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M. Shirvani

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ON CONJUGACY SEPARABILITY OF FUNDAMENTAL GROUPS OF GRAPHS OF GROUPS

M. SHIRVANI

ABSTRACT. A complete determination of when the elements of a fundamental group of a (countable) graph of profinite groups are conjugacy distinguished is given. By embedding an arbitrary fundamental group G into one with profinite vertex groups and making use of the above result, questions on conjugacy separability of G can be reduced to the solution of equations in the vertex groups of G .

1. INTRODUCTION

An element g of a group G is said to be *conjugacy distinguished* (or conjugacy separable) in G if for every element h of G not conjugate to g there exists $N \triangleleft_f G$ such that gN and hN are nonconjugate in G/N . A group is *conjugacy separable* if all its elements are conjugacy distinguished. The best-known classes of conjugacy separable groups are polycyclic-by-finite groups [10], profinite groups [11], free-by-finite groups [5], and certain Fuchsian groups [19]. Results are also known on conjugacy separability of certain amalgamated free products (e.g., of free groups [18]), and some one-relator groups with torsion [1]. In the last mentioned paper the authors ask whether every one-relator group with torsion is conjugacy separable. More generally, one can ask when the fundamental group of a graph of groups is conjugacy separable.

In order to investigate these problems one might adopt the following strategy: let G be a residually finite group, with \widehat{G} its profinite completion (i.e., \widehat{G} is the inverse limit of the system of finite images G/N of G). It is easily shown that if $g, h \in G$ are conjugate in every finite image of G , then they are conjugate in \widehat{G} , and so G is conjugacy separable if and only if whenever $g, h \in G$ are conjugate in \widehat{G} , then they are conjugate in G . Unfortunately, if G is a non-trivial fundamental group of a graph of groups, then \widehat{G} is still not sufficiently well understood (cf. [7, 20]) to allow for the successful completion of the last step. A complete answer can be given when the vertex groups are themselves profinite groups. This, in turn, allows necessary and sufficient conditions to be derived for the general case.

Let X be a connected graph, let $\mathcal{G} = \{A_x, x \in VX, H_e, e \in EX\}$ be a graph of groups over X , and let $G = \pi_1(\mathcal{G}, X)$ be the fundamental group of

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this graph of groups (see [2, 4], or [14] for the relevant definitions and properties). We assume that G is residually finite, and denote by \widehat{A}_x the profinite completion of the vertex group A_x with respect to the induced topology $\{A_x \cap M : M \triangleleft_f G\}$. Let \overline{H}_e denote the topological closure of H_e in $\widehat{A}_{i(e)}$. The edge isomorphisms can be extended to these closures (cf. Lemma 3.1). The result is a graph of groups over X with fundamental group G^+ (for details see §3). The obvious maps $A_x \rightarrow \widehat{A}_x$ extend to an embedding of G into G^+ . If we denote the conjugacy class of g in G by g^G and $\bigcap_{M \triangleleft_f G} g^G M$ by $\text{cl}_G(g^G)$, then we have:

Theorem 3.4. *For every $g \in G$ we have $\text{cl}_G(g^G) = \text{cl}_{G^+}(g^{G^+}) \cap G$.*

The point of this result is that most elements of G^+ turn out to be conjugacy distinguished (i.e. $\text{cl}_{G^+}(g^{G^+}) = g^{G^+}$). If $g \in G$ is such an element, then g is conjugacy distinguished in G if and only if $g^{G^+} \cap G = g^G$. This condition is quite tractable, and amounts to whether certain equations hold in the vertex groups A_x and \widehat{A}_x . Assuming that we have enough information about the vertex groups and their completions, the conjugacy separability of G can be decided.

To state the results for G^+ , let Y be a maximal subtree of X , and write $A^+ = \langle \widehat{A}_x : x \in VX \rangle \leq G^+$. Then A^+ is the fundamental group of the graph of groups \mathcal{G} restricted to Y , and G^+ is an HNN-extension with base group A^+ . As a consequence of 4.4, 4.6, and 4.9 below we get

Theorem A. *Let $a \in A^+$.*

(i) *If either no conjugate of a belongs to a vertex group, or X is locally finite, $a \in \widehat{A}_x$, and no \widehat{A}_x -conjugate of a belongs to an edge subgroup, then a is conjugacy distinguished in A^+ .*

(ii) *Assume that X is locally finite, let $D = \bigcup_{e \in EX} \overline{H}_e$, and suppose $a \in D$ belongs to only a finite number of edge subgroups. Then a is conjugacy distinguished in A^+ if and only if there exist vertices x_1, \dots, x_m of Y such that $D \cap a^{A^+} = D \cap \{a^{a_1 \dots a_m} : a_i \in \widehat{A}_{x_i}\}$. \square*

For elements of G^+ we obtain from 5.1, 5.3, 5.6, and 5.8

Theorem B. (i) *All elements of $G^+ \setminus A^+$, and all elements of A^+ satisfying condition (i) of Theorem A, are conjugacy distinguished in G^+ .*

(ii) *Assume that X is locally finite, and let $D' = \bigcup_{e \in EX \setminus EY} \overline{H}_e$ and $C = \bigcup_{x \in VX} \widehat{A}_x$.*

Then $h \in D'$ is conjugacy distinguished in G^+ if and only if $h^{G^+} \cap C$ is a union of a finite number of sets of the form $C \cap \{h^g : g \in \widehat{A}_{x_1} t_{e_1}^{\varepsilon_1} \dots \widehat{A}_{x_n} t_{e_n}^{\varepsilon_n} \widehat{A}_{x_{n+1}}, \text{ for some } e_i \in EX \setminus EY, \varepsilon_i = \pm 1, \text{ and } x_i \in VX\}$.

This in particular settles the problem for the case when the vertex groups are profinite, since such a G can be residually finite if and only if $G \cong G^+$, by 3.3 below.

2. NOTATION

Let X be a directed connected graph, \mathcal{G} a graph of groups $\{A_x : x \in VX; H_e : e \in EX \text{ with isomorphisms } \theta_e \text{ from } H_e \leq A_{i(e)} \text{ to } H_{\bar{e}} \leq A_{t(e)}\}$ over

X , and $G = \pi_1(\mathcal{G}, X)$ the fundamental group of (\mathcal{G}, X) relative to the choice of a fixed maximal subtree Y (see [2, 4], or [14] for details). It is well known that if $A = \pi_1(\mathcal{G}|_Y, Y)$ then the obvious map of A to $\langle A_x : x \in VY \rangle \subseteq G$ is an isomorphism, and G is an HNN-extension with base group A and stable letters t_e corresponding to the edges $e \in EX \setminus EY$:

$$G = \langle A, t_e : t_e^{-1} h t_e = h \theta_e \text{ for all } h \in H_e, e \in EX \setminus EY \rangle.$$

In the notation of [16] let $I = I(\mathcal{G}, X)$ denote the set of all sequences $(P_x)_{x \in VX}$ satisfying the following conditions: (a) $P_x \triangleleft A_x$ and there exists an integer $m = m(P)$ such that $|A_x : P_x| \leq m$ for all $x \in VX$, (b) $(P_{i(e)} \cap H_e) \theta_e = P_{i(e)} \cap H_{\bar{e}}$ for all $e \in EX$. For $P, Q \in I$ we write $P \leq Q$ if $P_x \subseteq Q_x$ for all x . For $P \in I$ let \mathcal{G}_P denote the graph of groups $\{A_x/P_x, H_e P_{i(e)}/P_{i(e)}, \text{ induced isomorphisms } \theta_{e,P}\}$, and let $G_P = \pi_1(\mathcal{G}_P, X)$. The projections $A_x \rightarrow A_x/P_x$ clearly extend to an epimorphism $\pi_P : G \rightarrow G_P$. It is well known that G_P is free-by-finite (e.g., [14, Exercise 2 on p. 123]). Also note that if $M \triangleleft_f G$ then $P = (M \cap A_x)_{x \in VX} \in I$, and G/M is a homomorphic image of G_P . In particular, this implies that if G is residually finite then $\bigcap_{P \in I} P_x = \langle 1 \rangle$ for all $x \in VX$. We assume this from now on. For $g \in G$ write g^G for the conjugacy class of g in G .

Proposition 2.1. *Let $g \in G$. Then $\bigcap_{M \triangleleft_f G} g^G M = \bigcap_{P \in I} g^G \ker \pi_P$. In particular, $\bigcap_{M \triangleleft_f G} M = \bigcap_{P \in I} \ker \pi_P$.*

Proof. If $M \triangleleft_f G$ induces $P \in I$ then $M \supseteq \ker \pi_P$, so $g^G M \supseteq \bigcap_P g^G \ker \pi_P$. Conversely, given $P \in I$ let $P \uparrow = \{M \triangleleft_f G : M \cap A_x = P_x \text{ for all } x \in VX\}$. Now by a theorem of Dyer [5], G_P is conjugacy separable, which implies that

$$(g \pi_P)^{G_P} = \bigcap_{M \in P \uparrow} (g \pi_P)^{G_P} M \pi_P = \bigcap_{M \in P \uparrow} (g^G M) \pi_P,$$

whence $g^G \ker \pi_P = \bigcap_{M \in P \uparrow} g^G M$. The result follows. (For the final part take $g = 1$.) \square

3. PROFINITE CLOSURES

It is evident that if $P, Q \in I$, then $P \cap Q = (P_x \cap Q_x)_{x \in VX}$ also belongs to I . We refer to the topology on A_x with $\{P_x : P \in I\}$ as a fundamental system of open neighbourhoods of the identity as the I -topology. For each x let $\hat{A}_x = \varprojlim_{P \in I} (A_x/P_x)$, the inverse limit being formed relative to the partial order \leq of I introduced above. Let \overline{H}_e denote the topological closure of H_e in $\hat{A}_{i(e)}$. We have

Lemma 3.1. *For every $e \in EX$, the isomorphism $\theta_e : H_e \rightarrow H_{\bar{e}}$ extends to an isomorphism $\overline{\theta}_e : \overline{H}_e \rightarrow \overline{H}_{\bar{e}}$ such that $(\overline{H}_e \cap \overline{P}_{i(e)}) \overline{\theta}_e = \overline{H}_{\bar{e}} \cap \overline{P}_{i(e)}$.*

Proof. For each $P \in I$ there are canonical isomorphisms

$$\begin{aligned} \overline{H}_e / (\overline{H}_e \cap \overline{P}_{i(e)}) &\cong \overline{H}_e \overline{P}_{i(e)} / \overline{P}_{i(e)} = H_e \overline{P}_{i(e)} / \overline{P}_{i(e)} \\ &\cong H_e / (H_e \cap \overline{P}_{i(e)}) = H_e / H_e \cap P_{i(e)}, \end{aligned}$$

and \overline{H}_e is naturally isomorphic to the inverse limit $\varprojlim_{P \in I} (H_e / H_e \cap P_{i(e)})$ [12]. The isomorphisms induced by θ_e on the quotients $H_e / H_e \cap P_{i(e)}$ are compatible

with the inverse limit structure, so the existence of $\bar{\theta}_e$ follows. (If one thinks of the inverse limit \hat{A}_x as a subgroup of the cartesian product $\prod_{x \in VX} (A_x/P_x)$, then the image of the element $h = (h_P \pi_P) \in \bar{H}_e$ is $h\bar{\theta}_e = (h_P \theta_e \pi_P) \in \bar{H}_{\bar{e}}$.) The intersection property of $\bar{\theta}_e$ is also trivial from the construction. \square

Thus we have a graph of groups $\mathcal{G}^+ = \{\hat{A}_x, \bar{H}_e, \bar{\theta}_e\}$ over X . Let $G^+ = \pi_1(\mathcal{G}^+, X)$. The natural embeddings $A_x \rightarrow \hat{A}_x$ extend to a homomorphism $\mu: G \rightarrow G^+$. The next few results exploit the relationship between G^+ and the residual properties of G .

Lemma 3.2. *Let G , G^+ , and μ be as above. Then G^+ is residually finite and $\ker \mu = \bigcap_{M \triangleleft_f G} M$.*

Proof. The edge subgroups of G^+ are closed in their vertex groups by definition, and 3.1 implies that if $P \in I(\mathcal{G})$ then $\bar{P} = (\bar{P}_x) \in I(\mathcal{G}^+)$. The residual finiteness of G^+ is therefore a consequence of the theorem of [16]. For the second part let $P \in I$, and consider the following diagram with exact top row:

$$\begin{array}{ccc} \ker \mu \hookrightarrow G & \xrightarrow{\mu} & G^+ \\ \pi_P \downarrow & & \downarrow \pi_P^+ \\ G_P & \xrightarrow{\mu_P} & G_P^+ \end{array}$$

The diagram is easily seen to commute, and $A_x/P_x \cong \hat{A}_x/\bar{P}_x$ implies that μ_P is an isomorphism. Then $1 = (\ker \mu)\mu\pi_P^+ = (\ker \mu)\pi_P\mu_P$ implies that $\ker \mu \subseteq \ker \pi_P$ for all P , so $\ker \mu \subseteq \bigcap_P \ker \pi_P = \bigcap_{M \triangleleft_f G} M$ by 2.1. On the other hand, $G/\ker \mu$ is a subgroup of G^+ ; being residually finite, the reverse inclusion follows. \square

Before proceeding further we mention that, in general, profinite closedness of the edge subgroups in their vertex groups is not necessary for the residual finiteness of G (cf. [15, 17]). In the case of profinite vertex groups, however, we have the following result.

Corollary 3.3. *Let $G = \pi_1(\mathcal{G}, X)$, and assume that $A_x = \varprojlim_{P \in I} (A_x/P_x)$ for all $x \in VX$. Then G is residually finite if and only if every edge subgroup is profinitely closed in its vertex group.*

Proof. Suppose there exists an element $a \in \bar{H}_e \setminus H_e$ for some $e \in EX$. If e is an edge in the maximal subtree Y let $g = a^{-1}(a\bar{\theta}_e)$; otherwise let $g = a^{-1}t_e(a\bar{\theta}_e)t_e$. Then $g \neq 1$ in G , while $g\mu = 1$, so G is not residually finite by 3.2. The converse follows from 3.2 and the fact that when the edge subgroups are closed we have $G = G^+$. \square

If S is a subset of a group H , write $\text{cl}_H(S)$ for $\bigcap_{M \triangleleft_f H} SM$.

Theorem 3.4. *Let G , G^+ , and μ be as above. If $g \in G$ then $\text{cl}_G(g^G)\mu = \text{cl}_{G^+}((g\mu)^{G^+}) \cap G\mu$.*

We are not assuming that μ is injective.

Proof. Let $S = \text{cl}_G(g^G) = \bigcap_{P \in I} g^G \ker \pi_P$ (by 2.1). Then using $\ker \mu \subseteq \ker \pi_P$ (cf. the proof of 3.2) we have

$$S\mu = \bigcap_{P \in I} (g^G \ker \pi_P)\mu = \bigcap_{P \in I} (g\mu)^{G\mu}(\ker \pi_P)\mu.$$

Now it is easy to see from the commutative diagram in the proof of 3.2 and the fact that μ_P is an isomorphism, that $(\ker \pi_P)\mu = \ker \pi_P^+ \cap G\mu$. The proof of 2.1 applies without change to show that $\text{cl}_{G^+}(y^{G^+}) = \bigcap_{P \in I} y^{G^+} \ker \pi_P^+$ for all $y \in G^+$. Thus

$$S\mu = \bigcap_{P \in I} (g\mu)^{G\mu}(\ker \pi_P^+ \cap G\mu) = \bigcap_{P \in I} (g\mu)^{G\mu} \ker \pi_P^+ \cap G\mu,$$

which is certainly contained in $\bigcap_{P \in I} (g\mu)^{G^+} \ker \pi_P^+ \cap G\mu$. Conversely, suppose we have an element $z\mu \in \bigcap_{P \in I} (g\mu)^{G^+} \ker \pi_P^+$, where $z \in G$. Then for each P there exists $w_P \in G^+$ such that $z\mu\pi_P^+ = (w_P^{-1}(g\mu)w_P)\pi_P^+$. Since μ_P is an isomorphism, there exists $u_P \in G$ such that $u_P\pi_P\mu_P = w_P\pi_P^+$. The above equation now becomes $z\pi_P\mu_P = (u_P^{-1}gu_P)\pi_P\mu_P$, and since μ_P is injective we obtain $z \in \bigcap_{P \in I} g^G \ker \pi_P = S$. We have now shown that

$$S\mu = \bigcap_{P \in I} (g\mu)^{G\mu} \ker \pi_P^+ \cap G\mu \subseteq \bigcap_{P \in I} (g\mu)^{G^+} \ker \pi_P^+ \cap G\mu \subseteq S\mu,$$

as required. \square

Since conjugacy separable groups are residually finite, we henceforth assume that $\ker \mu = \langle 1 \rangle$, and identify G with $G\mu$. The statement of 3.4 can then be written more simply as $\text{cl}_G(g^G) = \text{cl}_{G^+}(g^{G^+}) \cap G$. What is the point of this result? Let g and g' be elements of G which are conjugate in every finite image of G . By 2.1 this means that for every $P \in I$ there exists $w_P \in G$ such that $g'\pi_P = (w_P^{-1}gw_P)\pi_P$, and at first glance it is conceivable that the “length” of the elements w_P (in whatever sense) might be unbounded. Now if we have $\text{cl}_{G^+}(g^{G^+}) = g^{G^+}$, then 3.4 implies that $g' \in \text{cl}_G(g^G) = g^{G^+} \cap G$, so $g' = w^{-1}gw$, where $w \in G^+$. This, of course, means that the above w_P can be chosen to have bounded length. It turns out that conjugacy separability is commonplace for elements of G^+ . Before we proceed with this, we state the following, which does not require $\ker \mu = \langle 1 \rangle$.

Corollary 3.5. *If g is conjugacy distinguished in G , then*

$$(g\mu)^{G^+} \cap G\mu = g^G\mu.$$

Proof. For if $(g\mu)^{G^+} \cap G\mu = S\mu$, where $S \neq g^G$, then

$$g^G\mu = \text{cl}_G(g^G)\mu = \text{cl}_{G^+}((g\mu)^{G^+}) \cap G\mu \supseteq (g\mu)^{G^+} \cap G\mu = S\mu,$$

and so $\text{cl}_G(g^G) \supseteq S \supsetneq g^G$. \square

It is necessary to study conjugacy separability of the base group of G^+ first. The next section is devoted to the study of $A^+ = \pi_1(\mathcal{G}|_Y, Y)$, where Y is a maximal subtree of X .

4. CONJUGACY SEPARABILITY OF A^+

We begin with the following general fact.

Proposition 4.1. *Let \mathcal{G} be a graph of groups over X , and assume that*

$$(1) \quad \bigcap_{P \in I} H_e P_{i(e)} = H_e \quad \text{for all } e \in EX,$$

with $G = \pi_1(\mathcal{G}, X)$. Let Y be a connected subgraph of X , and put $G(Y) = \langle A_x : x \in VY \rangle \leq G$. Then $G(Y) = \bigcap_{P \in I} G(Y) \ker \pi_P$. If Y is finite then $G(Y) = \bigcap_{M \triangleleft_f G} G(Y)M$.

Proof. Fix a vertex $x_0 \in VY$, and for each edge $e \in EX$ let $T_e \ni 1$ be a left transversal of H_e in $A_{i(e)}$. Then every element of G is uniquely represented by a normal word $t_1 \cdots t_n a$, where $a \in A_{x_0}$, $t_i \in T_{e_i}$ for $1 \leq i \leq n$, (e_1, \dots, e_n) is a closed path at x_0 , and if $e_i = \bar{e}_{i-1}$ then $t_i \neq 1$ [8, Corollary 1]. Moreover, the proof shows that the elements of $G(Y)$ are represented by paths that entirely belong to Y . Now condition (1) means that, given any element $g = t_1 \cdots t_n a$ in normal form in G , we can find $P \in I$ such that $(t_1 \pi_P) \cdots (t_n \pi_P)(a \pi_P)$ is the normal form of $g \pi_P$ in G_P (relative to suitable transversals of the $H_e \pi_P$ in the $A_{i(e)} \pi_P$). In particular, if $g \in G \setminus G(Y)$ then g is represented by a path that goes outside Y , and since $g \pi_P$ is represented by the same path, we have $g \pi_P \notin G_P(Y) = \langle A_x \pi_P : x \in VY \rangle = G(Y) \pi_P$. The first part follows.

Now suppose that Y is finite. Then for any $P \in I$ we have the finitely generated subgroup $G(Y) \pi_P$ of the free-by-finite group G_P . Such subgroups are profinitely closed (e.g., [2, p. 229]). This evidently means that $G(Y) \ker \pi_P = \bigcap_{M \in P \uparrow} G(Y)M$, and the second part follows (cf. the proof of 2.1). \square

To simplify the notation let Y be a tree, \mathcal{G} a graph of groups over Y , and $A = \pi_1(\mathcal{G}, Y)$. Also write $C = \bigcup_{x \in VY} A_x$ and $D = \bigcup_{e \in EY} H_e$, the subsets of vertex elements and edge elements respectively in A .

Proposition 4.2. *Let g be an element of A . Then the following assertions are true:*

(i) *If $g \in D$, then there exists a subtree Y_g of Y such that $g \in H_e$ if and only if e is an edge of Y_g .*

(ii) *If $g \notin D$, then there exists a FINITE subtree Y_g of Y such that $g \in A(Y_g)$, and if Z is any subtree of Y with $g \in A(Z)$, then $Y_g \subseteq Z$.*

Proof. (i) Let Y_g consist of all vertices x of Y such that $g \in A_x$, and all geodesics between them. If $g \in A_x \cap A_y$, and p is the Y -geodesic from x to y , then g belongs to every vertex group of p (since adjacent vertex groups in A generate their amalgamated free product). In other words, g belongs to every vertex group of Y_g . Moreover, $H_e = A_{i(e)} \cap A_{t(e)}$ for any edge e , so $g \in H_e$ if and only if $i(e)$ and $t(e)$ belong to Y_g .

(ii) Among all subtrees Z with $g \in A(Z)$ pick one with the fewest number of vertices, and call it Y_g . Fix a vertex x_0 in Y_g . Then the normal form $t_1 \cdots t_n a$ of g can be represented by a closed path p at x_0 , with p entirely contained in Y_g (since $g \in A(Y_g)$). Now let Z be a subtree such that $g \in A(Z)$. Suppose first that $Y_g = \{x_0\}$. If $x_0 \notin VZ$ then $g \in A_{x_0} \cap A(Z)$ must be an edge element (consider the geodesic from x_0 to $A(Z)$), a contradiction. So in this case $Y_g \subseteq Z$.

There remains the case where Y_g has more than one vertex. If Z contains every end vertex of Y_g then $Y_g \subseteq Z$, so suppose Z does not contain some end vertex x of Y_g , and let e be the edge of Y_g that ends at x . Now the path p must visit x at least once (since otherwise $g \in A(Y_g \setminus \{e, x\})$), and if $t_i \in A_x$ then $t_i \neq 1$ (since p looks like $\dots, e, x, \bar{e}, \dots$ at x , and we have a normal form). If Z and Y_g were disjoint then $g \in A(Z) \cap A(Y_g)$ would be an edge element, which is not the case. Thus $Z' = Z \cup Y_g \setminus \{e, x\}$ is a subtree of Y such that $g \in A(Z')$. If x_0 is on Z' then we have the immediate contradiction that g can be represented in normal form by a path q in Z' , and $q \neq p$ since x is on p but not on q . So we must have $x_0 = x$. But since Z and Y_g are not disjoint we can always choose x_0 to be a common vertex. This final contradiction shows that $Z \supseteq Y_g$, as required. \square

To define the notion of a reduced form for elements of A we need a definition. Let Δ' be a nonempty subtree of a finite tree Δ . By a *reduction process* R from Δ to Δ' we mean a sequence $\Delta_1, \dots, \Delta_m$ of subtrees of Δ such that $\Delta_1 = \Delta$, $\Delta_m = \Delta'$, and Δ_{i+1} is obtained from Δ_i by deleting an end vertex x_i (for $i = 1, \dots, m-1$). We let y_i be the vertex of Δ_i adjacent to x_i , so when $\Delta \subseteq Y$ we have

$$A(\Delta_i) = \langle A(\Delta_{i+1}) * A_{x_i} : H_{y_i x_i} = H_{x_i y_i} \text{ via } \theta_{y_i x_i} \rangle, \quad i = 1, \dots, m-1.$$

Let $\Delta \supseteq \Delta'$ be finite subtrees of Y , and $R = \{\Delta_1, \dots, \Delta_m\}$ a reduction process from Δ to Δ' . Say g is reduced (resp. cyclically reduced) relative to R if either $g \in A(\Delta')$ or $g \in A(\Delta_i) \setminus A(\Delta_{i+1})$ for some $i \leq m-1$, and then g is reduced (resp. cyclically reduced) in the amalgamated free product $A(\Delta_i)$ of $A(\Delta_{i+1})$ and A_{x_i} . Every element of $A(\Delta)$ can be written in reduced form, and is conjugate, in $A(\Delta)$, to a cyclically reduced element of $A(\Delta)$. Note that an element may be cyclically reduced relative to one reduction process, but not another.

Lemma 4.3. *Assume that (1) holds. Let g be a nonvertex element of A such that g is cyclically reduced relative to some reduction process from Y_g to a single vertex. Then $\text{cl}_A(g^A) = \bigcup_{\alpha} g_{\alpha}^A$, where each $g_{\alpha} \in A(Y_g)$.*

Proof. There exists $P_0 \in I$ such that $Y_{g\pi_P} = Y_g$ for all $P \subseteq P_0$. To begin with, we can choose P_1 such that $g\pi_P$ is cyclically reduced in $A(Y_g)\pi_P$ relative to R , for all $P \subseteq P_1$. This is because (1) can be used to ensure that the finitely many vertex elements in the cyclically reduced form of g are excluded from the finitely many edge subgroups encountered by R . In particular, if $P \subseteq P_1$ then $g\pi_P$ is not an edge element of $A\pi_P$. For such P we have $Y_{g\pi_P} \subseteq Y_g$ (since $g\pi_P \in A(Y_g)\pi_P$), and if $Q \subseteq P$ then $Y_{g\pi_Q} \supseteq Y_{g\pi_P}$. Since Y_g is finite, there exists a subtree $\Delta \subseteq Y_g$ and $P_0 \subseteq P_1$ such that for all $P \subseteq P_0$ we have $Y_{g\pi_P} = \Delta$. Then $g \in \bigcap_{P \subseteq P_0} A(\Delta) \ker \pi_P = A(\Delta)$ by 4.1, and then $\Delta = Y_g$ by definition of Y_g .

Now suppose $P \subseteq P_0$. Then $g\pi_P$ is a cyclically reduced nonvertex element of $A(Y_g)\pi_P$, so the conjugacy theorem for amalgamated free products [9] implies that the conjugacy class of $g\pi_P$ in $A(Y_g)\pi_P$ contains no vertex elements. If $\Delta \supseteq Y_g$ is a finite subtree then the conjugacy class of $g\pi_P$ in $A(\Delta)\pi_P$ contains no vertex elements (consider a reduction process from Δ to Y_g). Now

let $g' \in \text{cl}_A(g^A)$, and let $\Delta \supseteq Y_g$ be a finite subtree with $g' \in A(\Delta)$. Let $R = \{\Delta_1, \dots, \Delta_m\}$ be a reduction process from Δ to Y_g . Replacing g' by a conjugate in $A(\Delta)$ we may assume that g' is cyclically reduced relative to R , and we may choose P_0 such that $g'\pi_P$ is cyclically reduced for all $P \subseteq P_0$. If $P \subseteq P_0$ then $g\pi_P$ has minimal length in its $A(\Delta)\pi_P$ -conjugacy class, and belongs to $A(\Delta_2)\pi_P \setminus H_{y_1x_1}\pi_P$. The conjugacy theorem for amalgamated free products [9] implies that $g'\pi_P \in A(\Delta_2)\pi_P$. Continuing in this way we finally obtain $g'\pi_P \in A(Y_g)\pi_P$, whence $g' \in \bigcap_{P \subseteq P_0} A(Y_g) \ker \pi_P = A(Y_g)$, as required. \square

We can now prove

Theorem 4.4. *Let g be a nonvertex element of A such that g is cyclically reduced relative to some reduction process R from Y_g to a vertex. Assume that (1) holds and that $H_{x_1y_1}$ is compact in the I -topology (where x_1y_1 is the first edge deleted in the reduction process). Then g is conjugacy distinguished in A .*

Proof. By 4.3 we have $\text{cl}_A(g^A) = \bigcup g_\alpha^A$, where each $g_\alpha \in A(Y_g)$, and we may assume that each g_α is cyclically reduced relative to R . Fix α , and choose P_0 such that for all $P \subseteq P_0$, $g\pi_P$, and $g_\alpha\pi_P$ are cyclically reduced in $A(Y_g)\pi_P$ relative to R . The conjugacy theorem for amalgamated free products can now be applied to deduce that $g_\alpha\pi_P$ is conjugate to some cyclic permutation of $g\pi_P$, via an element of $H_{x_1y_1}\pi_P$. Replacing g by a cyclic permutation if necessary we have $g_\alpha\pi_P = (h_P^{-1}gh_P)\pi_P$ for all $P \subseteq P_0$, where $h_P \in H_{x_1y_1}$. Consider the function $f: H_{x_1y_1} \rightarrow A(Y_g)$ given by $f(h) = g_\alpha^{-1}h^{-1}gh$. With the appropriate profinitely topologies it is easy to see that f is continuous and $\ker \pi_P \cap A(Y_g)$ is closed in $A(Y_g)$. Thus for each $P \subseteq P_0$, $f^{-1}(\ker \pi_P)$ is a nonempty closed subset of $H_{x_1y_1}$, and $P \subseteq Q$ implies that $f^{-1}(\ker \pi_P) \subseteq f^{-1}(\ker \pi_Q)$, so we have the finite intersection property. If

$$h \in \bigcap_{P \subseteq P_0} f^{-1}(\ker \pi_P) = f^{-1} \left(\bigcap_{P \subseteq P_0} \ker \pi_P \right) = f^{-1}(\langle 1 \rangle),$$

then $g_\alpha^{-1}h^{-1}gh = 1$, so $g_\alpha \in g^A$. The result follows. \square

The next lemma collects the information we need on conjugacy of edge and vertex elements.

Lemma 4.5. *Let $C = \bigcup_{x \in VY} A_x$ and $D = \bigcup_{e \in EY} H_e$.*

(i) *Let $h \in D$, and let $a \in C$ be such that $h^a \in C$. Then $a \in A_x$ for some $x \in VY_h$.*

(ii) *Let $a \in A_x$, $b \in A_y$, and $a^b \in C$. Then there exists a vertex z such that $a, b \in A_z$.*

(iii) *Let $h \in D$, and w be a nonvertex element of A such that $h^w \in C$. Let $w = a_1 \cdots a_r$ be the reduced form of w (relative to some reduction process R of Y_w), where the $a_i \in C$. Then $h^{a_1 \cdots a_i} \in D$ for $1 \leq i \leq r-1$.*

(iv) *Let $a \in A_x$ be such that $a^{A_x} \cap D = \emptyset$. Then $a^A \cap D = \emptyset$, and $a^A \cap C = a^{A_x}$.*

Proof. (i) Choose x as close to Y_h as possible subject to $a \in A_x$, and get a contradiction to $h^a \in C$ by assuming that $x \notin VY_h$ and considering the geodesic from x to Y_h .

(ii) Similar to (i): choose x_1 and y_1 as close to each other as possible (on the geodesic joining x and y) subject to $a \in A_{x_1}$ and $b \in A_{y_1}$.

(iii) First note that $Y_h \cap Y_w \neq \emptyset$, for otherwise a consideration of the geodesic from Y_h to Y_w gives the contradiction $h^w \notin C$. Thus $a = h^w \in A(Y_w)$. The proof is by induction on r , where $w = a_1 \cdots a_r$. Consider the first step in the reduction process: if h^{a_1} is not an edge element then h^w is reduced as written (in the amalgamated free product of $A(\Delta_2)$ and A_{x_1}) and so cannot be a vertex element. Therefore $h_1 = h^{a_1} \in D$, and $a = h_1^{a_2 \cdots a_r}$. The result follows by induction on r (using part (i)).

(iv) Suppose $a^w = h \in D$ for some nonvertex element $w = a_1 \cdots a_r$, in reduced form. Then $h^{a_r^{-1} \cdots a_1^{-1}} = a \in C$, and by part (iii) we get $a^{a_1} = h^{a_r^{-1} \cdots a_2^{-1}} \in D$. So $a^A \cap D \neq \emptyset$ implies that $a^b \in D$ for some vertex element b . By part (ii), a and b must belong to the same vertex group, which has to be A_x since a belongs to no other vertex group. But now we have $a^{A_x} \cap D \neq \emptyset$, a contradiction. It follows that $a^A \cap D = \emptyset$.

For the second part let $b = a^w \in A_y$, and choose y as close to x as possible subject to $b \in A_y$. Claim that $y = x$. For if not, then $Y_w \neq \{x\}$, and we have two cases to consider:

Case 1. $x \notin VY_w$. Then there exists a finite subtree $\Delta \supseteq Y_w$ and an edge yx with $y \in V\Delta$. Now $a \in A_x \setminus H_{xy}$, $w \in A(\Delta)$, and $b = w^{-1}aw \in A(\Delta)$ (as Δ has to contain y because $b = a^w \in A(\Delta \cup \{x\})$ and $y \neq x$). This forces $w \in H_{yx} = H_{xy}$, and $a^w \in A(\Delta) \cap A_x = H_{xy}$, contradicting $a^A \cap D = \emptyset$.

Case 2. $x \in VY_w$. Let $R = \{\Delta_1, \dots, \Delta_m\}$ be a reduction process from Y_w to a point, such that the first vertex deleted is $x_1 \neq x$ (this can be done since $Yw \neq \{x\}$). Write $w = a_1 \cdots a_r$ in reduced form relative to R . Then $b = a_r^{-1} \cdots a_1^{-1} a a_1 \cdots a_r \in A_y \subseteq A(\Delta_2) \cup A(x_1)$, so we must have $a_1 \in A_x$ and $a^{a_1} \in H_{y_1 x_1}$ (since $r \geq 2$). This is a contradiction.

We have now shown that $b = a^w \in A_x$. An argument similar to that in Case 2 now shows that $w \in A_x$, so $a^A \cap C \subseteq a^{A_x}$, are required. \square

We can now prove

Theorem 4.6. *Let $g \in A_x$ be such that $g^{A_x} \cap D = \emptyset$. Assume that X is locally finite, that every edge subgroup of A is compact in the I -topology, and that $\bigcap_{P \in I} g^{A_x} P_x = g^{A_x}$. Then g is conjugacy distinguished in A .*

Proof. If $\text{cl}_A(g^A)$ contains a nonvertex cyclically reduced element g_0 , then by 4.4 we have $\text{cl}_A(g^A) = \text{cl}_A(g_0^A) = g_0^A$, so $g^A = g_0^A = \text{cl}_A(g^A)$ and we are done. (Note that condition (1) is a consequence of the compactness assumption, so 4.4 is applicable.) We may therefore suppose that $\text{cl}_A(g^A) = \bigcup a_i^A$, where each a_i is a vertex element of A . Let $S = \bigcup_{i(e)=x} H_e$, and note that S is compact since X is locally finite. Put $T_P = g^{A_x} \pi_P \cap S \pi_P$. We claim that there exists P_0 such that $T_P = \emptyset$ for all $P \subseteq P_0$. Suppose not. Note that $Q \subseteq P$ implies that T_Q maps into T_P under the obvious map (from A_Q to A_P), so if $T_P \neq \emptyset$ then we have an inverse system of nonempty finite sets. Let $(h_P \pi_P) \in \varprojlim T_P$, where each $h_P \in S$. Since S is compact there exists $h \in S$ such that $h \pi_P = h_P \pi_P$ for all P . This clearly means that $h \in \bigcap_{P \in I} g^{A_x} P_x = g^{A_x}$, contradicting $g^{A_x} \cap D = \emptyset$. Hence there exists P_0 such that $g^{A_x} \pi_P \cap D \pi_P = \emptyset$ for all $P \subset P_0$. By 4.5(iv)

we have $g^A \pi_P \cap D \pi_P = \emptyset$, and $a_i \pi_P \in g^A \pi_P \cap C \pi_P = g^{A_x} \pi_P$. Thus first of all $a_i \in \bigcap_{P \subseteq P_0} A_x \ker \pi_P = A_x$, and then $a_i \in \bigcap_{P \subseteq P_0} g^{A_x} P_x = g^{A_x}$. Thus $\text{cl}_A(g^A) = g^A$, as required. \square

Remark 1. For g as in 4.6, the condition $\bigcap_P g^{A_x} P_x = g^{A_x}$ is implied by $\text{cl}_A(g^A) = g^A$. For if $b \in \bigcap g^{A_x} P_x$, then $b \in \text{cl}_A(g^A) = g^A$, and so $b \in g^A \cap C = g^{A_x}$ by 4.5(iv).

Remark 2. The usual proof of the conjugacy separability of the profinite group \widehat{A}_x in fact shows that for $g \in \widehat{A}_x$ we have $\bigcap_{P \in I} g^{\widehat{A}_x} \overline{P}_x = g^{\widehat{A}_x}$.

The next two results give partial information about conjugacy separability of edge elements.

Lemma 4.7. *Assume that (1) holds. Let h be an edge element of A , and assume that there exists $P_0 \in I$ such that $Y_{h\pi_{P_0}} = Y_h$. Then $\text{cl}_A(h^A) \cap D = \text{cl}_{A(Y_h)}(h^{A(Y_h)}) \cap D$.*

Proof. The condition implies that $h\pi_P \notin H_e \pi_P$ for all $P \subseteq P_0$, where e is an edge leading out of Y_h . Let $z \in \text{cl}_A(h^A) \cap D$, so $z\pi_P = h^w \pi_P$ for some $w \in A$. We claim that w can be chosen in $A(Y_h)$. For if not, let Δ be the smallest tree containing $h\pi_P$ and $w\pi_P$, and consider a reduction process $\{\Delta_1, \dots, \Delta_m\}$ from Δ to $\Delta \cap Y_h$. Now the first vertex $x_1 \notin V Y_h$, and $w\pi_P \notin A(\Delta_2)\pi_P$ by the minimal choice of Δ , and yet $(h\pi_P)^{w\pi_P}$ belongs to a vertex group. By the conjugacy theorem for amalgamated free product, we must have $h\pi_P \in H_{y_1 x_1} \pi_P$, contradicting the choice of P . We have therefore shown that

$$h_0 \in \bigcap_{P \subseteq P_0} h^{A(Y_h)} \ker \pi_P \subseteq \bigcap_{P \subseteq P_0} A(Y_h) \ker \pi_P = A(Y_h),$$

and so

$$\begin{aligned} h_0 &\in \bigcap_{P \subseteq P_0} h^{A(Y_h)} \ker \pi_P \cap A(Y_h) \\ &= \bigcap_{P \subseteq P_0} h^{A(Y_h)} (\ker \pi_P \cap A(Y_h)) = \text{cl}_{A(Y_h)}(h^{A(Y_h)}). \quad \square \end{aligned}$$

Let h be an edge element of A . For any sequence $\sigma = (x_1, \dots, x_r)$ of vertices of A put $D_\sigma(h) = D \cap \{h^{a_1 \dots a_r} : a_i \in A_{x_i}\}$. We have

Theorem 4.8. *Assume that X is locally finite and D is compact. Let h be an edge element of A^+ . Then h is conjugacy distinguished in A^+ if and only if $h^{A^+} \cap D = D_\sigma(h)$ for some finite sequence σ of vertices.*

Proof. We know that $\text{cl}_{A^+}(h^{A^+}) = \bigcup h_\alpha^{A^+}$ with each $h_\alpha \in D$. Let $K = h^{A^+} \cap D = \bigcup_\sigma D_\sigma(h)$, the union being over all finite sequences σ . If $z \in \text{cl}_{A^+}(h^{A^+}) \cap D$ and $P \in I$ then there exists a sequence $\sigma(P)$ such that $z\pi_P \in D_{\sigma(P)}(h)\pi_P \subseteq K\pi_P$, so $z \in K \ker \pi_P$. Thus

$$\text{cl}_{A^+}(h^{A^+}) \cap D \subseteq \bigcap_P K(\ker \pi_P) = \text{cl}_{A^+}(K).$$

Now D is a compact subset of A^+ , and so is closed. Therefore $\text{cl}_{A^+}(K) = \text{cl}_D(K)$. Conversely if $z \in \text{cl}_D(K)$ and $P \in I$ then $z\pi_P \in K\pi_P$, so $z\pi_P \in$

$D_\sigma(h)\pi_P$ for some sequence σ (depending on P). In other words, $z \in \text{cl}_{A^+}(h^{A^+})$, and we have shown that $\text{cl}_{A^+}(h^{A^+}) \cap D = \text{cl}_D(h^{A^+} \cap D)$. We also note that for a fixed sequence σ , the set $D_\sigma(h)$ is a complete subspace of the compact set D (since each vertex group \widehat{A}_x is compact), and is therefore compact. So if $h^{A^+} \cap D = D_\sigma(h)$ then $\text{cl}_{A^+}(h^{A^+}) \cap D = \text{cl}_D(K) = K = h^{A^+} \cap D$, which implies that every $h_\alpha \in h^{A^+}$, and so h is conjugacy distinguished in A^+ . Conversely, suppose $\text{cl}_{A^+}(h^{A^+}) = h^{A^+}$. Then $K = h^{A^+} \cap D = \text{cl}_D(K)$, so the compact (since closed) set K is the (countable) union of the closed sets $D_\sigma(h)$. Enumerate the finite sequences $\{\sigma_1, \sigma_2, \dots\}$, and put $T_i = \bigcup_{1 \leq j \leq i} D_{\sigma_j}(h)$. Then $K = \bigcup_{i=1}^\infty T_i$, and each T_i is closed. By the Baire Category Theorem (or rather its proof, using compactness), some T_{i_0} has nonempty interior, and so contains a set of the form $hP_x \cap D$, where $h \in D$ and $P \in I$. Since $P_x \triangleleft_f A_x$ and K is compact, it is easy to see that K is the union of finitely many T_i , whence $K = T_j$ for some j . Let σ be the concatenation of the sequences $\sigma_1, \dots, \sigma_j$. Then it is clear that $D_{\sigma_i}(h) \subseteq D_\sigma(h)$ for $1 \leq i \leq j$, and so $h^{A^+} \cap D = K = T_j \subseteq D_\sigma(h)$, as required. \square

The following are immediate consequences of 4.7 and 4.8.

Corollary 4.9. *Let X be locally finite, and let h be an edge element of A^+ such that Y_h is finite. Then h is conjugacy distinguished in A^+ if and only if for some finite sequence σ of vertices of Y_h we have $h^{A^+} \cap D = D_\sigma(h)$. \square*

Corollary 4.10. *Let A^+ be the amalgamated free product of countably many groups. Then an element h of the associated subgroup H is conjugacy distinguished in A^+ if and only if $h^{A^+} \cap H = D_\sigma(h)$ for some finite sequence σ of vertices. \square*

5. CONJUGACY SEPARABILITY OF G^+

If X is not a tree then G^+ is a nontrivial HNN-extension over the base group $A^+ = \langle \widehat{A}_x : x \in V X \rangle$. By the length $|g|$ of an element $g \in G^+$ we mean the number of t -symbols appearing in an HNN-reduced form for g . Thus $g = a_1 t_{e_1}^{\varepsilon_1} \cdots t_{e_n}^{\varepsilon_n} a_{n+1}$, where all $a_i \in A^+$, a_{n+1} is arbitrary, each $\varepsilon_i = \pm 1$, each $e_i \in EX \setminus EY$ (where Y is a fixed maximal subtree of X), and if $e_{i-1} = e_i$ and $\varepsilon_{i-1} + \varepsilon_i = 0$ then $a_i \notin H_{e_i, e_i}$ (we write H_{-e} instead of $H_{\bar{e}}$ for ease of notation). The reduced element g is cyclically reduced if $a_{n+1} = 1$ and $t_{e_n}^{\varepsilon_n} a_1 t_{e_1}^{\varepsilon_1} \cdots a_n$ is also reduced.

Theorem 5.1. *Let $g \in G^+$ be cyclically reduced and of length at least one. Then g is conjugacy distinguished in G^+ .*

Proof. Let z be a cyclically reduced element in $\text{cl}_{G^+}(g^{G^+})$. It is easy to see that the edge subgroups \overline{H}_{ee} are closed in A^+ , so we can find $P_0 \in I$ such that $|g| = |g\pi_P|$, $|z| = |z\pi_P|$, and both $g\pi_P$ and $z\pi_P$ are cyclically reduced in G_P for all $P \subseteq P_0$. The conjugacy theorem for HNN-extensions (e.g. [3]) now implies that $|g\pi_P| = |z\pi_P|$, so $|g| = |z|$. Modulo the same P we know that $g\pi_P$ and some cyclic permutation of $z\pi_P$ have the same sequence of t -symbols with the same exponents. Replacing z by a suitable cyclic permutation we may assume that $g = a_1 t_{e_1}^{\varepsilon_1} \cdots a_n t_{e_n}^{\varepsilon_n}$ and $z = b_1 t_{e_1}^{\varepsilon_1} \cdots b_n t_{e_n}^{\varepsilon_n}$. For $P \subseteq P_0$ the conjugacy theorem in G_P implies the existence of elements $h_{0,P}, \dots, h_{n,P}$ such that, modulo P

$$\begin{aligned}
h_{0,P}b_1 &\equiv a_1h_{1,P}, & h_{1,P} &\in \overline{H}_{e_1e_1}, \\
(h_{1,P}\overline{\theta}_{e_1}^{e_1})b_2 &\equiv a_2h_{2,P}, & h_{2,P} &\in \overline{H}_{e_2e_2}, \\
&\vdots \\
(h_{n-1,P}\overline{\theta}_{e_{n-1}}^{e_{n-1}})b_n &\equiv a_nh_{n,P}, & h_{n,P} &\in \overline{H}_{e_ne_n}, \\
(h_{n,P}\overline{\theta}_{e_n}^{e_n}) &\equiv h_{0,P}, & \text{so } h_{0,P} &\in \overline{H}_{-\varepsilon_ne_n} = \overline{H}_{e_0e_0}.
\end{aligned}$$

Since $\bigcup_{i=0}^n \overline{H}_{e_i e_i}$ is compact, there exist elements $h_i \in \overline{H}_{e_i e_i}$ such that $h_i \pi_P = h_{i,P} \pi_P$ for all P . Then we have the equations $h_0 b_1 = a_1 h_1$, $(h_1 \overline{\theta}_{e_1}^{e_1}) b_2 = a_2 h_2$, \dots , $h_n \overline{\theta}_{e_n}^{e_n} = h_0$, and so $z = h_0^{-1} g h_0$. The result follows. \square

To deal with conjugacy separability of elements of A^+ in G^+ , we begin with the following trivial observation, which can be proved by using a length argument (as in the proof of 5.1) and the conjugacy theorem for HNN-extensions.

Lemma 5.2. *If $a \in A^+$ then $\text{cl}_{G^+}(a^{G^+}) = \bigcup a_\alpha^{G^+}$, where each $a_\alpha \in A^+$. In particular, a is conjugacy distinguished in G^+ if and only if $\text{cl}_{G^+}(a^{G^+}) \cap A^+ = a^{G^+} \cap A^+$. \square*

The conjugacy theorem for elements of the base of an HNN-extension states that if $a, b \in A^+$ are conjugate in G^+ , then there exist elements $h_1, \dots, h_m \in \bigcup_{e \in EX \setminus EY} \overline{H}_e$, such that $a \sim h_1$ in A^+ (i.e., conjugate in A^+), $b \sim h_m$ in A^+ , and for $1 \leq i \leq m-1$, either $h_{i+1} \sim h_i$ in A^+ or $h_{i+1} = h_i \theta_e$ for some $e \in EX \setminus EY$. Now we have

Proposition 5.3. *Let $a \in A^+$ be a nonvertex cyclically reduced element (relative to some reduction process of Y_a). Then $\text{cl}_{G^+}(a^{G^+}) \cap A^+ = a^{A^+}$, and so a is conjugacy distinguished in G^+ .*

Proof. Choose P_0 such that for all $P \subseteq P_0$ we have $Y_{a\pi_P} = Y_a$ and $a\pi_P$ is cyclically reduced (cf. the proof of 4.4). Then $(a\pi_P)^{A^+\pi_P} \cap D\pi_P = \emptyset$ by 4.5(iii) and using the fact that no nonvertex conjugate of an edge element can be cyclically reduced. Now let $b \in \text{cl}_{G^+}(a^{G^+}) \cap A^+$. Then $b\pi_P$ is conjugate to $a\pi_P$ in G_P . Since no $A^+\pi_P$ -conjugate of $a\pi_P$ can belong to an edge subgroup in G_P , we must have $b\pi_P$ conjugate to $a\pi_P$ in $A^+\pi_P$, and so $b \in \text{cl}_{A^+}(a^{A^+}) = a^{A^+}$, the latter by 4.4. The final part follows from 5.2. \square

For vertex elements of A^+ we have the following (recall that $C = \bigcup_{x \in VX} \widehat{A}_x$):

Proposition 5.4. *Assume that X is locally finite. Let a be a vertex element of A^+ , and let $b \in \text{cl}_{G^+}(a^{G^+}) \cap C$. Then there exists a sequence c_0, c_1, \dots of elements of C such that (i) $c_0 = a$; (ii) for each $i \geq 1$, either $c_i \sim c_{i-1}$ in C , or $c_i = c_{i-1} \theta_{e_i}$ for some $e_i \in EX \setminus EY$; and (iii) $c_i \rightarrow b$ as $i \rightarrow \infty$.*

Proof. For $P \in I$ let $S_P(a, b)$ denote the set of all sequences $\sigma = \{a_{i,P}\pi_P : i \geq 0\}$, where each $a_{i,P} \in C$, $a_{0,P}\pi_P = a\pi_P$, $a_{i,P}\pi_P = b\pi_P$ for all sufficiently large i , and each $a_{i,P}\pi_P$ is either $C\pi_P$ -conjugate to, or the $\theta_{e,P}$ -image of, $a_{i-1,P}\pi_P$ (in other words, each $\sigma \in S_P(a, b)$ represents a possible conjugacy $a\pi_P \sim b\pi_P$ in G_P^+). For fixed i and P write $T_i(P)$ for the set of i th components of all $\sigma \in S_P(a, b)$. Suppose $a \in \widehat{A}_x$, and let Δ_i denote the (finite)

subtree of radius i with centre x . A simple inductive argument shows that $T_i(P) \subseteq \bigcup_{y \in V\Delta_i} \hat{A}_y \pi_P$, so each $T_i(P)$ is finite. If $Q \subseteq P$ then every sequence in $S_Q(a, b)$ maps to a sequence in $S_P(a, b)$ under the induced map $G_Q^+ \rightarrow G_P^+$, and so $T_i(Q)$ maps into $T_i(P)$.

We construct the elements c_i inductively, beginning with $c_0 = a$. Assume that c_1, \dots, c_{i-1} have been chosen subject to (ii) and $c_j \pi_P \in T_j(P)$ for all P , and $1 \leq j \leq i-1$. Let $K_P = \{c' \pi_P \in T_i(P) : \text{there exists } \sigma \in S_P(a, b) \text{ such that } \sigma_{i-1} = c_{i-1} \pi_P \text{ and } \sigma_i = c' \pi_P\}$. Then each $K_P \neq \emptyset$, and if $Q \subseteq P$ then K_Q maps into K_P under the induced map $G_Q \rightarrow G_P$. Let $(d_P \pi_P)$ belong to the inverse limit of the sets K_P . Then each $d_P \in \bigcup_{y \in V\Delta_i} \hat{A}_y$, and the latter set is compact, so there exists $c_i \in \bigcup_{y \in V\Delta_i} \hat{A}_y$ such that $c_i \pi_P = d_P \pi_P$ for all P . Therefore, for every P there exists $\sigma \in S_P(a, b)$ such that $\sigma_{i-1} = c_{i-1} \pi_P$ and $\sigma_i = c_i \pi_P$. Also $c_i \pi_P \in T_i(P)$ for all P . To verify (ii), note that for each P the element $c_i \pi_P$ is either conjugate to $c_{i-1} \pi_P$ by some element of $\bigcup_{y \in V\Delta_{i-1}} \hat{A}_y \pi_P$, or is the image of $c_{i-1} \pi_P$ under some $\theta_{e,P}$, where $e \in E\Delta_i$. The number of possibilities is therefore finite and independent of P , and so at least one must hold for all P . If $c_i \pi_P = c_{i-1} \pi_P \theta_{e,P}$ for all P , then $c_{i-1} \in \overline{H}_e$ and we get $c_i \pi_P = (c_{i-1} \theta_e) \pi_P$ for all P , whence $c_i = c_{i-1} \theta_e$. If $c_i \pi_P \sim c_{i-1} \pi_P$ in $\bigcup_{y \in V\Delta_i} \hat{A}_y \pi_P$ for all P , then $c_i \sim c_{i-1}$ in $\bigcup_{y \in V\Delta_i} \hat{A}_y$, using compactness. This proves (ii) for c_i , and so the inductive construction goes through. Finally given P it is clear that $c_i \pi_P = b \pi_P$ for all sufficiently large i , so $c_i \rightarrow b$ as $i \rightarrow \infty$. \square

Corollary 5.5. *Let $a \in A^+$ be a vertex element. Then a is conjugacy distinguished in G^+ if and only if $a^{G^+} \cap C$ is a closed subset of A^+ .*

Proof. By 5.2 and the proof of 5.3 we see that $\text{cl}_{G^+}(a^{G^+}) = \bigcup a_\alpha^{G^+}$, where each $a_\alpha \in C$. By 5.4, for each a_α there exists a sequence $c_i \in C$ such that $c_i \sim a$ in G^+ and $c_i \rightarrow a_\alpha$ as $i \rightarrow \infty$. Thus each $a_\alpha \in \text{cl}_{A^+}(a^{G^+} \cap C)$, and so $\text{cl}_{G^+}(a^{G^+}) \cap C \subseteq \text{cl}_{A^+}(a^{G^+} \cap C)$. The reverse inclusion being trivial, we have $\text{cl}_{G^+}(a^{G^+}) \cap C = \text{cl}_{A^+}(a^{G^+} \cap C)$. The result follows. \square

Corollary 5.6. *Let a be a vertex element of A^+ , and let $D' = \bigcup_{e \in EX \setminus EY} \overline{H}_e$. If $a^{A^+} \cap D' = \emptyset$, then a is conjugacy distinguished in G^+ if and only if a is conjugacy distinguished in A^+ .*

Proof. Since no A^+ -conjugate of a belongs to an associated subgroup, we have $a^{G^+} \cap A^+ = a^{A^+}$ by the conjugacy theorem for HNN-extensions, and so $a^{G^+} \cap C = a^{A^+} \cap C$. The result follows from 5.5 and the fact that $\text{cl}_{A^+}(a^{A^+}) = \bigcup a_\alpha^{A^+}$, with all $a_\alpha \in C$ (this is a consequence of 4.4). \square

To deal with elements of $D' = \bigcup_{e \in EX \setminus EY} \overline{H}_e$ we need the following:

Lemma 5.7. *If D is compact, then so is $C = \bigcup_{x \in VX} \hat{A}_x$.*

Proof. It is sufficient to prove that C is complete, since we can always regard C as a subspace of the profinite completion \hat{A}^+ . So let $\{a_P : P \in I\}$ be a subset of C such that for all $Q \subseteq P$ we have $a_Q \pi_P = a_P \pi_P$. If every $a_P \in D$, then the compactness of D implies the existence of $h \in D$ such that $a_P \pi_P = h \pi_P$ for all P , and we are done. Now suppose that $a_{P_1} \pi_{P_0} \notin D \pi_{P_0}$ for some P_0 . Then $a_P \pi_P \notin D \pi_P$ for all $P \subseteq P_0$, for if $a_P \in \hat{A}_y \neq \hat{A}_x$ for some $P \subseteq P_0$,

then $a_{P_0}\pi_{P_0} = a_P\pi_{P_0} \in \widehat{A}_y\pi_{P_0} \cap \widehat{A}_x\pi_{P_0} \subseteq D\pi_{P_0}$, contrary to assumption. But now the compactness of \widehat{A}_x gives the result. \square

Consider the set of all finite sequences $\sigma = (x_1, e_1^{e_1}, \dots, e_n^{e_n}, x_{n+1})$, where $x_i \in VX$, $e_i \in EX \setminus EY$, and $\varepsilon_i = \pm 1$. For such a sequence let

$$B_\sigma(h) = C \cap \{h^g : g = a_1 t_{e_1}^{e_1} \cdots t_{e_n}^{e_n} a_{n+1}, a_i \in \widehat{A}_{x_i} \text{ for all } i\}.$$

Theorem 5.8. *Assume that X is locally finite and D is compact. Then $h \in D'$ is conjugacy distinguished in G^+ if and only if there exist finite sequences $\sigma_1, \dots, \sigma_m$ such that $h^{G^+} \cap C = \bigcup_{i=1}^m B_{\sigma_i}(h)$.*

Proof. By 5.5 the element h is conjugacy distinguished in G^+ if and only if $K = h^{G^+} \cap C$ is closed in A^+ . Since C is compact (by 5.7) and therefore closed, this is equivalent to K being closed in C , i.e., to K being compact. Now $K = \bigcup_\sigma B_\sigma(h)$, the union being over the countably many sequences σ . An application of Baire's Category Theorem (cf. the proof of 4.8) shows that K is compact if and only if it is the union of finitely many $B_{\sigma_i}(h)$. (One needs to show that each $B_\sigma(h)$ is closed. For this, use the compactness of the \widehat{A}_x to show that $B_\sigma(h)$ is complete.) \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, T6G 2G1
CANADA