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On the Conjugacy Separability of Certain Graphs of Groups

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1. INTRODUCTION

A group G is said to be *conjugacy separable* if, whenever x and y are nonconjugate elements of G, there exists a finite quotient group of G in which the images of x and y are not conjugate. The importance of this notion was pointed out by Mal'cev, who proved in [7] that if a finitely presented group G is conjugacy separable, then G has solvable the conjugacy problem, that is, there exists an algorithm to decide whether or not any two given elements of G are conjugate.

Polycyclic-by-finite groups ([5, 12]), free groups ([13, 16]), free-by-finite groups [3] are conjugacy separable. It is known that free products of conjugacy separable groups are again conjugacy separable ([16, 13]). However, the property is not preserved in general by the formation of free products with amalgamation or *HNN*-extensions.

For example, one of the simplest type of *HNN*-extensions, the Baumslag–Solitar group $\langle a, t \mid t^{-1}a^7t = a^{13} \rangle$ is not even residually finite; note that if a group G is conjugacy separable then it must be residually finite. C. F. Miller in [8] gave examples of *HNN*-extensions which are residually finite and not conjugacy separable.

In this paper we are concerned with the conjugacy separability of fundamental groups of certain graphs of groups. Our main results are:

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THEOREM A. *Let* $G = \langle t, K \mid t^{-1}At = B, \phi \rangle$ *be the HNN-extension with base group K a finitely generated abelian group and associated subgroups* A, B *with* $A \neq K$, $B \neq K$. *Then G is residually finite iff G is conjugacy separable.*

THEOREM B (Corollary 2.4). *Let* $G = \langle t, K \mid t^{-1}At = B, \phi \rangle$ *be the HNN-extension with base group K a finitely generated torsion free nilpotent group and associated subgroups* A, B *of finite index in K, with* $A \neq K$, $B \neq K$. *Then the following statements are equivalent for G*

(i) *G is residually finite*

(ii) *G is conjugacy separable*

(iii) *G is* $\mathbf{Z}$*-linear.*

Note that in [11] we provide an easily computable criterion for deciding about the $\mathbf{Z}$-linearity of G.

We also establish

THEOREM C. *Let G be the fundamental group of a finite graph* $(\mathscr{G}, X)$ *of groups, where X is a tree and all vertex and edge groups are finitely generated torsion free nilpotent groups of the same Hirsch number. Then G is conjugacy separable.*

For the proof of Theorem A we consider three cases. We first show that if the conjugacy class, x^G, of an element x of G is such, that $x^G \cap K = \varnothing$ and if y is not conjugate to x, then there is an epimorphic image of G, which is a certain *HNN*-extension with a finite base group, in which the image of x is not conjugate to the image of y. By a theorem of J. Dyer [4] such an *HNN*-extension is conjugacy separable, which of course implies that x and y are conjugacy distinguished in a finite epimorphic image of G.

The other two cases relate the conjugacy class x^G of an element x of G to a certain normal subgroup of G whose properties characterize the residual finiteness of G [1]. In one of these cases we find a polycyclic group Γ such that $x^G = x^\Gamma$ and use the fact that polycyclic groups are conjugacy separable. In the other case, we first show that $x^G \cap K$ is finite and then we proceed as in the first case. In all the preceding cases, Proposition 1.1, which gives conditions equivalent to G being residually finite and D. Collins' Conjugacy Theorem for *HNN*-extensions play a crucial role.

If G is as in Theorem B and residually finite or if G is as in Theorem C, then there exists a "very big" normal subgroup H of G. The proofs of Theorems B and C are based on Lemma 2.1 which states that if a certain subset of H is closed in the profinite topology of H then G is conjugacy separable.

Notation.

If A is a subgroup of G of finite index then we write $A \leq_f G$.

If A is a normal subgroup of G of finite index then we write $A \lhd_f G$.

If x is conjugate to y in G then we write $x \sim_G y$.

If A is a polycyclic group then its Hirsch number is denoted by $h(A)$.

2. PROOF OF THEOREM A

We first recall some facts about *HNN*-extensions and state D. Collins' conjugacy theorem.

Let $G = \langle t, K \mid t^{-1}At = B, \phi \rangle$ be the HNN-extension with base group K, A and B associated subgroups and $\phi\colon A \to B$ the isomorphism effected by conjugation by t within G. Each element of G may be written as a *t-reduced* product $k_0 t^{e(1)} k_1 \cdots t^{e(n)} k_n$, where $k_i \in K$, $e(i) = \pm 1$ and no subwords $t^{-1}at (a \in A)$ or $tbt^{-1} (b \in B)$ occur. Such a product has length n and the length is well defined in G. A t-reduced product is *cyclically reduced* if it is of length zero or it is of the form $t^{e(1)}k_1 t^{e(2)} \cdots t^{e(n)} k_n$ all of those cyclic permutations $t^{e(i)}k_i \cdots t^{e(n)} k_n t^{e(1)} k_1 \cdots t^{e(i-1)} k_{i-1}$ are also t-reduced.

Note that all cyclic permutations of a t-reduced product are in the same conjugacy class and that every conjugacy class contains cyclically reduced elements.

D. Collins' Conjugacy Theorem for HNN-extensions [4] states that if x and y are cyclically reduced elements of $G \setminus K$ and $x \sim_G y$ then:

(i) length of x = length of y.

(ii) $z^{-1}yz = x^*$ for some $z \in A \cup B$, where x^* is a cyclic permutation of x. Moreover if $x^* = t^{e(1)} u_1 t^{e(2)} u_2 \cdots t^{e(n)} u_n$ and $y = t^{\varepsilon(1)} v_1 t^{\varepsilon(2)} v_2 \cdots t^{\varepsilon(n)} v_n$, then $z^{-1}yz = x^*$ for some $z \in A \cup B$ iff $\varepsilon(i) = \varepsilon(i)$ $i = 1, \dots, n$ and there is a finite sequence of elements $z_0, z_1, \dots, z_n \in A \cup B$ for which

(1) $t^{-e(i)} z_{i-1} t^{e(i)} = w_i \in A \cup B$ for $i = 1, \dots, n$.

(2)
$$\begin{aligned} u_1 &= w_1^{-1} v_1 z_1 \\ u_2 &= w_2^{-1} v_2 z_2 \\ &\vdots \\ u_n &= w_n^{-1} v_n z_n. \end{aligned}$$

(3) $z_n = z_0 = z$.

Now let $M \triangleleft K$ such that $\phi(M \cap A) = M \cap B$. Then we obtain $G_M = \langle \tau, K/M \,|\, \tau^{-1}(AM/M)\tau = BM/M \rangle$ and an epimorphism $\vartheta_M\colon G \to G_M$ where $t \to \tau$ and $x \to xM$ for $x \in K$. Note that if $M = N \cap K$ with $N \triangleleft G$ then $\phi(M \cap A) = M \cap B$.

PROPOSITION 1.1. *Let $G = \langle t, K \,|\, t^{-1}At = B, \phi \rangle$ be the HNN-extension with base group K a finitely generated abelian group and associated subgroups A, B with $A \neq K$ and $B \neq K$. Then the following statements are equivalent*

(i) *G is residually finite*

(ii) *A, B are closed in the profinite topology of G*

(iii) *Every subgroup of K is closed in the profinite topology of G*

(iv) *If $H = \{x \in K \,|\, t^{-\rho}xt^{\rho} \in K \ \forall \rho \in \mathbf{Z}\}$ then H is of finite index in D, where $D = \{x \in K \,|$ for every $n \in \mathbf{Z}$ there is $m_n \in N$ such that $t^{-n}x^{m_n}t^n \in K\}$.*

Proof. (i) $\Rightarrow$ (ii) Assume A is not closed and let $x \in \bigcap\{NA \,|\, N \triangleleft_f G\} \setminus A$, if $y \in K \setminus B$ then $[t^{-1}xt, y] \neq 1$. if $N \triangleleft_f G$ then since $x \in NA$ and K is abelian it follows that $[t^{-1}xt, y]N = N$ hence $[t^{-1}xt, y] \in \bigcap\{N \,|\, N \leq_f G\}$ a contradiction since $[t^{-1}xt, y] \neq 1$. Similarly we prove that B is closed in the profinite topology of G.

(ii) $\Rightarrow$ (iii) Since A, B are closed in the profinite topology of G and the torsion subgroups of $K/A, K/B$ are finite there are $N_a \triangleleft_f G$ and $N_b \triangleleft_f G$ such that if $M_a = N_a \cap K$ and $M_b = N_b \cap K$ then $M_a A/A, M_b B/B$ are torsion free. Hence if $N = N_a \cap N_b$ and $M = N \cap K$ then $N \triangleleft_f G$ and $M/(M \cap A), M/(M \cap B)$ are free abelian of finite rank, which implies that $M \cap A, M \cap B$ are direct summands of M and so

$$\phi(M^n \cap A) = M^n \cap B \quad \text{for all } n \in N. \qquad (*)$$

We will first show that every subgroup of M is closed in the profinite topology of G.

Let $\Lambda \leq M$ and $x \in G \setminus \Lambda$. Clearly it is enough to find a subgroup separable group G_n and an epimorphism $\vartheta_n\colon G \to G_n$ such that $\vartheta_n(x) \notin \vartheta_n(\Lambda)$. If $x \in K$ then since $\Lambda = \bigcap_{n \in N} M^n\Lambda$, there is an $n_0 \in N$ such that $x \notin M^{n_0}\Lambda$. By $(*)$ the *HNN*-extension $G_{n_0} = \langle t_{n_0}, K/M^{n_0} \,|\, t_{n_0}^{-1}M^{n_0}A/M^{n_0}t_{n_0} = M^{n_0}B/M^{n_0} \rangle$ is defined and let $\vartheta_{n_0}\colon G \to G_{n_0}$ be the obvious epimorphism. Clearly $\vartheta_{n_0}(x) \notin \vartheta_{n_0}(\Lambda)$ and since $|K : M^{n_0}| < \infty$, G_{n_0} is free-by-finite, hence subgroup separable. If $x \notin M$ and $x = k_0 t^{e_1} k_1 t^{e_2} \cdots k_r t^{e_r}$ t-reduced, then since $A = \bigcap_{n \in N} M^n A$ and $B = \bigcap_{n \in N} M^n B$ there is an $m \in N$ such that if $G_m = \langle t_m, K/M^m \,|\, t_m^{-1}M^m A/M^m t_m = M^m B/M^m \rangle$ then $\vartheta_m(x) = (k_0 M^m)t_m^{e_1}(k_1 M^m)t_m^{e_2} \cdots (k_r M^m)t_m^{e_r}$ is t_m-reduced. The result now follows. So we proved that every subgroup of M is closed in the profinite topology of G. Now let Γ be a subgroup of K, then $\Gamma \cap M$ is closed in the profinite topology of G and since $|\Gamma\colon \Gamma \cap M| < \infty$, Γ is closed in the profinite topology of G.

(ii) $\Rightarrow$ (i) In [1] and [15].

(i) $\Leftrightarrow$ (iv) In [1]. ∎

COROLLARY 1.2. *Let G be as in Proposition 1.1 and assume that G is residually finite. If $g \in G$ is cyclically reduced then there is $M \lhd_f K$ with $\phi(M \cap A) = M \cap B$ such that $\vartheta_M(g)$ is cyclically reduced and length of $g =$ length of $\vartheta_M(g)$.*

Proof. By Proposition 1.1 A, B are closed in the profinite topology of G. The result now follows easily. ∎

COROLLARY 1.3. *Let G be as in Proposition 1.1 and assume that G is residually finite. If $gA \cap B = \varnothing$ for some $g \in G$ then there is an $N \lhd_f G$ such that $gA(N \cap K) \bigcap B(N \cap K) = \varnothing$.*

Proof. By Proposition 1.1 AB is closed in the profinite topology of G. The result now follows easily. ∎

THEOREM A. *Let $G = \langle t, K \,|\, t^{-1}At = B, \phi \rangle$ be the HNN-extension with base group K a finitely generated abelian group, and associated subgroups A, B with $A \neq K$, $B \neq K$. Then G is residually finite iff G is conjugacy separable.*

Proof. (α) Let $x \in G$ and assume that $x^G \cap K = \varnothing$. Now let $y \in G$ with $x \nsim_G y$. Clearly we may assume that x, y are cyclically reduced. We will show that there is $M \lhd_f K$ with $\phi(M \cap A) = M \cap B$ such that $\vartheta_M(x) \nsim_{G_M} \vartheta_M(y)$. Since by [4] G_M is conjugacy separable it follows that there is a finite quotient group of G in which the images of x and y are not conjugate.

(1) If length of $x \neq$ length of y, then by Corollary 1.2 there is an $M \lhd_f K$ with $\phi(M \cap A) = M \cap B$ and length of $\vartheta_M(x) \neq$ length of $\vartheta_M(y)$. By D. Collins' Conjugacy Theorem for *HNN*-extension $\vartheta_M(x) \nsim_{G_M} \vartheta_M(y)$.

(2) If length of $x =$ length of y and $x = t^{e_1}u_1t^{e_2}u_2 \cdots t^{e_n}u_n$ and $y = t^{\varepsilon_1}v_1t^{\varepsilon_2}v_2 \cdots t^{\varepsilon_n}v_n$ assume that $(\varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_{i-1}) \neq (e_1, \dots, e_n)$ for all $1 \leq i \leq n$. Then again by Corollary 1.2 and D. Collins' Conjugacy Theorem there is an $M \lhd_f K$ with $\phi(M \cap A) = M \cap B$ and $\vartheta_M(x) \nsim_{G_M} \vartheta_M(y)$.

(3) Let now $x = t^{e_1}u_1t^{e_2}u_2 \cdots t^{e_n}u_n$ and $y = t^{e_1}v_1t^{e_2}v_2 \cdots t^{e_n}v_n$. Without loss of generality we may assume that $x = tu_1tu_2$ and $y = tv_1tv_2$.

D. Collins' Conjugacy Theorem states that in this case $x \sim_G y$ iff there are $z_0, z_1, z_2 \in A$ such that

(i) $t^{-1} z_0 t = w_1$
$t^{-1} z_1 t = w_2,\ w_1, w_2 \in B$

(ii) $u_1 = w_1^{-1} v_1 z_1$
$u_2 = w_2^{-1} v_2 z_2$ and

(iii) $z_2 = z_0$

Clearly $z_1 \in (u_1 v_1^{-1}) B \cap A$. We are given that $x \nsim_G y$. Now if $(u_1 v_1^{-1}) B \cap A = \emptyset$, then by Corollary 1.3 there is an $M \lhd_f K$ with $\phi(A \cap M) = B \cap M$ and $(u_1 v_1^{-1}) BM \cap AM = \emptyset$, which implies that $\vartheta_M(x) \nsim_{G_M} \vartheta_M(y)$. Let assume that $(u_1 v_1^{-1}) B \cap A \neq \emptyset$. Then $(u_1 v_1^{-1}) B \cap A = a_0(A \cap B)$ for some $a_0 \in A$ and $z_1 \in a_0(A \cap B)$. Then $w_1 \in (u_1^{-1} v_1) a_0 (A \cap B) = d(A \cap B)$ which implies that $z_0 \in \phi^{-1}(d) \phi^{-1}(A \cap B) = d' \phi^{-1}(A \cap B)$ and $z_2 \in k_0 \phi(A \cap B)$ where $k_0 = u_2 v_2^{-1} \phi(a_0)$. Now $z_2 = (u_2 v_2^{-1}) t^{-1} (u_1 v_1^{-1}) t^{-1} z_0 t^2$ and since $x \nsim_G y$ it follows that $z_2 \neq z_0$ for all possible values of z_0. This implies that $t^{-2} z_0 t^2 z_0^{-1} \neq (u_2 v_2^{-1} t^{-1} u_1 v_1^{-1} t)^{-1} = g \in K$ for every z_0 which is in $d' \phi^{-1}(A \cap B)$. If $\Omega = \{t^{-2} z_0 t^2 z_0^{-1};\ z_0 \in d' \phi^{-1}(A \cap B)\}$ then $\Omega = wY$ where $w = t^{-2} d' t^2 (d')^{-1}$ and $Y = \{t^{-2} z_0 t^2 z_0^{-1};\ z_0 \in \phi^{-1}(A \cap B)\}$ is a subgroup of K. By Proposition 1.1 Y is closed in the profinite topology of G, hence wY is closed in the profinite topology of G. Hence there is an $M \lhd_f K$ with $\phi(M \cap A) = M \cap B$ and $gM \notin \Omega M$, which implies that $\vartheta_M(x) \nsim_{G_M} \vartheta_M(y)$.

(β) Let $H = \{x \in K \mid t^{-\rho} x t^{\rho} \in K\ \forall \rho \in \mathbf{Z}\}$ and let $x \in G$ with $x^G \subseteq H$. We shall show that x^G is closed in the profinite topology of G. We consider the split extension $\Gamma = H]\langle t \rangle$. Clearly $x^G = x^\Gamma$ and since Γ is polycyclic x^Γ is closed in the profinite topology of Γ by [5]. Hence $x^G = x^\Gamma = \bigcap\{x^G M \mid M \lhd_f \Gamma\}$. It is easy to see that

$$x^G = \bigcap\left\{x^G M \mid M \lhd_f \Gamma\right\} = \bigcap\left\{x^G H^n \mid n \in \mathbf{N}\right\}. \tag{1}$$

Now since $x^G \subseteq H$ and H is closed in G it follows that $\bigcap\{x^G N \mid N \lhd_f G\} \subseteq \bigcap\{HN \mid N \lhd_f G\} = H$, hence,

$$\bigcap\left\{x^G N \mid N \lhd_f G\right\} = \bigcap\left\{x^G N \cap H \mid N \lhd_f G\right\}$$
$$\subseteq \bigcap\left\{x^G (N \cap H) \mid N \lhd_f G\right\}. \tag{2}$$

However, by Proposition 1.1 G/H^n is residually finite and since H/H^n is finite there is $N_n \lhd_f G$ such that $N_n \cap H \subseteq H^n$, which implies that $\bigcap\{N \cap H \mid \mathrm{N} \lhd_f G\} \subseteq \bigcap\{H^n \mid n \in \mathbf{N}\}$.

From (2) it follows that $\bigcap\{x^G N \mid N \lhd_f G\} \subseteq \bigcap\{x^G (N \cap H) \mid N \lhd_f G\} \subseteq \bigcap\{x^G H^n \mid n \in \mathbf{N}\}$ and from (1) $\bigcap\{x^G H^n \mid n \in \mathbf{N}\} = x^G$, hence the result.

(γ) Let now $x^G \cap K \neq \emptyset$ and $x^G \cap H = \emptyset$. We may assume that $x \in K \setminus H$. If $x^G \cap K$ is infinite, then it is easy to see that either there is a negative integer ν_1 such that

(i) $\{t^{-\rho}xt^{\rho} \mid \rho \geq \nu_1\} \subseteq x^G \cap K$.

or there is a positive integer ν_2 such that

(ii) $\{t^{-\rho}xt^{\rho} \mid \rho \leq \nu_2\} \subseteq x^G \cap K$.

If we assume that exactly one of the (i) and (ii) holds then w.l.g. there is a negative integer ν_1 such that $\{t^{-\rho}xt^{\rho} \mid \rho \geq \nu_1\} \subseteq x^G \cap K$ and $t^{-(\nu_1-1)}xt^{\nu_1-1} \notin K$. Consider now $\Lambda = \langle t^{-\rho}xt^{\rho} \mid \rho \geq \nu_1 \rangle$. This is a subgroup of K and $t^{-1}\Lambda t < \Lambda$ since $t^{-\nu_1}xt^{\nu_1} \notin t^{-1}\Lambda t$. Because $t^{-1}\Lambda t \leq_f \Lambda$ and Λ is closed there is $N \lhd_f G$ such that $N \cap \Lambda \leq t^{-1}\Lambda t$. If $M = N \cap \Lambda$ then clearly $tMt^{-1} = M$ hence $t^{-1}\Lambda t = \Lambda$ contradiction. Hence, $\{t^{-\rho}xt^{\rho} \mid \rho \cap \mathbf{Z}\} \subseteq x^G \cap K$ and thus $x \in H$ contradiction. Hence $x^G \cap K$ is finite.

Let now $y \in G$ with $x \nsim_G y$. Clearly we may assume that $y^G \cap K \neq \emptyset$ and $y^G \cap H = \emptyset$ hence $y^G \cap K$ is finite. Let now $x^G \cap K = \{t^{-\rho_0}xt^{\rho_0}, \dots, x, \dots, t^{-\rho_i}xt^{\rho_i}\}$ and $y^G \cap K = \{t^{-\xi_0}yt^{\xi_0}, \dots, y, \dots, t^{-\xi_j}yt^{\xi_j}\}$. Clearly $t^{-\rho_0}xt^{\rho_0} \notin B$, $t^{-\xi_0}yt^{\xi_0} \notin B$, and $t^{-\rho_i}xt^{\rho_i} \notin A$, $t^{-\xi_j}yt^{\xi_j} \notin A$. Since $x \nsim_G y$, it follows that $(x^G \cap K) \bigcap (y^G \cap K) = \emptyset$. Since A, B are closed there is $N \lhd_f G$ such that if $M = N \cap K$ then $t^{-\rho_0}xt^{\rho_0}M \notin BM/M$, $t^{-\xi_0}yt^{\xi_0}M \notin BM/M$, $t^{-\rho_i}xt^{\rho_i}M \notin AM/M$, $t^{-\xi_j}yt^{\xi_j}M \notin AM/M$, and $x_\lambda y_\sigma^{-1} \notin M$ for all $x_\lambda \in x^G \cap K$ and all $y_\sigma \in y^G \cap K$. It is easy to see now that in G_M we have $(\vartheta_M(x))^{G_M} \bigcap K/M = \vartheta_M(x^G \cap K)$ and $(\vartheta_M(y))^{G_M} \bigcap K/M = \vartheta_M(y^G \cap K)$ which implies that $\vartheta_M(x) \nsim_{G_M} \vartheta_M(y)$ and the result follows sine G_M is conjugacy separable. ∎

The following corollary is now immediate.

COROLLARY 1.4. *Let $G = \langle t, K \mid t^{-1}At = B, \phi \rangle$ be the HNN-extension with base group K a finitely generated abelian group, associated subgroups A, B with $A \neq K$ and $B \neq K$. Then the following statements are equivalent for the group G*

(i) *G is residually finite*

(ii) *G is conjugacy separable*

(iii) *A, B are closed in the profinite topology of G.*

3. THE CONJUGACY SEPARABILITY OF CERTAIN GROUPS IN $\mathscr{X}_n$

We recall that a graph of groups $(\mathscr{G}, X)$ is a connected graph X together with a function which assigns:

(i) to each vertex v of X a group G_v (*vertex group*).

(ii) to each edge e of X a group G_e (*edge group*) such that $G_e = G_{\bar{e}}$ and

(iii) to each edge e of x a monomorphism $\iota_{\tau(e)} G_e \to G_{\tau(e)}$.

If $(\mathcal{G}, X)$ is a graph of groups then $\pi(\mathcal{G}, X)$ denotes the fundamental group of the graph of groups $(\mathcal{G}, X)$. For the relevant definitions we refer to [14].

Let $\mathcal{X}_n$ denote the class of all groups which are fundamental groups of a finite graph of groups with all vertex and edge groups polycyclic of Hirsch number n-by-finite.

Clearly a Baumslag–Solitar group is in $\mathcal{X}_1$. One relator groups with nontrivial center are in $\mathcal{X}_1$ [9].

Let G be a group in $\mathcal{X}_n$ and assume that there is a polycyclic normal subgroup H of G with $h(H) = n$. It is not difficult to see that if $G = \pi(\mathcal{G}, X)$ then H is contained in every edge group of X. Since H is polycyclic, H^n is of finite index in H and it follows that G/H^n is the fundamental group of a finite graph of finite groups. Let $b \in G$ and Δ_b be the pre-image of $C_{G/H}(bH)$ in G. If $[b, \Delta_b] = \{b^{-1}x^{-1}bx; x \in \Delta_b\}$ then clearly $[b, \Delta_b] \subseteq H$.

LEMMA 2.1. *If the set $[b, \Delta_b]$ is closed in the profinite topology of H for all $b \in G$, then G is conjugacy separable.*

Proof. Let b, c be elements of G and assume that bN is conjugate to cN in G/N for every $N \triangleleft_f G$. The group G/H^n being the fundamental group of a graph of finite groups is conjugacy separable by [3], hence it follows that bH^n is conjugate to cH^n for all $n \in \mathbf{N}$.

Let $bH^n = g_n c g_n^{-1} H^n$, $n \in \mathbf{N}$. If $g_1 := g$ then $gg_n^{-1} \in \Delta_b$ for every $n \in \mathbf{N}$ and $b^{-1}gcg^{-1} \in \bigcap\{[b, \Delta_b]H^n \mid n \in \mathbf{N}\}$. Since $[b, \Delta_b]$ is closed in the profinite topology of H it follows that $\bigcap\{[b, \Delta_b]H^n \mid n \in N\} = [b, \Delta_b]$, hence $b^{-1}gcg^{-1} = b^{-1}x^{-1}bx$ for some $x \in \Delta_b$ and so b is conjugate to c in G. ∎

LEMMA 2.2. *Let K be a polycyclic-by-finite group, $A \leq_f K$, $B \leq_f K$, and ϕ: $A \to B$ an isomorphism. Suppose that there is a polycyclic-by-finite group $\overline{K}$ and $\vartheta \in$ aut $\overline{K}$ so that $K \leq_f \overline{K}$ and $\vartheta|_A = \phi$. Then the HNN-extension $G = \langle t, K \mid t^{-1}At = B, \phi\rangle$ is conjugacy separable.*

Proof. Since $\vartheta \in \operatorname{Aut} \overline{K}$ and $\vartheta|_A = \phi$ we have that $|\overline{K}\colon A| = |\overline{K}:B|$. Hence if $H = \bigcap\{\Lambda \leq \overline{K}\colon |\overline{K}:\Lambda| = |\overline{K}:A|\}$ then the $H \triangleleft K$ with $H \leq_f A$ and $\phi(H) = H$. Let now $L = \overline{K}]\langle\vartheta\rangle$ be the split extension of $\overline{K}$ by $\langle\vartheta\rangle$. Clearly there is a homomorphism λ: $G \to L$ with $\lambda(k) = k$ for all $k \in K$ and $\lambda(t) = \vartheta$. For $b \in G$ let Δ_b be the subgroup of G defined by $\Delta_b/H = C_{G/H}(bH)$ and consider the derivation d: $\lambda(\Delta_b) \to \lambda(H)$ where

$d(\lambda(x)) = [\lambda(b), \lambda(x)]$. By Theorem D in [6] $d(\lambda(\Delta_b)) = [\lambda(b), \lambda(\Delta_b)]$ is closed in the profinite topology of $\lambda(H)$. By definition of the homomorphism λ we have that $[\lambda(b), \lambda(\Delta_b)] = \lambda([b, \Delta_b]) = [b, \Delta_b]$ since $[b, \Delta_b] \subseteq H$ and the result follows from Lemma 2.1. ∎

THEOREM 2.3. *Let $G = \langle t, K \mid t^{-1}At = B, \phi \rangle$ be the HNN-extension with base group K a finitely generated torsion free nilpotent group and associated subgroups, A, B of finite index in K with $A \neq K$, $B \neq K$. Then G is residually finite if and only if it is conjugacy separable.*

Proof. If G is residually finite then by Corollary 5.1 in [10] there is an $H \lhd G$ with $H \leq_f K$. It then follows that if $\mathscr{M}$ is the Mal'cev completion of K then we can find $\overline{K} \leq \mathscr{M}$ with $K \leq_f \overline{K}$ and $\vartheta \in \operatorname{Aut}(\overline{K})$ with $\vartheta|_A = \phi$ [2]. The result now follows from Lemma 2.2. ∎

COROLLARY 2.4. *Let K be a finitely generated torsion-free nilpotent group, A, B subgroups of finite index in K with $A \neq K$, $B \neq K$ and $\phi\colon A \to B$ an isomorphism. Then the following statements are equivalent for the HNN-extension $G = \langle t, K \mid t^{-1}At = B, \phi \rangle$*

(i) *G is residually finite*

(ii) *G is conjugacy separable*

(iii) *G is **Z**-linear*

(iv) *There is an $H \lhd G$ with $h(H) = h(K)$.*

Proof. It follows from Theorem 1.3 in [11] and the proof of Theorem 2.3. ∎

THEOREM C. *Let $G \in \mathscr{X}_n$ and $G = \pi(\mathscr{G}, X)$ with X a tree and every vertex and edge group of X finitely generated torsion-free nilpotent. Then G is conjugacy separable.*

Proof. Let G_{v_0} be a vertex group of X and $\varepsilon_{v_0}\colon G_{v_0} \to \mathscr{M}$ an embedding of G_{v_0} into its Mal'cev completion. Since X is a tree and every edge group is of finite index in the corresponding vertex group it follows that there is a homomorphism $\varepsilon\colon G \to \mathscr{M}$ such that for every vertex v of X $\varepsilon_v = \varepsilon|_{G_v}\colon G_v \to \mathscr{M}$ is an embedding. Now since X is finite it follows that there is $H \leq G$ with $H \lhd G$, $H \leq G_e$ for every edge e of X and $h(H) = n$ [11].

Let $b \in G$ and Δ_b the pre-image of $C_{G/H}(bH)$ in G and consider the derivation $d\colon \varepsilon(\Delta_b) \to \varepsilon(H)$ which is given by $d(\varepsilon(x)) = [\varepsilon(b), \varepsilon(x)]$. Then by Theorem D in [6] $\operatorname{Im} d = [\varepsilon(b), \varepsilon(\Delta_b)]$ is closed in the profinite topology of $\varepsilon(H)$, hence by the definition of ε, the set $[b, \Delta_b]$ is closed in the profinite topology of H and the result follows from Lemma 2.1. ∎

If we allow the vertex groups to be abelian then Theorem C holds without the restriction to the torsion free groups, i.e., we have:

THEOREM 2.6. *Let $G \in \mathscr{X}_n$ and $G = \pi(\mathscr{G}, X)$ with X a tree and every vertex and edge group of X finitely generated abelian group. Then G is conjugacy separable.*

Proof. Let $H = Z(G)$. Then clearly $h(H) = n$. Now if $b \in G$ and Δ_b is the pre-image of $C_{G/H}(bH)$ in G, then the set $[b, \Delta_b]$ is a subgroup of H hence it is closed in the profinite topology of H and the result follows from Lemma 2.1. ∎

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