showed subgroup separability for surface groups. Brunner et al. [6] generalised the above results by showing that free products of two free groups with cyclic amalgamation are subgroup separable.

Further work on the subject has been done by Tretkoff, Niblo, Gitik and Wise (see [12, 13, 21] and the references cited there) using topological rather than purely algebraic techniques.

The present work gives a complete characterisation for subgroup separability of HNN-extensions with finitely generated abelian base groups. The counterexample of Burns et al. found in [8] is a key point for our results. A similar characterisation was given by Wong [22] for a special case. In fact, Wong, using the results in [3], gives a characterisation for subgroup separability of HNN-extensions where the associated subgroups have finite index as subgroups of the base group.

We should mention here that another class of examples not related to that of [8], but strongly related to the result in [5], can be found in the work of Leary et al. in [15].

2. MAIN RESULTS

For the sequel, G is the HNN-extension

$$G = \langle t, K \mid t^{-1}At = B \rangle,$$

where K is a finitely generated abelian group and A, B are finitely generated isomorphic subgroups of K. With the above notation we prove the following.

LEMMA 1. *If $A = K (B = K)$ and $B \neq K (A \neq K)$ then G is not subgroup separable.*

The proof is an immediate consequence of the following theorem of Blass and Neumann.

THEOREM 1 [5]. *Let G be a group and let H be a subgroup of G which is the intersection of subgroups of finite index in G. Then H is not a conjugate in G to a proper subgroup of itself.*

Notice that if $A = K = B$ then K is a normal subgroup of G and $G/K \cong \langle t \rangle$. Therefore G is a polycyclic group and, by [16], G is subgroup separable. For the sequel we shall assume that the case $A = K = B$ does not occur.

Lemma 1 provides examples of HNN-extensions with subgroup separable base group which are not subgroup separable.

LEMMA 2. *Let $A \cap B$ be a subgroup of finite index in both A and B, such that $A \neq K \neq B$. Then the following statements are equivalent*:

1. *G is subgroup separable.*
2. *There is a subgroup of finite index in $A \cap B$, say H, such that $H \lhd G$.*

Proof. If G is subgroup separable then G is residually finite and the result follows from [3].

Conversely, let $H^k = \{h^k | h \in H\}$. Then H^k is a characteristic subgroup of H for every $k \in \mathbb{N}$. Using standard results one can show that $G/H^k = \langle \tau, K/H^k | \tau^{-1}(A/H^k)\tau = B/H^k \rangle$ where the isomorphic subgroups A/H^k and B/H^k are finite. We shall show first that G/H^k is subgroup separable. The group $G/H^k (k \in \mathbb{N})$ is an HNN-extension with base an abelian group. By Corollary 1 in [4] G/H^k is residually finite. So for every non-trivial $gH^k \in G/H^k$ there is a subgroup of finite index in G, say N, such that $gH^k \notin N/H^k$. Notice that the base group of $G/H^k, K/H^k$, is finitely generated (since K is) and abelian, so its finite part is a finite subgroup of K/H^k. Let $g_i H^k, i = 1, \ldots, n$ be the elements of K/H^k of finite order. Then we can always find subgroups of finite index in G, say $M_i/H^k, i = 1, \ldots, n$, such that $g_i H^k \notin M_i/H^k$. Let $M/H^k = \bigcap_{i=1}^n M_i/H^k$. Then M/H^k is a subgroup of finite index in G/H^k and $P/H^k = M/H^k \cap (K/H^k)$ is of finite index in K/H^k. Moreover, P/H^k is torsion-free, hence free-abelian.

Now, let $(\mathcal{G}, X)$ be a graph of groups with a single vertex v and single edge e with $G_v = K/H^k$ and $G_e = A/H^k$. It is well known that the fundamental group $\pi_1((\mathcal{G}, X)) = G/H^k$. From the structure theorem of Bass–Serre theory (see for example [9]), we know that M/H^k is the fundamental group of a graph of groups where the vertex groups are $M/H^k \cap aG_v a^{-1} = M/H^k \cap a(K/H^k)a^{-1}$, a runs over a suitable finite set of $(M/H^k, K/H^k)$ double coset representatives and edge groups $M/H^k \cap bG_e b^{-1}$, where b runs over a suitable finite set of $(M/H^k, A/H^k)$ double coset representatives. But from the choice of $M/H^k, M/H^k \cap a(K/H^k)a^{-1}$ is some conjugate of P/H^k and

$M/H^k \cap b(A/H^k)b^{-1} = \{1\}$. Hence, M/H^k is the free product of a finite number of free abelian groups and (possibly) a free group. But by [7, 17] such a product is subgroup separable. Moreover, M/H^k is of finite index in G/H^k and by [18], G/H^k is also subgroup separable.

Let L be a finitely generated subgroup of G. Then it suffices to show that $\bigcap_{N \in \mathcal{N}} LN = L$ where $\mathcal{N}$ is the set of all normal subgroups of finite index in G. We claim that is enough to show that $\bigcap_{n \in \mathbb{N}} LH^n = L$. Indeed, we know that the quotient group LH^n/H^n is a finitely generated subgroup of G/H^n since L is finitely generated. Moreover, G/H^n is subgroup separable so if $\mathcal{V}$ is the set of all normal subgroups of finite index in G/H^n then

$$\bigcap_{N/H^n \in \mathcal{V}} \frac{N}{H^n} \frac{LH^n}{H^n} = \frac{LH^n}{H^n}.$$

But the above is equivalent to

$$\bigcap_{N \in \mathcal{N}} \frac{N}{H^n} \frac{LH^n}{H^n} = \frac{LH^n}{H^n}.$$

and so $\bigcap_{N \in \mathcal{N}} LN \subseteq LH^n$ for every $n \in \mathbb{N}$. Therefore $\bigcap_{N \in \mathcal{N}} LN$ is a subgroup of $\bigcap_{n \in \mathbb{N}} LH^n$ and that completes the proof of the claim.

Now, let $U = \bigcap_{n \in \mathbb{N}} LH^n$. Obviously L is a subgroup of U. Then $U \cap H = (\bigcap_{n \in \mathbb{N}} LH^n) \cap H = \bigcap_{n \in \mathbb{N}} (L \cap H)H^n = L \cap H$ since $L \cap H$ is a subgroup of H and H is subgroup separable as a finitely generated abelian group. Let $w \in \bigcap_{n \in \mathbb{N}} LH^n$. Then there is an $l \in L$ such that for every $n \in \mathbb{N}$ there is an $h_n \in H^n$ with $w = lh_n$. So $l^{-1}w = h_n \in H$. But $l^{-1}w = h_n \in \bigcap_{n \in \mathbb{N}} LH^n$. Hence, $l^{-1}w \in (\bigcap_{n \in \mathbb{N}} LH^n) \cap H = L \cap H$. Thus, there is an $l_1 \in L$ such that $l^{-1}w = l_1$ which implies that $w = ll_1 \in L$. So $\bigcap_{n \in \mathbb{N}} LH^n \subseteq L$. But $L \subseteq \bigcap_{n \in \mathbb{N}} LH^n$ and therefore $\bigcap_{n \in \mathbb{N}} LH^n = L$. Consequently, G is subgroup separable. ∎

THEOREM 2. *Let K be a finitely generated abelian group and let A, B be subgroups of K such that G is the HNN-extension*

$$G = \langle t, K \mid t^{-1}At = B \rangle.$$

Then G is subgroup separable (LERF) if and only if $A \cap B$ is a subgroup of finite index in both A and B and there is a finitely generated normal subgroup of G, say H, such that H has finite index in $A \cap B$.

Proof. Assume that G is subgroup separable. Then G is residually finite and by [4] either $A = K$ or $B = K$ or there exist a finitely generated normal subgroup H of G such that H is a subgroup of finite index in D where D is the group

$$D = \{x \in K \mid \text{for every } n \in \mathbb{Z} \text{ there exists a } \lambda_n \in \mathbb{N} \text{ such that } t^{-n} x^{\lambda_n} t^n \in K\}.$$

If $A = K(B \neq K)$ or $B = K(A \neq K)$ we get a contradiction by using Lemma 1. On the other hand, if the subgroup H described above exists then in view of Lemma 2 it suffices to show that $|A : A \cap B| < \infty$ and $|B : A \cap B| < \infty$. Assume that $|A : A \cap B| = \infty$. Since $H \leq A \cap B, H$ has infinite index in A. But H has finite index in D. Therefore we can find an element of infinite order in A, say x, such that $\langle x \rangle \cap D = \{1\}$. This choice of x implies that there is a $n \in \mathbb{N}$ such that $t^{-n}x^{\lambda}t^{n} \notin K$ for every $\lambda \in \mathbb{N}$ and $t^{-n+1}xt^{n-1} \in K$. For if $t^{-n}xt^{n} \in K$ for every $n \in \mathbb{N}$, together with the assumption that K is finitely generated implies that $t^{-n}xt^{n} \in K$ for every $n \in \mathbb{Z}$, a contradiction to the choice of x. Let $y = t^{-n+1}xt^{n-1}$. Then $\{x, y\}$ freely generate a subgroup of K. Indeed, if there is some $m \in \mathbb{N}$ such that $x^{m} = y^{k} = t^{-n+1}x^{k}t^{n-1} (k \in \mathbb{N})$ then $t^{-1}x^{m}t = t^{-1}y^{k}t = t^{-n}x^{k}t^{n}$. But $t^{-1}x^{m}t \in B < K$ and so $t^{-n}x^{k}t^{n} \in K$, a contradiction to the above discussion.

Now let P be the subgroup of G generated by $\{t^{n-1}, x, y\}$. Then P has a presentation of the form

$$P = \langle t^{n-1}, x, y \mid [x, y] = 1, t^{n-1}xt^{n-1} = y \rangle.$$

By setting $\alpha = t^{n-1}$ we get the equivalent presentation

$$P = \langle \alpha, x, y \mid [x, y] = 1, \alpha x \alpha^{-1} = y \rangle.$$

Set again $y = \beta x$ and use Tietze transformations to get the equivalent presentation

$$P = \langle \alpha, \beta, x \mid x\beta x^{-1} = \beta, x\alpha x^{-1} = \alpha\beta \rangle.$$

But P is not subgroup separable by the results in [8]. Hence, G contains a subgroup which is not subgroup separable although G is subgroup separable and that contradicts the results of Scott in [18, 19]. Therefore $|A : A \cap B| < \infty$ and $|A : H| < \infty$. Moreover, since $A/H \cong B/H$ we have $|B : A \cap B| < \infty$.

The converse of the theorem is obvious from Lemma 2. ∎

Apparently, the case $K = \langle a \rangle$ is a special case of a result proven in [21].

COROLLARY 1. *The Baumslag–Solitar groups*

$$\mathrm{BS}(p, q) = \langle a, t \mid t^{-1}a^{p}t = a^{q} \rangle$$

are subgroup separable if and only if $|p| = |q|$.

Another consequence of Theorem 2 is the following:

COROLLARY 2. *If G is subgroup separable then* $|K : A| = |K : B|$.

Finally, an application of our theorem gives a partial answer to Problem 5 set by Scott in [11]. The problem is as follows: let F be a free group of rank n and let G be the semidirect product $G = F \rtimes_\phi \mathbb{Z}$ with $\phi \in \operatorname{Aut} F$. Is G subgroup separable? The above problem was first answered negatively by Burns et al. in [8]. Here, we show that the members of a certain family of free-by-$\mathbb{Z}$ groups contain the example of Burns et al. [8] as a subgroup and therefore can never be subgroup separable.

More specifically, suppose that ϕ is an automorphism of the free group F of rank n and that $X = \{x_1, \ldots, x_n\}$ is a basis for F; then X is a right layered basis for ϕ if

$$\phi(x_1) = x_1 \quad \text{and} \quad \phi(x_i) = x_i w_i, \quad w_i \in F(x_1, \ldots, x_{i-1}) \quad \text{for } 2 \le i \le n.$$

The existence of a right layered basis for an automorphism ϕ is strongly connected with the rank of $\operatorname{Fix}(\phi)$. In fact, it was shown in [10] that if F is a free group of rank n and ϕ is an automorphism of F such that rank $(\operatorname{Fix}(\phi)) = n$ then there is a right layered basis for ϕ such that all the words w_i lie in $\operatorname{Fix}(\phi)$.

Corollary 3. *Let G be as above and let $1 \neq \phi \in \operatorname{Aut} F$ such that there is a right layered basis for ϕ. Then G is subgroup separable if and only if ϕ is the identity automorphism.*

Proof. Let $1 \neq \phi \in \operatorname{Aut} F$ such that there is a right layered basis $\{x_1, \ldots, x_n\}$ for ϕ. Then,

$$\phi(x_1) = x_1 \quad \text{and } \phi(x_i) = x_i w_i, \quad w_i \in F(x_1, \ldots, x_{i-1}) \quad \text{for } 2 \le i \le n.$$

So G has a presentation

$$G = \langle \alpha, x_1, \ldots, x_n \mid \alpha x_1 \alpha^{-1} = x_1, \alpha x_2 \alpha^{-1} = x_2 w_2, \ldots, \alpha x_n \alpha^{-1} = x_n w_n \rangle,$$

where w_i are words as above.

Since $\phi \neq 1$, at least one of the w_i is a non-trivial word. Assume that w_s is the first such word and let H be the subgroup of G generated by $\{\alpha, w_s, x_s\}$. Then one can easily show that H has a presentation of the form

$$H = \langle \alpha, x_s, w_s \mid [\alpha, w_s] = 1, \alpha x_s \alpha^{-1} = x_s w_s \rangle$$

or equivalently

$$H = \langle \alpha, w_s, x_s \mid [\alpha, w_s] = 1, x_s^{-1} \alpha x_s = \alpha w_s \rangle.$$

So H is an HNN-extension with base K a free abelian group of rank two generated by $\{\alpha, w_s\}$, and stable letter x_s. The isomorphic subgroups of K, A and B, are the infinite cyclic subgroups generated by $\{\alpha\}$ and $\{\alpha w_s\}$. It is obvious that $A \cap B = \{1\}$ and by Theorem 2, H is not subgroup separable. Therefore G cannot be subgroup separable because it contains H.

On the other hand if $\phi = 1$ then $G = F_n \times \mathbb{Z}$ and G is known to be subgroup separable by the results in [1, 18]. ∎

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