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ON RESIDUALLY FINITE GRAPH PRODUCTS

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We provide a simple set of sufficient conditions for the residual finiteness of a graph product of groups, which is a generalization of G. Baumslag's residual finiteness criterion for an amalgamated free product of two groups.

Let $\mathcal{G}$ be a graph of groups over a connected graph C. We prove a residual finiteness criterion for the fundamental group $\pi_1(\mathcal{G}, C)$ (see [3] for the definition), which generalizes the well-known criterion of Baumslag [1, Proposition 2] for the amalgamated free product of two groups, and provides an alternative proof of one of the criteria in [5]. This result arose out of a conversation with I.M. Chiswell, to whom I am indebted for many helpful comments.

Let C be a graph with vertex set V and edge set E, and let $\mathcal{G}$ be a graph of groups over C with vertex groups A_v $(v \in V)$, edge subgroups $H_e \subseteq A_{o(e)}$ $(e \in E)$, and edge isomorphisms $\theta_e : H_e \to H_{\bar{e}}$ where $\bar{e}$ is the inverse edge of e. (In this context it is customary to consider monomorphisms $\rho_e : \tilde{H}_e \to A_{o(e)}$ and $\rho_{\bar{e}} : \tilde{H}_e \to A_{t(e)}$. Our edge subgroups are then $H_e = \tilde{H}_e \rho_e$ and $H_{\bar{e}} = \tilde{H}_e \rho_{\bar{e}}$, with θ_e the induced isomorphism between the two. This is notationally more convenient here.) The criterion is stated in terms of certain families of subgroups indexed by V. To be precise, define

$$I_0 = \{(P_v)_{v \in V} : P_v \lhd_f A_v \text{ for all } v,\ (P_{o(e)} \cap H_e)\theta_e = P_{t(e)} \cap H_{\bar{e}} \text{ for all } e\}$$

and

$$I = \{P \in I_0 : \text{there exists } m \geq 1 \text{ with } |A_v : P_v| \leq m \text{ for all } v\}. \tag{1}$$

Note that if P, Q belong to I, then so does $P \cap Q = (P_v \cap Q_v)$. The criterion is as follows:

Theorem. *Let $\mathcal{G}$ be a graph of groups over C, and let I be as in* (1). *Assume that*

(i) $\displaystyle\bigcap_{P \in I} H_e P_{o(e)} = H_e$ *for all* $e \in E$;

(ii) $\displaystyle\bigcap_{P \in I} P_v = \langle 1 \rangle$ *for all* $v \in V$

Then the fundamental group $\pi_1(\mathcal{G}, C)$ is residually finite.

Proof. Let $A = *_{v\in V} A_v$, and consider the HNN-extension

$$G = \langle A, t_e, e \in E: t_e^{-1}H_e t_e = H_{\bar{e}} \text{ via } \theta_e \text{ for all } e \in E\rangle .$$

It is well known [3, Proposition 5.20] that $\pi_1(\mathcal{G}, C)$ embeds into G. We show that under the assumptions of the theorem, G is residually finite. As in [4] let

$$\mathfrak{N} = \{M: M \lhd_f A \text{ and } (M \cap H_e)\theta_e = M \cap H_{\bar{e}} \text{ for all } e \in E\} . \tag{2}$$

Let $P = (P_v) \in I$. Then $A_P = *_{v\in V}(A_v/P_v)$ is a free product of finite groups of bounded order, and is therefore free-by-finite and residually finite [3, Exercise 2 on p. 123]. Let $\pi_P: A \to A_P$ denote the obvious extension of the canonical epimorphisms $A_v \to A_v/P_v$. If $M \lhd_f A$ is such that $M \supseteq \ker \pi_P$ and $M\pi_P$ is a free subgroup of A_P, then clearly $M \cap A_v = P_v$ for all $v \in V$. Thus $(M \cap H_e)\theta_e = (P_{o(e)} \cap H_e)\theta_e = P_{t(e)} \cap H_{\bar{e}} = M \cap H_{\bar{e}}$ for all $e \in E$. In other words, $M \in \mathfrak{N}$. Let $e \in E$, and choose $x \in A_P \backslash H_e\pi_P$. Since A_P is residually finite, the finite set of elements $x^{-1}(H_e\pi_P)$ of A_P may be excluded from some $M\pi_P$ as above. In other words, $x \not\in (H_e\pi_P(M\pi_P) = (H_eM)\pi_P$ for some $M \in \mathfrak{N}$, and hence $H_e \ker \pi_P = \bigcap H_eM$, where the intersection is taken over all $M \in \mathfrak{N}$ with $M \supseteq \ker \pi_P$. Therefore

$$\bigcap_{M\in\mathfrak{N}} H_eM \subseteq \bigcap_{P\in I} H_e \ker \pi_P \quad \text{for all } e \in E . \tag{3}$$

If a is any element of A, then condition (ii) implies that there exists $P \in I$ such that a and $a\pi_P$ have the same syllable length with respect to the appropriate free product decompositions. Further, if $a \in A_{o(e)}\backslash H_e$, then by (i) there exists $P \in I$ such that $a \not\in H_eP_{o(e)}$. Consequently $\bigcup_{P\in I} H_e \ker \pi_P \subseteq H_e$, which in view of (3) implies that

$$\bigcap_{M\in\mathfrak{N}} H_eM = H_e \quad \text{for all } e \in E .$$

Essentially the same argument employed in establishing (3), with $\langle 1\rangle$ in place of H_e, shows that $\bigcap_{M\in\mathfrak{N}} M \subseteq \bigcap_{P\in I} \ker \pi_P$. In view of the remark about the length of the images of the elements of A, the latter intersection is trivial. Hence $\bigcap_{M\in\mathfrak{N}} M = \langle 1\rangle$. By [2, Theorem 4.2] (or rather its immediate extension to an arbitrary number of stable letters; cf. [4]), the HNN-extension G is residually finite, and hence so is $\pi_1(\mathcal{G}, C) \subseteq G$. $\square$

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