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CONJUGACY SEPARABILITY OF CERTAIN FREE PRODUCTS WITH AMALGAMATION

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Abstract. Let G be a group. An element g of G is called conjugacy distinguished or c.d. in G if and only if given any element h of G either h is conjugate to g or there is a homomorphism ξ from G onto a finite group such that $\xi(h)$ and $\xi(g)$ are not conjugate in $\xi(G)$. Following A. Mostowski, a group G is conjugacy separable or c.s. if and only if every element of G is c.d. in G. In this paper we prove that every element conjugate to a cyclically reduced element of length greater than 1 in the free product of two free groups with a cyclic amalgamated subgroup is c.d. We also prove that a group formed by adding a root of an element to a free group is c.s.

In [4], A. Mostowski defined conjugacy separable groups and showed that the conjugacy problem is soluble for conjugacy separable groups. S. Lipschutz [1] has solved the conjugacy problem for the free products of free groups with cyclic amalgamated subgroups.

In this paper the problem of conjugacy separability of free products of free groups with a cyclic amalgamated subgroup is considered. It is shown that every element conjugate to a cyclically reduced element of length greater than 1 in the free product of two free groups with a cyclic amalgamated subgroup is c.d. Also, it is shown that a group formed from a free group F by adding a new generator x and a single relation $x^n = g$ for some $g \in F$ is a conjugacy separable group.

A general reference for theorems in infinite group theory is the book by W. Magnus, A. Karrass and D. Solitar [3]. References to this book are given as M.K.S. followed by the page number or the number of the theorem or corollary cited.

The proof that a cyclically reduced element of length greater than one in the free product of two free groups with a cyclic amalgamated subgroup is conjugacy distinguished depends on certain properties of free groups. The set of lemmas to follow explains these properties.

LEMMA 1. *Let F be a free group. Let g be an element of F and let u be an integer. If $g \neq 1$, there is a homomorphism χ from G onto a finite group such that $\chi(g)$ has order u. If u is a power of a prime q, χ may be chosen so that $\chi(G)$ is a q-group.*

Proof. According to a theorem of W. Magnus, the intersection of the groups F^k of the lower central series of F is the identity. Since $g \neq 1$, there is an index n

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such that g is an element of F^{n-1} but g is not an element of F^n. Let N be the subgroup of F generated by g^u and F^n. N is normal in F since F^n is normal in F and the image of g in F/F^n is central. If g^v is an element of N, $g^v = g^{uk}f$, where f is an element of F^n. Since F/F^n is torsion free, g^v is an element of N only if u divides v. Thus g has order u modulo N. Since $N \supset F^n$, F/N is nilpotent. Let η be the natural homomorphism from F onto F/N. Then $\eta(g)$ has order u and $\eta(F)$ is nilpotent.

By a theorem of K. Hirsch, $\eta(F)$ is residually finite. Let M be a normal subgroup of finite index in $\eta(F)$ not containing $g, g^2, \ldots, g^{u-1}$. Let ξ be the natural homomorphism from $\eta(F)$ onto $\eta(F)/M$. Then $\xi\eta(F)$ is finite and $\xi\eta(g)$ has order u.

Now suppose $u = q^e$ for q a prime. The group $\xi\eta(F)$ is nilpotent since it is an image group of the group F/N. Since $\xi\eta(F)$ is finite, it is the direct product of its Sylow p-subgroups. Since $\xi\eta(g)$ has order q^e, there is a direct factor Q of $\xi\eta(F)$ such that $\xi\eta(g) \in Q$ and Q is a finite q group. There is a homomorphism δ from $\xi\eta(F)$ onto Q, such that δ restricted to Q is the identity. Thus $\delta\xi\eta(F)$ is a finite q-group and $\delta\xi\eta(g)$ has order q^e.

The proof of Lemma 1 was suggested by D. S. Passman.

LEMMA 2. *Let $a_1, \ldots, a_k$ and b be nonidentity elements of a free group F. Let p be a given prime number. If $a_i \neq b^z$ for each i and all integers z, there is a normal subgroup N of finite index in F such that $a_i \not\equiv b^z$* mod *N for each i and all integers z and the order of b modulo N is a power of p.*

Proof. Suppose there is an a_i such that $(a_i, b) \neq 1$. For each i such that $(a_i, b) \neq 1$ let M_i be a normal subgroup of index a power of p in F such that $(a_i, b) \notin M_i$. Let M be the intersection of all the M_i. The group F/M is a p-group. If all a_i commute with b, let M be a normal subgroup of index a power of p in F such that $b \notin M$. Let the order of b modulo M be p^e.

The element b and all the a_i that commute with b generate a cyclic subgroup C of F. Let f generate C. Thus $b = f^s$, $a_i = f^{r_i}$ for each a_i commuting with b, and s divides no r_i. Let R be a normal subgroup of finite index in F such that f has order sp^e modulo R.

Let $N = M \cap R$. The element b has order p^e modulo each of M and R so b has order p^e modulo N. If a_i commutes with b, $a_i \equiv b^z$ mod N implies $a_i \equiv b^z$ mod R and this congruence implies that s divides r_i, contrary to hypothesis. If $(a_i, b) \neq 1$ then $(a_i, b) \not\equiv 1$ mod N so $a_i \not\equiv b^z$ mod N. Thus, $a_i \not\equiv b^z$ mod N for all i and each z.

LEMMA 3. *Let F be a finitely generated free group. Let a, b and c be elements of F. If the equation $a = b^n c^m$ has no solution for integral n and m, there is a normal subgroup N of finite index in F such that the congruence $a \equiv b^n c^m$* mod *N has no integral solutions.*

Proof. The proof is divided into several cases.

Case 1. The elements b and c commute. In this case b and c generate a free cyclic subgroup of F. Let f generate the cyclic subgroup generated by a and b. The equation $a=b^nc^m$ is equivalent to the equation $a=f^n$. By Lemma 2, there is a normal subgroup N of finite index in F such that $a\neq f^n$ mod N for all integers n. Thus $a\not\equiv b^nc^m$ mod N for all integers n, m.

Case 2. Assume that the elements b and c do not commute. In this case the subgroup generated by b and c is free of rank 2, and hence is freely generated by b and c. Let H be the subgroup generated by b and c. According to a theorem of A. Karrass and D. Solitar [2], H is a free factor of a subgroup M of finite index in F. Since M is of finite index in a finitely generated free group, M is finitely generated. Since b and c freely generate a free factor of M, we may take $x_1=b$, $x_2=c$, $x_3, \ldots, x_n$ to be the free generators of M.

Suppose a is an element of M. Let w be the reduced word in the generators x_i of M representing a. Let v be an integer greater than any exponent in the expression $w=x_{n_1}^{e_1}\cdots x_{n_k}^{e_k}$ with $n_i\neq n_{i+1}$. Let ξ be the homomorphism from M onto $G=(a_1, \ldots, a_n; a_i^v=1)$ defined by assigning $\xi(x_i)=a_i$. Now $\xi(a)\neq\xi(b)^s\xi(c)^t$ for all integers s and t, since the image of a in G is represented by $a_{n_1}^{e_1}\cdots a_{n_k}^{e_k}=w$, and w is a reduced word in G. G is the free product of the cyclic groups of order v generated by the a_i. Now G is residually finite since it is the free product of a set of finite groups. G contains only finitely many elements $\xi(b)^s\xi(c)^t$. There is a homomorphism η from $\xi(M)$ onto a finite group so that $\eta\xi(a)\neq\eta\xi(b)^s\eta\xi(c)^t$. Let U be the kernel of $\eta\xi$.

If a is not an element of M, let $U=M$. In either case, U is of finite index in M and hence F. Let N be the intersection of all the conjugates of U. N is a normal subgroup of finite index in F. Since $a^{-1}b^nc^m$ is not an element of U for all n, m and $U\supset N$, $a\not\equiv b^nc^m$ mod N for all n and all m.

LEMMA 4. *Let F be a free group. Let g, h be elements of F. If $(g, h)\neq 1$, then $(h^{-1}gh, g)\neq 1$.*

Proof. Let S be the subgroup of F generated by g and h. S is free of rank 1 or 2. If S has rank 1, it is abelian and $(g, h)=1$, contrary to hypothesis. If S is free of rank 2, it is freely generated by g and h, so that $(h^{-1}gh, g)\neq 1$. Q.E.D.

LEMMA 5. *Let F_1 and F_2 be free groups. For each i, let N_i be a normal subgroup of finite index in F_i. For each i, let g_i be a nonidentity element of F_i. If g_i has order n_i modulo N_i, there is a normal subgroup M_i of finite index in F_i such that N_i contains M_i and g_i has order n_1n_2 modulo M_i.*

Proof. Let U_i be a normal subgroup of finite index in F_i such that g_i has order n_1n_2 modulo U_i. Let $M_i=N_i\cap U_i$. Since $g_i^{n_1n_2}$ is an element of N_i, g_i has order n_1n_2 modulo M_i.

LEMMA 6. *Let G be a group. Let g and h be elements of G. Let p be a prime. Let g have order p^e in G. If $(g, h^{-1}gh)\neq 1$ and $h^{-1}g^rh=g^s$, then p divides r and s.*

Proof. If r were relatively prime to the order of g, we would have $h^{-1}gh=(h^{-1}g^rh)^v=g^{vs}$ for some integer v, so that $h^{-1}gh$ would commute with g. Thus p

divides r. If s were relatively prime to the order of g, we would have $g=g^{sv}=(h^{-1}g^r h)^v=(h^{-1}gh)^{rv}$ for some integer v, so that $h^{-1}gh$ would commute with g. Thus p divides s.

LEMMA 7. *Let G be a group. Let N_i be a normal subgroup of G for $i=1,\ldots,k$. Let b be an element of G. Let p be a prime number. If the order of b modulo N_i is a power of p for each i, b has order a power of p modulo the intersection of the N_i.*

Proof. Since k is finite, we need only the case of two N_i. Let b have order p^{e_1} modulo N_1 and order p^{e_2} modulo N_2. Assume without loss of generality that e_1 is greater than or equal to e_2. Now $b^{p^{e_1}}$ is an element of both N_1 and N_2 and hence of $N_1 \cap N_2$. Thus the order of b modulo $N_1 \cap N_2$ divides p^{e_1} and so is a power of p.

The next lemma is about the free product of any two groups with a cyclic amalgamated subgroup.

LEMMA 8. *Let G be the free product of two groups A and B with a cyclic amalgamated subgroup C generated by an element c. Let g and h be elements of G. Let $g=t_1\cdots t_n$ and $h=s_1\cdots s_m$, where each t_i is in A or B, each s_i is in A or B, consecutive t_i are in different factors of G and consecutive s_i are in different factors of G. The equation $c^{-z}hc^z=g$ is valid for an integer z if and only if there exist integers $u_0,\ldots,u_m$ such that $t_i^{-1}c^{u_{i-1}}s_i=c^{u_i}$ for $i=1,\ldots,m$ and $u_0=u_m$.*

Proof. If the equations have a solution, clearly $g=c^{u_0}hc^{-u_0}$. Suppose $g=c^{u_0}hc^{-u_0}$. Then $t_1\cdots t_n=c^{u_0}s_1\cdots s_mc^{-u_0}$. The left-hand side of the equation has syllable length n while the right-hand side of the equation has syllable length m since c is in the amalgamated subgroup. Thus $n=m$. Also

$$t_2\cdots t_n = t_1^{-1}c^{u_0}s_1s_2\cdots s_nc^{-u_0}.$$

The left-hand side of the equation has syllable length $n-1$ so that $t_1^{-1}c^{u_0}s_1$ must be in the same factor of G as s_2. This is possible only if $t_1^{-1}c^{u_0}s_1$ is in the amalgamated subgroup. Thus $t_1^{-1}c^{u_0}s_1=c^{u_1}$ for u_1 an integer. The process can clearly be continued, so that an induction based on this process will prove the lemma.

In the statements and proofs of Lemmas 9 through 16 we make the following conventions. Let F_1 and F_2 be finitely generated free groups. Let c_i be an element of F_i, $i=1, 2$. Let G be the free product of the F_i with the cyclic subgroups generated by the c_i amalgamated. Let c be the generator of the amalgamated subgroup of G.

LEMMA 9. *Let N_1 and N_2 be normal subgroups of F_1 and F_2 respectively. If the order of c_1 modulo N_1 equals the order of c_2 modulo N_2, there is a homomorphism ξ from G onto the free product of F_1/N_1 and F_2/N_2 with the images of c_1 and c_2 amalgamated. The homomorphism ξ acts as the natural homomorphism from F_i onto F_i/N_i.*

Proof. Lemma 9 is trivial.

LEMMA 10. *Let $F_i\supset N_i\supset M_i$ where N_i and M_i are normal in F_i for $i=1, 2$. Let the order of c_1* mod N_1 *equal the order of c_2* mod N_2 *and let the order of c_1* mod M_1

equal the order of c_2 mod M_2. *If* α *is the homomorphism constructed as in Lemma* 9 *with the* N_i *and* β *is the homomorphism constructed as in Lemma* 9 *with the* M_i *then the kernel of* α *contains the kernel of* β.

Proof. Clearly $\alpha(G)$ is a factor group of $\beta(G)$. Thus the kernel of α contains the kernel of β.

LEMMA 11. *Let* g *and* h *be elements of* G. *Let* $g=t_1\cdots t_n$, $h=s_1\cdots s_m$ *be expressions for* g *and* h *in terms of syllables* t_i, s_i, *where consecutive* t_i *and consecutive* s_i *are in different factors of* G. *Let* $n>1$. *If* $m\neq n$ *or one of the equations* $t_i^{-1}c^u s_i=c^v$ *has no integral solution* u, v, *there is a homomorphism* ξ *corresponding to normal subgroups of the* F_i *as in Lemma* 9 *such that each* $\xi(F_i)$ *is finite and* $\xi(c)^{-z}\xi(g)\xi(c)^z\neq\xi(h)$ *for all integers* z.

Proof. First we consider the case $n=m$. Suppose $t_i^{-1}c^u s_i=c^v$ has no solutions and that t_i and s_i are elements of the same factor of G. Without loss of generality, let the factor of G be F_1. Let N_1 be a normal subgroup of finite index in F_1 such that

(1) $t_i^{-1}c^u s_i c^{-v}\notin N_1$ for all integers u, v.

(2) If $t_j\in F_1$, $t_j^{-1}c^z\notin N_1$ for all integers z.

(3) If $s_j\in F_1$, $s_jc^z\notin N_1$ for all integers z.

The subgroup N_1 is the intersection of subgroups provided by Lemmas 2 and 3. Let N_2 be a normal subgroup of finite index in F_2 such that

(1) If $t_j\in F_2$, $t_jc^z\notin N_2$ for all integers z.

(2) If $s_j\in F_2$, $s_jc^z\notin N_2$ for all integers z.

The subgroup N_2 is the intersection of normal subgroups provided by Lemma 2.

If $t_i^{-1}c^u s_i=c^v$ has no solution for t_i and s_i in different factors of G, we omit the property (1) from the properties of N_1.

Let $N_1\supset M_1$, $N_2\supset M_2$, where M_1 and M_2 are normal subgroups of finite index in F_1 and F_2 respectively such that the order of c_1 mod M_1 equals the order of c_2 mod M_2. Let ξ be the homomorphism of G corresponding to M_1 and M_2 according to the construction of Lemma 9.

Suppose there is an integer z such that $\xi(c)^{-z}\xi(g)\xi(c)^z=\xi(h)$. Then by Lemma 8 there are integers $u_0,\ldots,u_n$ such that

$$\xi(t_i)^{-1}\xi(c)^{u_{i-1}}\xi(s_i) = \xi(c)^{u_i}, \qquad u_0 = u_n,$$

for $\xi(t_i)$ and $\xi(s_i)$ are the syllables of $\xi(g)$ and $\xi(h)$. Thus

$$\xi(t_i^{-1}c^{u_{i-1}}s_ic^{-u_i}) = 1.$$

But this is impossible, so the result follows in the case $n=m$.

If $n\neq m$, choose N_i to be a normal subgroup of finite index in F_i such that

(1) If $m=1$ and $s_1\in F_i$, then $s_1\notin N_i$.

(2) If $m\neq 1$ and $s_j\in F_i$, then $s_jc^{-z}\notin N_i$ for all z.

(3) If $t_j\in F_i$, then $t_jc^{-z}\notin N_i$ for all z.

Let the subgroups M_i and the homomorphism ξ be constructed as above. The syllable length of $\xi(g)$ is n. The syllable length of $\xi(h)$ is m. Since $m\neq n$, Lemma 8 implies that $\xi(c)^{-z}\xi(h)\xi(c)^z\neq\xi(g)$.

LEMMA 12. *Let $g=t_1\cdots t_n$, $h=s_1\cdots s_n$ where $n>1$ and the s_i and t_i are syllables of g and h as in Lemma 11. Let $(t_i, c)\neq 1$ for all i and let each equation $t_i^{-1}c^{u_{i-1}}s_i=c^{v_i}$ have an integral solution. If $c^{-z}hc^z\neq g$ for all integers z there is a homomorphism ξ as described in Lemma 9 such that each $\xi(F_i)$ is finite and $\xi(c^{-z}hc^z)\neq\xi(g)$ for all z.*

Proof. According to Lemma 4, $(t_i^{-1}ct_i, c)\neq 1$ for all i. Let p be a prime number dividing no nonzero difference u_1-v_n, $u_2-v_1,\ldots, u_n-v_{n-1}$. Let N_i be a normal subgroup of index a power of p in F_i such that

(1) If $t_j\in F_i$, then $(t_j^{-1}ct_j, c)\notin N_i$.

(2) If $s_j\in F_i$, then $(s_j^{-1}cs_j, c)\notin N_i$.

The subgroup N_i is the intersection of subgroups of index a power of p in F_i. Lemma 1 is used repeatedly. Since the equation $t_i^{-1}c^{u_i}s_i=c^{v_i}$ is valid for integers u_i, v_i, we have $(s_j, c)\neq 1$.

Let M_i be a normal subgroup of finite index in F_i such that M_i is contained in N_i, the order of c_i modulo M_i is a power of p, and the order of c_1 modulo M_1 equals the order of c_2 modulo M_2. Let ξ be the homomorphism defined as in Lemma 8 from G onto the free product of F_1/M_1 and F_2/M_2 with the image of C amalgamated.

Suppose $\xi(t_i)^{-1}\xi(c)^v\xi(s_i)=\xi(c)^w$. Since $\xi(t_i)^{-1}\xi(c)^{u_i}\xi(s_i)=\xi(c)^{v_i}$, we have $\xi(c)^{w-v_i}=\xi(t_i)^{-1}\xi(c)^{v-u_i}\xi(t_i)$. Now $\xi((t_i^{-1}ct_i, c))\neq 1$ so that $w-v_i$ and $v-u_i$ are divisible by p according to Lemma 6. If $\xi(c)^{-z}\xi(h)\xi(c)^z=\xi(g)$, there exist integers $w_0,\ldots, w_n$ such that $\xi(t_i)^{-1}\xi(c)^{w_{i-1}}\xi(s_i)=\xi(c)^{w_i}$ with $w_0=w_n$. Thus $0=w_n-w_0=u_1-v_n \bmod p$, $0=w_{n-1}-w_{n-1}=u_n-v_{n-1} \bmod p$ etc. Since by choice of p at least one of the differences on the right is incongruent to zero mod p, we have $\xi(c)^{-z}\xi(h)\xi(c)^z\neq\xi(g)$. Q.E.D.

LEMMA 13. *Let $g=t_1\cdots t_m$, $h=s_1\cdots s_m$ where h and g are elements of G and the t_i and the s_i are syllables of g and h respectively. Let m be greater than 1. Let at least one $(t_i, c)\neq 1$. If each of the equations $t_i^{-1}c^{u_i}s_i=c^{v_i}$ has an integral solution but $c^{-z}hc^z\neq g$ for all integers z, there is a homomorphism ξ as given by Lemma 9 such that each $\xi(F_i)$ is finite and $\xi(c^{-z}hc^z)\neq\xi(g)$ for all integers z.*

Proof. For each i such that $(t_i, c)=1$ we have $t_i^{-1}s_i=c^{v_i-u_i}$. We may set $h=s_1\cdots s_m$ where $s_i=t_i$ if t_i commutes with c, no two adjacent s_i are in the same factor of G, and no s_i is in the amalgamated subgroup. The equations $t_i^{-1}c^{u_i}s_i=c^{v_i}$ have solutions u_i, v_i with $u_i=v_i$ if t_i commutes with c. Let $t_{n_1},\ldots, t_{n_k}$ be the t_i not commuting with c and let $n_k>n_{k-1}>\cdots>n_1$. If each of the differences

$$u_{n_1}-v_{n_k},\quad u_{n_2}-v_{n_1},\quad \ldots,\quad u_{n_k}-v_{n_{k-1}}$$

are zero, there is an integer z such that $c^{-z}hc^z=g$. Let p be a prime relatively prime to at least one of the nonzero differences. By Lemma 4, $(t_i, c)\neq 1$ implies $(t_i^{-1}ct_i, c)$

$\neq 1$ since t_i is in a factor of G. By Lemmas 1 and 2 a normal subgroup N_i of F_i can be found with the properties:

(1) N_i is of finite index in F_i.

(2) c_i has order a power of p modulo N_i.

(3) If $t_j \in F_i$, $t_j \not\equiv c_i^z$ mod N_i for all z.

(4) If $s_j \in F_i$, $s_j \not\equiv c_i^z$ mod N_i for all z.

(5) If $t_j \in F_i$, $(t_j, c_i) \neq 1$, $(t_j^{-1}c_it_j, c_i) \notin N_i$.

(6) If $s_j \in F_i$, $(s_j, c_i) \neq 1$, $(s_j^{-1}c_is_j, c_i) \notin N_i$.

The subgroups N_i are found by intersecting the normal subgroups of index a power of p given by Lemma 1 for properties (5) and (6) with normal subgroups of finite index given by Lemma 2 for properties (3) and (4) and the prime p. By Lemma 7, c_i has order a power of p modulo N_i. Let M_i be a normal subgroup of finite index in F_i such that N_i contains M_i and the order of c_1 modulo M_1 equals the order of c_2 modulo M_2. By Lemma 5, the M_i can be chosen so that the order of c_i modulo M_i is a power of p. Let ξ be the homomorphism from G onto the free product of the groups F_i/M_i with the images of the c_i amalgamated, as given by Lemma 9. Let $\boldsymbol{g}$ denote the image $\xi(g)$ of $g \in G$.

Suppose $(t_i, c) \neq 1$. By the properties of the M_i we have $(\boldsymbol{t}_i^{-1}\boldsymbol{c}\boldsymbol{t}_i, \boldsymbol{c}) \neq 1$. If $\boldsymbol{t}_i^{-1}\boldsymbol{c}^{a_i}\boldsymbol{t}_i = \boldsymbol{c}^{b_i}$, then $\boldsymbol{c}^{b_i - v_i} = \boldsymbol{t}_i^{-1}\boldsymbol{c}^{a_i - u_i}\boldsymbol{t}_i$. By Lemma 6, $a_i - u_i$ and $b_i - v_i$ are divisible by p. If $(t_i, c) = 1$, then $t_i = s_i$ so that $a_i = b_i$.

Now $\boldsymbol{t}_i$ and $\boldsymbol{s}_i$ are the syllables of $\boldsymbol{g}$ and $\boldsymbol{h}$ respectively, so that $\boldsymbol{c}^{-z}\boldsymbol{h}\boldsymbol{c}^z = \boldsymbol{g}$ if and only if the equations $\boldsymbol{t}_i^{-1}\boldsymbol{c}^{a_i}\boldsymbol{s}_i = \boldsymbol{c}^{b_i}$ have integral solutions a_i, b_i with $\boldsymbol{c}^{a_1} = \boldsymbol{c}^{b_n}$, $\boldsymbol{c}^{a_2} = \boldsymbol{c}^{b_1}, \ldots, \boldsymbol{c}^{a_n} = \boldsymbol{c}^{b_{n-1}}$. Suppose there are such solutions to these equations. The elements of the list $a_1 - b_n, a_2 - b_1, \ldots, a_n - b_{n-1}$ are each congruent to zero modulo the order of $\boldsymbol{c}$ and hence congruent to zero modulo p. It follows from the last paragraph that $a_1 \equiv a_{n_1}$, $b_n \equiv b_{n_k}$ and $a_1 - b_n$ is congruent modulo p to $u_{n_1} - v_{n_k}$. In general each of the differences $u_{n_i} - v_{n_{i-1}}$ is congruent to an element of $a_1 - b_n, \ldots, a_n - b_{n-1}$ and at least one of the differences $u_{n_i} - v_{n_{i-1}}$, $u_{n_1} - v_{n_k}$ is not congruent to zero modulo p. Thus $\boldsymbol{c}^{-z}\boldsymbol{h}\boldsymbol{c}^z \neq \boldsymbol{g}$ for all z.

LEMMA 14. *Let $g = t_1 \cdots t_n$, $h = s_1 \cdots s_n$, where g and h are elements of G and the t_i and s_i are syllables of g and h respectively. Let m be greater than one. Let every t_i commute with c. If each of the equations $t_i^{-1}c^{u_i}s_i = c^{v_i}$ has an integral solution u_i, v_i but $c^{-z}gc^z \neq h$ for all integers z, there is a homomorphism ξ as given by Lemma* 9 *such that each $\xi(F_i)$ is finite and $\xi(c^{-z}gc^z) \neq \xi(h)$ for all integers z.*

Proof. Let $f = gh^{-1}$. Since $c^{-z}gc^z \neq h$ for all $z, f \neq 1$.

Suppose f is in a factor of G. Without loss of generality assume f is an element of F_1. Let N_1 be a normal subgroup of finite index in F_1 not containing f. Let N_2 be a normal subgroup of finite index in F_2 such that the order of c_2 modulo N_2 equals the order of c_1 modulo N_1. Let ξ be the homomorphism constructed according to Lemma 9 using the N_i. Clearly $\xi(f) \neq 1$.

If f is not in a factor of G, let $f=u_1\cdots u_k$ where each u_i is in a factor of G, adjacent u_i are in different factors of G, and no u_i is in the amalgamated subgroup of G. Let N_i be a normal subgroup of finite index in F_i such that if $u_j\notin F_i$, then $u_jc_i^{-z}\notin N_i$ for all integers z. Let M_i be a normal subgroup of finite index in F_i such that $N_i\supset M_i$ and the orders of c_i modulo M_i are equal. Let ξ be the homomorphism constructed according to Lemma 9 using the M_i. Clearly $\xi(f)\neq 1$.

Since each t_i commutes with c, g commutes with c and hence $\xi(g)$ commutes with $\xi(c)$. Thus $\xi(c)^{-z}\xi(g)\xi(c)^z=\xi(h)$ if and only if $\xi(g)=\xi(h)$ or $\xi(f)=1$. Thus the lemma is proven.

LEMMA 15. *Let g be a cyclically reduced element of length greater than one in G. Let $g=t_1\cdots t_m$ where $m>1$, t_i is an element of a factor of G, consecutive t_i are in different factors of G, no t_i is in the amalgamated subgroup and t_1 and t_m are in different factors of G. Let h be a cyclically reduced element of G. Let $h=s_1\cdots s_n$ where $n\geqq 1$, consecutive s_i are from different factors of G, and if $n>1$, no s_i is in the amalgamated subgroup and s_1 and s_n are from different factors of G. If for each cyclic permutation φ of $1,\ldots,m$ and all integers z we have $c^{-z}t_{\varphi(1)}\cdots t_{\varphi(m)}c^z\neq h$, there is a homomorphism ξ from G onto the free product of two finite groups with a cyclic amalgamated subgroup such that $\xi(t_i)$ are the syllables of $\xi(g)$, $\xi(s_i)$ are the syllables of $\xi(h)$, $\xi(g)$ is cyclically reduced, $\xi(h)$ is cyclically reduced, $\xi(c)$ generates the amalgamated subgroup of $\xi(G)$ and $\xi(c^{-z}t_{\varphi(1)}\cdots t_{\varphi(m)}c^z)\neq\xi(h)$ for all integers z and each cyclic permutation φ of $1,\ldots,m$.*

Proof. By Lemmas 11 through 14 there is for each φ a homomorphism ξ_φ from G onto the free product of $\xi_\varphi(F_1)$ and $\xi_\varphi(F_2)$ with the images of the c_i amalgamated such that $\xi\varphi(c^{-z}t_{\varphi(1)}\cdots t_{\varphi(m)}c^z)\neq\xi_\varphi(h)$ for all integers z. Let K_φ be the kernel of ξ_φ. Let $K_{\varphi,i}=K_i\cap F_i$. Let $K_i=\bigcap K_{\varphi,i}$ so that each K_i is a normal subgroup of finite index in F_i. Let M_i be a normal subgroup of finite index in F_i such that $M_i\subset K_i$, $t_jc^{-z}\notin M_i$, $s_jc^{-z}\notin M_i$ for all j and all integers z and the orders of c_i modulo M_i are equal. Let ξ be the homomorphism obtained by Lemma 9 using the M_i. Let K be the kernel of ξ. By Lemma 10, $K\subset K_\varphi$ for each φ, so that ξ is the required homomorphism.

LEMMA 16. *Let G be the free product of two finite groups with an amalgamated subgroup. If g is a cyclically reduced element of length greater than one in G, g is c.d. in G.*

Proof. According to B. H. Neumann [5, p. 532], there is a homomorphism ξ from G onto a finite group such that the kernel of ξ meets each factor of G only in the identity. According to a theorem of H. Neumann, M.K.S., Corollary 4.9.2, the kernel of ξ is free. Thus G is a finite extension of a free group. Since g is cyclically reduced and has syllable length greater than one, g is of infinite order in G. It follows from a theorem of the author [7] that g is c.d. in G.

THEOREM 1. *If G is the free product of two free groups with a cyclic amalgamated subgroup, every element of G conjugate to a cyclically reduced element of length greater than one is c.d. in G.*

Proof. Clearly we need only consider g cyclically reduced in G. Let h be a cyclically reduced element of G. Let $g=t_1\cdots t_m$, $h=s_1\cdots s_n$ where $m>1$, $n\geqq 1$, consecutive t_i are elements of different factors of G, consecutive s_i are in different factors of G, no t_i is in the amalgamated subgroup of G, t_1 and t_m are from different factors of G, s_1 and s_n are from different factors of G if $n>1$, and no s_i is in the amalgamated subgroup if $n>1$. According to a theorem of D. Solitar, M.K.S., Theorem 4.6, g is conjugate to h if and only if there is a cyclic permutation φ of $1,\ldots,m$ and an integer z such that $c^{-z}t_{\varphi(1)}\cdots t_{\varphi(m)}c^z=h$. Since this equation is untrue for all φ and z, there is, by Lemma 15, a homomorphism ξ from G onto the free product of two finite groups such that $\xi(g)$ is cyclically reduced, $\xi(t_i)$ are the syllables of $\xi(g)$, $\xi(g)$ has syllable length greater than 1, $\xi(s_i)$ are the syllables of $\xi(h)$ and for each cyclic permutation φ of $1,\ldots,m$, $\xi(c^{-z}t_{\varphi(1)}\cdots t_{\varphi(m)}c^z)\neq\xi(h)$ for all z. Since the quoted theorem of D. Solitar applies to $\xi(G)$, $\xi(g)$ and $\xi(h)$ are not conjugate in $\xi(G)$. By Lemma 16, $\xi(g)$ is c.d. in $\xi(G)$ so there is a homomorphism χ from $\xi(G)$ onto a finite group such that $\chi\xi(g)\nsim\chi\xi(h)$.

If h is not cyclically reduced, let $h'=h^x$ where h' is cyclically reduced. Since $h\nsim g$, $h'\nsim g$ so that by the last paragraph there is a homomorphism $\chi\xi$ from G onto a finite group so that $\chi\xi(g)\nsim\chi\xi(h')$. But then $\chi\xi(g)\nsim\chi\xi(h)$. Thus g is c.d. in G.

In the next lemma we consider the group G formed by adding a single relation $x^n=g$ to the free product of a free group F and the free cyclic group generated by a generator x. We always let g be an element of F and call G the group formed by adding a root of an element to a free group. The notation for G as constructed above is $G=(F,x;x^n=g)$.

Note that if F is a free group $G=(F,x;x^n=g)$ is a free product of two free groups with a cyclic amalgamated subgroup. By Theorem 1, every element of G conjugate to a cyclically reduced element of length greater than one is c.d. in G. Thus to prove that G is c.s. we need only consider elements of length one in G.

LEMMA 17. *Let F be a free group and let $G=(F,x;x^n=g)$ with $g\in F$. If g_1 and g_2 are nonconjugate elements of F or distinct powers of x, there is a homomorphism ξ from G onto a finite group such that $\xi(g_1)$ is not conjugate to $\xi(g_2)$ in $\xi(G)$.*

Proof. The proof is divided into two cases.

Case 1. Let g_1 and g_2 be two nonconjugate elements of F. Since F is c.s., there is a normal subgroup N of finite index in F such that $g_1\not\equiv h^{-1}g_2h \bmod N$ for $h\in F$. Let g have order m modulo N and ξ be the natural homomorphism from F onto F/N. Let ψ be the homomorphism from G onto $H=(F/N,y;y^n=\xi(g))$ defined as follows: $\psi(u)=\xi(u)$ for $u\in F$, $\psi(x^r)=y^r$, $\psi(ab)=\psi(a)\cdot\psi(b)$. Now ψ is a homomorphism since $\psi(x^n)=y^n=\xi(g)$ and $\psi(g)=\xi(g)$. Let M be the set of matrices with entries in the integral group ring R of F/N. We set $\varphi(u)=\text{diag}\,(u,\ldots,u)$ for $u\in F/N$ and $\varphi(y)=\text{diag}\,(1,\ldots,1,\xi(g))\cdot P$ where P is the $n\times n$ permutation matrix corresponding to the cycle $(1,2,\ldots,n)$. Now $\varphi(u)^{-1}=\text{diag}\,(u^{-1},\ldots,u^{-1})$ for $u\in F/N$, $\varphi(y^n)=\varphi(\xi(g))$, $\psi(y)^{-1}=P^{n-1}\,\text{diag}\,(1,\ldots,1,\xi(g)^{-1})$, so that the matrices $\psi(H)$ generate a group U in M, and ψ is a homomorphism from H onto the group U. If D is a diagonal matrix, $DP=PD^*$ where D^* is a diagonal matrix whose entries are, up to order along the diagonal, the same as those of D. Thus if T is an element of U, $T=P^r\,\text{diag}\,(d_1,\ldots,d_n)$ where the d_i are elements of F/N. Thus U is finite.

Suppose $\varphi\psi(g_1)$ is conjugate to $\varphi\psi(g_2)$ in U. Then

$$\text{diag}\,(\xi(g_1),\ldots,\xi(g_1)) = \text{diag}\,(d_1^{-1},\ldots,d_n^{-1})P^{-t}\,\text{diag}\,(\xi(g_2),\ldots\,\xi(g_2))P^t\,\text{diag}\,(d_1,\ldots,d_n).$$

Now P^t commutes with $\text{diag}\,(\xi(g_2),\ldots,\xi(g_2))$ so that one has $\xi(g_1)=d_i^{-1}\xi(g_2)d_i$ for each d_i, $d_i\in S/N$. But if $h\in\xi^{-1}(d_1)$ we have $g_1\equiv h^{-1}g_2h \bmod N$, contrary to hypothesis. Thus $\varphi\psi(g_1)$ is not conjugate to $\varphi\psi(g_2)$ in the finite group U.

Case 2. Let $g_1=x^i$ and $g_2=x^j$, and let g_1 not be conjugate to g_2. One has $i\neq j$. If $g_1^n=g^i$ is not conjugate to $g_2^n=g^j$ in F, according to Case 1, there is a homomorphism γ from G onto a finite group such that $\gamma(g_1^n)$ is not conjugate to $\gamma(g_2^n)$ in $\gamma(G)$. Now $\gamma(g_1)=h^{-1}\gamma(g_2)h$ implies $\gamma(g_1)^n=h^{-1}\gamma(g_2)^nh$ so $\gamma(g_1)$ and $\gamma(g_2)$ are not conjugate. Let there be an h in F such that $h^{-1}g^ih=g^j$ for $i\neq j$. The subgroup S generated by g and h must be free of rank $\leqq 2$. If S has rank 2, g and h are free generators of S and $i=j=0$, contrary to hypothesis. If S has rank 1, S is abelian and $g^i=g^j$. Since F is torsion free and $i\neq j$, g is the identity. But if g is the identity, G is a free product of F and the cyclic group of order n generated by x. Thus G is c.s. by Theorem 2 of [7] and the result follows.

REMARK. The matrix construction used here is based on a construction in the book by A. Speiser [6]. D. S. Passman has remarked that a representation of G as a wreath product would be sufficient.

THEOREM 2. *Let F be a free group. Let g be an element of F. If $G=(F,x;\,x^n=g)$, then G is c.s.*

Proof. Let f be an element of G and let h be an element of G not conjugate to f. If either f or h is conjugate to a cyclically reduced element of length greater than one then it is c.d. in G by Theorem 1 so that there is a homomorphism ξ from G onto a finite group with the property $\xi(f)\nsim\xi(h)$. Thus we may assume that f and h are conjugate to elements of length one in G, and we need only consider f and h cyclically reduced.

If f and h are in the same factor of G, Lemma 17 implies that there is a homomorphism ξ from G onto a finite group such that $\xi(f)\nsim\xi(h)$.

If f and h are in different factors of G, neither is in the amalgamated subgroup. Thus one of f and h is a power of x but not a power of g. Let ξ be the homomorphism from G onto the group $(y; y^n=1)$ determined by the assignments $x \to y$, $F \to 1$. Clearly ξ is a homomorphism from G onto a finite group and $\xi(f) \nsim \xi(h)$.

Thus every element of G is c.d. in G so G is c.s.

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