HOUSTON JOURNAL OF MATHEMATICS
Volume 22, No. 2, 1996

THE RESIDUAL NILPOTENCE OF THE FUNDAMENTAL GROUP OF CERTAIN GRAPHS OF GROUPS

D. VARSOS
Communicated by the editors.

1. Introduction.

The residual nilpotence of a free product $G = \star_{i \in I} G_i$ is studied in [6]. There Malc'ev gives (separate) necessary and sufficient conditions, which, in general, do not constitute a characterization. A characterization for the residual nilpotence of a free product has been given by Lichtman (Th. 1 of [5]). On the other hand Raptis and Varsos in [8] give a characterization for the residual nilpotence of HHN-extensions with base group a finite or a finitely generated ($f.g.$) abelian group.

Here we study the residual nilpotence (via the residual finiteness-p) of the fundamental group of a graph of groups. First we deal with the case where the vertex groups are finite (Th. 6), where we reduce the general case to the case of finite p-groups. If the graph is finite we give a characterization (Prop. 7), which in the case where the vertex groups are finite abelian p-groups is given by an internal condition on the edge groups (Prop. 11). Second, we study the case where the vertex groups are $f.g.$ abelian groups and give a necessary condition (Prop. 14). Finally we study the case of a specific kind of tree products and give a characterization if the vertex groups are finite abelian p-groups (Th. 13) and a sufficient condition if the vertex groups are $f.g.$ generated free abelian groups (Cor. 15.1).

These results extend the results of [4] and [8].

2. Definitions and preliminary results.

Definition. A group G has the property $\mathcal{X}$ residually if to every element $g \in G$, $g \neq 1$, there exists a normal subgroup N of G such that $g \notin N$ and

G/N has the property $\mathcal{X}$. We also say that G is residually $\mathcal{X}$ abbreviated to $(\mathcal{RX})$.

It is clear that a group G is $\mathcal{RX}$ if and only if the normal subgroups whose factor groups have property $\mathcal{X}$ intersect in the trivial group. It follows that a group G is residually nilpotent $(\mathcal{RN})$ if and only if $\cap_n \gamma_n(G) = 1$, where $\gamma_n(G), n \in \mathbb{N}$, is the n^{th} term of the lower central series of G.

If a group G is residually finite-p $(\mathcal{RF}_p)$, p-prime, then it is $\mathcal{RN}$. But a $\mathcal{RN}$ group is not necessarily a $\mathcal{RF}_p$ group. In the class of finitely generated groups residual nilpotence and residual p-finiteness coincide for certain primes p described in [3] (Th 2.1 (ii)).

A graph of groups $\mathcal{L}(\mathcal{G}, X)$ is a connected graph X, where to each vertex $v \in V(X)$ is assigned a (vertex) group G_v and each edge $e \in E(X)$, with initial and terminal vertices $\iota(e)$ and $\tau(e)$ respectively, is assigned two isomorphic (edge) subgroups $H_{\iota(e)} \leq G_{\iota(e)}, H_{\tau(e)} \leq G_{\tau(e)}$ via an isomorphism $\theta_e : H_{\iota(e)} \to H_{\tau(e)}$ $(\theta_e^{-1} = \theta_{\overline{e}}$, where $\overline{e}$ is the inverse of e with $\iota(\overline{e}) = \tau(e)$, $\tau(\overline{e}) = \iota(e))$. The definition of the fundamental group $G = \pi(\mathcal{G}, X)$ of a graph of groups $\mathcal{L}(\mathcal{G}, X)$, relative to a maximal tree T of X, is given in [10] (§1.5), where it is also proved that the structure of G is independent of the choice of the maximal tree T.

There may exist edges $e \in E(X)$ for which the assigned edge groups are trivial $(H_{\iota(e)} = H_{\tau(e)} = 1)$. We consider two sets of edges with trivial edge groups:

(i) Let $EC(X)$ be a maximal set of edges e of X such that $H_{\iota(e)} = H_{\tau(e)} = 1$ and $\overline{X} = X \setminus EC(X)$ is connected. It may be that $EC(X) = \emptyset$. The fundamental group of $\mathcal{L}(\mathcal{G}, X)$ has the structure $\pi(\mathcal{G}, X) = F \star \pi(\mathcal{G}, \overline{X})$, where F is a free group with (free) generators the (free) generators of $\pi(\mathcal{G}, X)$ which correspond to the edges of $EC(X)$.

(ii) Let $ET(X)$ be the set of edges e of X such that $H_{\iota(e)} = H_{\tau(e)} = 1$, e is not in $EC(X)$ and e disconnects $\overline{X}$. It may be that $ET(X) = \emptyset$. If $e \in ET(X)$, then $\overline{X} = X_{\iota(e)} \cup \{e\} \cup X_{\iota(e)}$, where $X_{\iota(e)}$ and $X_{\tau(e)}$ are connected components of $\overline{X}$ joined by e, and the fundamental group has the structure $\pi(\mathcal{G}, \overline{X}) = \pi(\mathcal{G}, X_{\iota(e)}) \star \pi(\mathcal{G}, X_{\iota(e)})$.

Therefore according to the discussion above the fundamental group has the structure $\pi(\mathcal{G}, X) = F \star (\star \pi(\mathcal{G}, X_i))$ where F is a free group with rank $|EC(X)|$ and $\star \pi(\mathcal{G}, X_i)$ is a free product with $|ET(X)| + 1$ factors and in each factor $\pi(\mathcal{G}, X_i)$ none of the edge group is the trivial group. Since for a free product of groups there exists a characterization of the residual nilpotence (Th. 2.1 of [5]), in the following we suppose, without any further mention, that the edge groups are not trivial. We can also suppose that each edge group is a proper subgroup of the corresponding vertex group, unless we have a loop ($\iota(e) = \tau(e)$), where it may happen $H_{\iota(e)} = G_{\iota(e)}$. In the case where the graph is a loop and the vertex group is a finitely generated abelian group we have a characterization (Th. 2.4 of [7]).

Let $\mathcal{L}(\mathcal{G}, C)$ be a graph of groups, over the circuit C with vertices $v_1, \dots, v_n$ and edges $e_1, \dots, e_n$, where $\iota(e_{i+1}) = \tau(e_i) = v_{\iota+1}$, $i = 1, \dots, n-1$ and $\iota(e_1) = \tau(e_n) = v_1$. Let $G_{\iota(e_\iota)}$ be the vertex groups and $H_{\iota(e_\iota)}\theta_{e_i} = H_{\tau(e_i)}$ the corresponding edge groups for $i = 1, \dots, n$ (cf [10] p. 15). In the fundamental group $G = \pi(\mathcal{G}, C)$ we have the relations $H_{\iota(e_i)} = H_{\iota(e_i)} = \theta_{ei} = H_{\tau(e_i)}, i = 1, \dots, j-1, j+1, \dots, n$ and $t^{-1}H_{\iota(e_j)}t = H_{\iota(e_j)}\theta_{e_j} = H_{\tau(e_j)}$. The generator t of $\pi(\mathcal{G}, C)$ corresponds to the edge $e_j \in E(X)$, which is omitted to obtain a maximal tree. Let $\theta_i = \theta_{e_i} \circ \theta_{e_{i+1}} \circ \cdots \circ \tau_t \circ \cdots \circ \theta_{e_{i-1}}$, where in the composition each θ_{e_r} acts as the restriction of the initial θ_{e_r} on a subgroup of $H_{\iota(e_r)}$ and τ_t is the inner automorphism induced by t. For $i = 1, \dots, n$ let $\mathcal{N}_i = \{K \leq G_{\iota(e_i)}$ such that $(K)\theta_i = K\}$. For $K, L \in \mathcal{N}_i$, it is easy to see that the subgroup $M = \langle K, L \rangle$ belongs to N_i. So the set $H_i = \cup\{K \in N_i\}$ is a subgroup of $G_{\iota(e_i)}$ with the property; $(H_i)\theta_i = H_i$ and if $N \leq G_{\iota(e_i)}$ with $(N)\theta_i = N$, then $N \leq H_i$. In this sense the subgroup H_i is the "largest" subgroup of $G_{\iota(e_i)}$ with this property. From the way the subgroups H_i are defined and from the relations of the fundamental group $\pi(\mathcal{G}, C)$, it is easy to see that $H_i = H_r$ for every $i, r \in \{1, \dots, n\}$. The homomorphisms θ_i are automorphisms of $H = H_i$. Moreover, we have $H\theta_i = H_i\theta_i = H_i\theta_{e_i} \circ \theta_{i+1} \circ \theta_{e_i}^{-1}$. The subgroup H is called the *core* of the circuit C.

Remark. If the graph Y is a part of a larger graph of groups $\mathcal{L}(\mathcal{G}, X)$, then $\mathcal{L}(\mathcal{G}, Y)$ is a subgraph of groups of $\mathcal{L}(\mathcal{G}, X)$ and the fundamental group $\pi(\mathcal{G}, Y)$ of $\mathcal{L}(\mathcal{G}, Y)$ is embeddable in the fundamental group $\pi(\mathcal{G}, X)$ of $\mathcal{L}(\mathcal{G}, X)$ (cf. [2] §2).

If $\mathcal{L}(\mathcal{G}, X)$ is a graph of groups and T is a maximal tree of X, chosen to obtain a presentation of the fundamental group $G = \pi(\mathcal{G}, X)$, then there is an $1-1$ and onto correspondence between the edges of X omitted to obtain the maximal tree T and the (free) generators of G. Let $G_T = \pi(\mathcal{G}, T)$ be the fundamental group over T with vertex groups $G_{v(i)}$, $v(i) \in V(X) = V(T)$ and edge groups $H_{\iota(e)}\theta_e = H_{\tau(e)}$, $e \in E(T)$. By contracting the maximal tree T to a point (cf. [10] or [2] §2), the resulting graph of groups is a bouquet of cycles, where to its single vertex is assigned the group G_T and to its loops are assigned the (isomorphic) subgroups $(H_{\iota(e_j)})\theta_{e_j} = H_{\tau(e_j)}$, where $e_j \in E(X) \setminus E(T)$. So the fundamental group $G = \pi(\mathcal{G}, X)$ has the presentation $G = \pi(\mathcal{G}, X) = \langle t_j, G_T \mid t_j^{-1} H_{\iota(e_j)} t_j = H_{\tau(e_j)},\ e_j \in E(X) \setminus E(T) \rangle$ of an HNN-extension. The core H_j of each loop in the presentation of G as an HNN-extension is the core of an original circuit C_j in X with $e_j \in E(C_j)$.

Lemma 1. *Let G be a group. Suppose that for some $g \in G$, $g \neq 1$, the equation $x^{p^n} = g$, p-prime, has solutions for infinitely many $n \in \mathbb{Z}^+$. Then the group G is not $\mathcal{RF}_p$.*

Proof. Let N be a normal subgroup of G such that $g \notin N$ and $|G/N| = p^m$. For $n > m$ there exists $x \in G$ with $x^{p^n} = g$, which imiplies $g \in N$. So G is not $\mathcal{RF}_p$. □

A simple example of a group which is abelian but not $\mathcal{RF}_p$, is the quasicyclic p-group $C_{p}\infty = \langle a_1, a_2, \dots \mid a_1^p = 1, a_{i+1}^p = a_i \rangle$.

As a Corollary of Proposition 2 of [8] we have:

Proposition 2. *The semi-direct product $G = K]kH$ of a finite p-group K by a $\mathcal{RF}_p$ group H is $\mathcal{RF}_p$ if and only if it is $\mathcal{RN}$.*

Proof. Suppose that G is $\mathcal{RN}$. Let $\pi : H \to \mathrm{Aut}K$ be the induced homomorphism via the semi-direct product $G = K]H$. Let $h \in H$ be such that $h\pi \in \mathrm{Aut}K$ has order $o(h\pi) = p^\xi \cdot q$, $q \neq 1, (p, q) = 1$. Then for every $k \in K$ we have $(k)h\pi = h^{-1}kh$, but there exist $k \in K$ such that $h^{-p^\xi} k h^{p^\xi} \neq k$. So $[k, h^{p^\xi}] \neq 1$ with $[k, h^{p^\xi}]^q \equiv [k, h^{p^\xi q}] \mod \gamma_3(G)$ and $[k, h^{p^\xi}]^{o(k)} \equiv [k^{o(k)}, h^{p^\xi q}] \mod \gamma_3(G)$. Therefore $[k, h^{p^\xi}] \in \gamma_3(G)$. Similarly $[k, h^{p^\xi}] \in \gamma_n(G)$ for every $n \in \mathbb{N}$, a contradiction. So every element of $H\pi$ has an order of power p and the result follows by Proposition 2 of [8]. □

The following proposition is a generalization of Lemma 3 of [8]. The proof is similar and it is omitted.

Proposition 3. *Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups over the connected finite graph X with vertex groups G_v, $v \in V(X)$ finite p-groups. The fundamental group $g = \pi(\mathcal{G}, X)$ is $\mathcal{RF}_p$ if and only if it is $\mathcal{RN}$.*

Corollary 3.1. *Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups as in Proposition 3 and C be a circuit in the graph X. Suppose that the fundamental group $\pi(\mathcal{G}, X)$ is $\mathcal{RN}$, then the automorphism θ_i of the core H of C defined above has order a power of p.*

Proof. It is immediate from Propositions above and Proposition 2 of [8]. □

Lemma 4. *Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups over the graph X with vertex groups G_v, $v \in V(X)$. Suppose that the equation $x^n = g$, for a $g \in \pi(\mathcal{G}, X)$, has solutions for infinitely many $n \in \mathbb{Z}^+$. Then g belongs to a conjugate of a vertex group. So every solution x_n of $x^n = g$ belongs to a conjugate of a vertex group.*

Proof. Without any loss of generality we can suppose that g is cyclically reduced with reduced length $l(g) \geq 2$. (For the definition of the length of g see [2] §1.11). Then for infnitely many $n \in \mathbb{Z}^+$ we have $g = r_n^{-1} x_n^n r_n$, with $r_n, x_n \in G$ and x_n cyclically reduced. If $l(x_n) > 1$, then we have $x_n^n = r_n g r_n^{-1}$ with x_n^n cyclically reduced and $l(x_n^n) = n \cdot l(x_n)$. Thus the cyclically reduced length of g is arbitrarily large, a contradiction. Whence $l(x_n) = 1$, which implies that x_n and g belong to a conjugate of a vertex group. □

3. The residual p-finiteness of the fundamental group of a graph of finite groups.

Proposition 5. *Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups over the connected graph X with finite vertex groups G_v, $v \in V(X)$. Suppose that the fundamental group $G = \pi(\mathcal{G}, X)$ is $\mathcal{RN}$. Then we have:*

(i) The vertex groups G_v, $v \in V(X)$ are nilpotent groups.

(ii) There exists a prime p such that $\left|G_v : H_{\iota(e)}\right| = p^{r(e)}$, $(v = \iota(e))$, for all $v \in V(X), e \in E(X)$.

(iii) There exist $p_1, \ldots, p_k$ primes such that for each $v \in V(X)$ $G_v = \prod_{s=1}^k S_{p_s}(G_v)$, where $S_{p_s}(G_v)$ is the p_s-Sylow subgroup of G_v.

Proof. (i) Since the group G is $\mathcal{RN}$ and the vertex groups finite, they must be nilpotent.

(ii) Let $H_{\iota(e)}$ be an edge group which is proper subgroup of the vertex group G_v, $(v = \iota(e))$. So $H_{\iota(e)} \cap S_p(G_v)$ is a proper subgroup of $S_p(G_v)$ for a prime p. Suppose that for another vertex group G_u there exists another prime q such that the edge group $H_{\iota(e')}$, $(u = \iota(e'))$, is proper and $H_{\iota(e')} \cap S_q(G_u)$ is a proper subgroup of $S_q(G_u)$. Let $a \in S_p(G_v) \setminus (H_{\iota(e)} \cap S_p(G_v))$, $a \neq 1$ and $b \in S_q(G_u) \setminus (H_{\iota(e')} \cap S_q(G_u)) \cap b \neq 1$. Then $[a,b] \neq 1$ and $[a,b]^{o(\alpha)} \equiv [a^{o(\alpha)}, b] \mod \gamma_3(G)$, $[a,b]^{o(b)} \equiv [a, b^{o(b)}] \mod \gamma_3(G)$. So $[a,b] \in \gamma_3(G)$, because $(o(a), o(b)) = 1$. Similarly, $[a,b] \in \gamma_n(G)$ for every $n \in \mathbb{N}$, a contradiction, since G is $\mathcal{RN}$. So $H_{\iota(e)} \cap S_p(G_v)$ is a proper subgroup of $S_p(G_v)$ for the same prime p for all $v \in V(X)$ and $e \in E(X), (\iota(e) = v)$.

(iii) Since $\left|G_v : H_{\iota(e)}\right| = p^{r(e)}$ for all $v \in V(X)$, $e \in E(X)$, $\iota(e) = v$, we have $H_{\iota(e)} = A_{\iota(e)} \times K_{\iota(e)}$, where $A_{\iota(e)} \leq S_p(G_v)$ and $K_{\iota(e)} = S_{p_2}(G_v) \times \cdots \times S_{p_k}(G_v)$ (if $p = p_1$). But $(H_{\iota(e)})\theta_e = H_{\tau(e)}$, so $H_{\tau(e)} = A_{\tau(e)} \times K_{\tau(e)}$ with $K_{\tau(e)} = S_{p_2}(G_u) \times \cdots \times S_{p_k}(G_u)$, $T(e) = u$ and $S_{pS}(G_v) \simeq S_{p_s}(G_u)$, $s = 2, \ldots, k$. Finally, since X is connected, $G_v = \prod_{s=1}^{k} S_{p_s}(G_v)$ for each $v \in V(X)$ with the same primes $p_1, \ldots, p_k$. $\square$

The proof above contains implicitly the proof of the:

Corollary 5.1. *With the assumptions of Proposition 5, for $e \in E(X)$ the subgroups $K_{\iota(e)}$ are normal in G.*

Proof. Let G_v be a vertex group. Then from the proposition above we have that $K_{\iota(e)} = K_{\iota(e')}$ for every $e, e' \in E(X)$ with $\iota(e) = \iota(e') = v$, let $K_v = K_{\iota(e)}$. So $(K_v)\theta_e = K_u$, where $u = T(e)$, for each $e \in E(X)$. Let T be a maximal tree of X, from the defining relations in G we have $K_v = (K_v)\theta_e = K_u$ for each $e \in E(T)$ with $\iota(e) = v$, $\tau(e) = u$, and $t_\iota^{-1} K_r t_e = K_s$, where t_e is the generator corresponding to the edge $e \in E(X) \setminus E(T)$ with $\iota(e) = r$, $\tau(e) = s$. But the graph is connected, so we have $K_v = K_u$ for every $v, u \in V(X)$ and finally $g^{-1} K_v g = K_v$ for $g \in G$, since each K_v is normal in the vertex group G_v. $\square$

It is easy to see that the normal subgroup $K = K_v$, for every $v \in V(X)$, belongs to the core of every circuit C of the graph X.

If we preserve the same underlying graph X and assign to each vertex $v \in V(X)$ the p_s-Sylow subgroup of G_v, say $S_{p_s}(G_v)$ we have for every $s = 1, \ldots, k$ a graph of groups $\mathcal{L}(\mathcal{G}_{p_s}, X)$ with edge subgroups $S_{p_s}(G_v) \cap$

$H_{\iota(e)} \simeq S_{p_s}(G_u) \cap H_{\tau(e)}$ for the edge $e \in E(X)$ with $\iota(e) = v$, $\tau(e) = u$. The fundamental groups $G_{p_s} = \pi(\mathcal{G}_{p_s}, X)$ for $s = 1, \dots, k$ are subgroups of $G = \pi(\mathcal{G}, X)$. From previous Corollary we have that each G_{p_s}, $s = 2, \dots, k$, is the semi-direct product of $S_{p_s}(G_v)$ by a free group $F = \langle t_m,\ m \in \Lambda\rangle$ of rank $r(F) = |E(X) \setminus E(T)|$, where T is a maximal tree of X. Since it is supposed that the fundamental group G is $\mathcal{RN}$, from Proposition 2 each G_{p_s} is $\mathcal{RF}_{p_s}$ for $s = 2, \dots, k$.

We can now state the converse of the Proposition 5.

Theorem 6. *The fundamental group $G = \pi(\mathcal{G}, X)$ of the graph of groups $\mathcal{L}(\mathcal{G}, X)$, where X is connected and the vertex groups G_v are finite, is $\mathcal{RN}$ if, and only, if we have:*

(i) *The vertex groups G_v, $v \in V(X)$ are nilpotent groups.*
(ii) *There exists a prime p such that $|G_v : H_{\iota(e)}| = p^{r(e)}$, $(v = \iota(e))$, for all $v \in V(X), e \in E(X)$.*
(iii) *There exist $p_1, \dots, p_k$ primes such that for each $v \in V(X)$ $G_v = \prod_{s=1}^{k} S_{p_s}(G_v)$, where $S_{p_s}(G_v)$ is the p_s-Sylow subgroup of G_v.*
(iv) *The group $G_{p_1} = \pi(G_{p_1}, X)$ is $\mathcal{RN}$, where $p = p_1$.*
(v) *For every circuit C of X with core H_c we have: The automorphism θ_i, defined above, has order divided only by the primes, which appear in the orders of the vertex groups of C.*

Proof. If the fundamental group $G = \pi(\mathcal{G}, X)$ is $\mathcal{RN}$, then (i)-(iii) follow from Proposition 5 and (iv) follows from the fact that the group $G_{p_1} = \pi(G_{p_1}, X)$ is a subgroup of $G = \pi(\mathcal{G}, X)$. Let C be a circuit of X with vertex groups $G_1, \dots, G_n$ and core H_c. For each $s = 1, \dots, k$ let C_{p_s} be the (sub)circuit of C with vertex groups $S_{p_s}(G_i)$ $i = 1, \dots n$. The fundamental group of each C_{p_s} for $s = 1, \dots, k$ is a subgroup of $\pi(\mathcal{G}, X)$, so it is $\mathcal{RF}_{p_s}$ by Proposition 3. It is easy to see that the core H_c of C is the direct product of the cores H_{p_s} of C_{p_s} $s = 1, \dots, k$ and (v) follows from Corollary 3.1.

Conversely, the normal subgroup $K = \prod_{s=2}^{k} S_{p_s}(G_v)$ (Corollary 5.1) has the property $G/K \simeq G_{p_1}$, therefore $\cap_n \gamma_n(G) \leq K$, Since G_{p_i} is $\mathcal{RN}$. If $N = \langle S_{p_1}(G_v), v \in V(X)\rangle^G$, then $G/N \simeq K]F$, where F is the free group $F = \langle t_m, m \in \Lambda\rangle$ defined above. From (v) and Proposition 2 (cf. the converse in the proof of Th. 4 of [8]) we have that $K]F$ if $\mathcal{RN}$, so $\cap_n \gamma_n(G) \leq N$ and finally $\cap_n \gamma_n(G) \leq K \cap N = 1$. $\square$

The previous Theorem reduces the problem of the residual nilpotence of the fundamental group of a graph of finite groups to the problem of the residual nilpotence of the fundamental group of a graph of finite p-groups.

In the case where the graph is a segment or a loop there is a characterization of the fundamental groups (Theorem of [4] and Th. 13 of [8]). In the general case of a finite graph of finite p-groups the residual nilpotence and the residual p-finiteness coincide (Prop. 3) and we have an analogous proposition of the Proposition 1 of [8]. The proof is similar and it is omitted.

Proposition 7. *Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups over the connected finite graph X with vertex groups G_v, $v \in V(X)$, finite p-groups. The fundamental group $G = \pi(\mathcal{G}, X)$ is $\mathcal{RF}_p$ if and only if there exists a finite p-group Y, such that the vertex groups G_v are subgroups of Y and for each generator t_e of G there is a $g_e \in Y$ such that $\tau_{g_e|H_{\iota(e)}} = \theta_e$, where $H_{\tau(e)} = H_{\iota(e)}\theta_e = t_e^{-1}H_{\iota(e)}t_e$, $e \in E(X) \setminus E(T)$, where T is a maximal tree of X.* □

A necessary condition for the $\mathcal{RF}_p$ of the fundamental group $G = \pi(\mathcal{G}, X)$, which depends upon an internal condition on the edge groups, is given by Corollary 3.1. In the case, where the vertex groups are finite abelian p-groups, we can give a sufficient condition, which depends on the isomormorphisms associating the edge groups.

Before stating this condition we shall give the following necessary background.

As it is known every finite abelian p-group K has a decomposition $K = C_1 \times C_2 \times \cdots \times C_n$ as a direct product of cyclic subgroups with $|C_i| = p^{m_i}$ and $m_1 \geq m_2 \geq \cdots \geq m_n$. The number n depends upon the group G and it is the p-rank of $K(r_p(K) = n)$.

Lemma 8. *Let K be a finite abelian p-group and $H \leq K$ with $r_p(K) = r_p(H) = r$. If $\varphi \in Aut(H)$, then there exists a finite abelian p-group X with $r_p(X) = r$ and $\overline{\varphi} \in Aut(X)$ such that $K \leq X, \overline{\varphi}_{|H} = \varphi$ and $o(\overline{\varphi}) = o(\varphi)$.*

Proof. This comes from the proof of Proposition 3 of [1] by taking $A = B = H$ where it is easy to see that $o(\overline{\varphi}) = o(\varphi)$. □

Lemma 9. *Let K be a finite abelian p-group, $A, B \leq K$ and $\varphi : A \to B$ an isomorphism. Then we can find a finite abelian p-group X and an* *automorphism* θ *of* X *such that* $K \leq X, o(\theta) = p^{\epsilon}$ *for some* $\epsilon \in \mathbb{N}$ *and* $\theta_{|A} = \varphi$, *if and only if* $o(\varphi_{|H}) = p^{v}$ *for some* $v \in \mathbb{N}$, *where* H *is the "largest" subgroup of* K *such that* $H\varphi = H$.

Proof. If $H = 1$, then we have Lemma 6 in [8]. Suppose that $H \neq 1$ and the order of $\varphi_{|H}$ is not a power of p, then it is clear that there is no group X such that $K \leq X$ and $\theta \in \operatorname{Aut}X$ with $o(\theta) = p^{\epsilon}$ and $\theta_{|H} = \varphi$. So we may assume that the order of $\varphi_{|H}$ must be a power of p. Let $\{k_1, \ldots, k_r\}$ be a set of generators of K such that $H = \langle k_i^{p^{m_1}}, \ldots, k_s^{p^{m_s}}\rangle$, $s \leq r$ and $L = \langle k_1, \ldots, k_r\rangle$. Namely L is a minimal direct factor of K such that $H \leq L(r_p(L) = r_p(H))$. The isomorphism φ induces an isomorphism $\overline{\varphi} : A/H \to B/H$, since the "largest" subgroup of K/H with the property $(M/H)\overline{\varphi} = M/H$ is the trivial group, the groups K/H, $A/H, B/H$ and the isomorphism $\overline{\varphi}$ satisfy the assumptions of Lemma 6 in [8]. So there exists a finite abelian p-group Y such that $K/H \leq Y$ and $\theta_1 \in \operatorname{Aut}(Y)$ with $o(\theta_1) = p^{\epsilon_1}$ and $\theta_{1|A/H} = \overline{\varphi}$. The groups L, H and the isomorphism φ satisfy the assumptions of the previous Lemma. So there exists a finite abelian p-group W such that $L \leq W$ and $\theta_2 \in \operatorname{Aut}(W)$ with $o(\theta_2) = p^{\epsilon_2}$ and $\theta_{2|H} = \varphi$.

Let $K = N \times L$, for every element $k \in K$ there exist unique $x \in N, y \in L$ such that $k = x \cdot y$. Since $H \leq L$, it is easy to see that the map $f : K \to K/H \times L$ with $kf = (kH, y)$ defines a monomorphism. So X can be chosen to be the direct product $X = Y \times W$ and θ be the automorphism $\theta = (\theta_1, \theta_2)$ which satisfy the requirements of the statements. $\square$

Remark 1. The hypotheses in the Lemma above are the same as in Proposition 7 in [8], but the result is sharper, since the group K is embedded in a finite abelian p-group X.

Remark 2. It follows from the proofs of the Proposition 3 in [1] and Lemma 6 in [8] that the group X has a decomposition as a direct product of cyclic subgroups of the same order, say p^{τ}, which is equal to the maximal order of the elements of the group K.

Proposition 10. *Let* K *be a finite abelian p-group,* $A_1, B_1, A_2, B_2 \leq K$ *and* $\varphi_1 : A_1 \to B_1, \varphi_2 : A_2 \to B_2$ *isomorphisms. Then there exists a finite abelian p-group* V *with* $K \leq V$, $\theta_1, \theta_2 \in Aut(V)$ *with* $o(\theta_1) = p^{\epsilon_1}, o(\theta_2) = p^{\epsilon_2}$ *and* $\theta_{1|A_1} = \varphi_1, \theta_{2|A_2} = \varphi_2$ *if and only if there exist* $v_1, v_2 \geq 0$ *such that*

$o(\varphi_{1|H_1}) = p^{v_i}, o(\varphi_{2|H_2}) = p^{v_2}$, where H_1, H_2 are the "largest" subgroups of K such that $H_1\varphi_1 = H_1, H_2\varphi_2 = H_2$.

Proof. The previous Lemma ensures the existence of a finite abelian p-group X and an automorphism $\theta_1 \in \mathrm{Aut}(X)$ such that $K \leq X$, $\theta_{1|A_1} = \varphi_1$ and $o(\theta_1) = p^{\epsilon_1}$ if and only if $o(\varphi_{1|H_1}) = p^{v_1}$. Again, by Lemma 9, we can embed the group X into a finite abelian p-group, say V, such that there exists an automorphism $\theta_2 \in \mathrm{Aut}(V)$ with $\theta_{2|A_2} = \varphi_2$, $o(\theta_2) = p^{\epsilon_2}$ if and only if $o(\varphi_{2|H_2}) = p^{v_2}$. From the remark 2 above the elements of maximal order of the group V have the same order p^r. So $V = X \times N$, whence the automorphism θ_1 of X can be extended easily to an automorphism $\overline{\theta}_1 \in \mathrm{Aut}(V)$ with the required properties and we have finished. □

Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups over the tree X, with vertex groups G_v, $v \in V(X)$, finite abelian p-groups of bounded p-rank. Let $G_{v(0)}$ be a vertex group of maximal p-rank ($k = r_p(G_{v(0)}) \geq r_p(G_{v(i)})$, $v(i) \in v(x)$). Let $R = C_1 \times \cdots \times C_k$ be the direct product of k copies of the quasi-cyclic p-group $C_p\infty$. Let $\epsilon_0 : G_{v(0)} \to R$ be the (obvious) embedding of $G_{v(0)}$ in R. In the fundamental group $G = \pi(\mathcal{G}, X)$, if a neighbouring vertex group of $G_{v(0)}$ is the $G_{v(1)}$, we have $H_{\iota(e)}\theta_e = H_{\tau(\theta)}$ for the edge groups ($i(e) = v(0), \tau(e) = v(1)$). So we have an embedding $\epsilon_1 = \theta_e^{-1} \circ \epsilon_0 : H_{\tau(e)} \to R$. Let $\overline{H}_{\tau(e)}$ be a minimal direct factor of $G_{v(1)}$ such that $H_{\tau(e)} \leq \overline{H}_{\tau(e)}((r_p(H_{\tau(e)}) = r_p(\overline{H}_\tau(e))$ and $G_1 = \overline{H}_{\tau(e)} \times L_1))$. Since R is divisible, ϵ_1 can be extended to a monomorphism $\epsilon_1 : \overline{H}_{\tau(e)} \to R$ and finally, since $r_p(G_{v(1)} \leq r_p(G_{v(0)}))$, we can have an embedding $\epsilon_1 : G_{v(1)} \to R$ (The complement L_1 of $\overline{H}_{\tau(e)}$ in $G_{v(1)}$ is embedded into M_1, a complement of $\overline{\overline{H}_{\tau(e)}\epsilon_1}$ in R, where $\overline{\overline{H}_{\tau(e)}\epsilon_1}$ is a minimal factor of R of $r_p(\overline{H}_{\tau(e)})$ many copies of $C_p\infty$ such that $\overline{H}_{\tau(e)}\epsilon_1 \leq \overline{\overline{H}_{\tau(e)}\epsilon_1}$, ($R = \overline{\overline{H}_{\tau(e)}\epsilon_1} \times M_1$)). Similarly continuing we can embed all the vertex groups $G_{v(i)}$, $v(i) \in V(X)$, with monomorphisms $\epsilon_i : G_{v(i)} \to R$. Since in the fundamental group we have the relations $H_{\tau(e)}\theta_e = H_{\tau(e)}$ for $e \in E(X)$, the monomorphisms ϵ_i can be extended to a common homomorphism $\epsilon : G = \pi(\mathcal{G}, X) \to R$. The pair $(\epsilon, G\epsilon)$ is a *realization* of $\pi(\mathcal{G}, X)$ in R with respect to the vertex groups $G_{v(i)}$, $v(i) \in V(X)$ (cf. [9]). It is easy to see that $Ker\ \epsilon \cap G_{v(i)} = 1$ for $v(i) \in V(X)$. Therefore the kernel of ϵ is a free group.

Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups over the connected finite graph X

with vertex groups finite abelian p-groups. Suppose that in each circuit C with core H the induced automorphism φ has order a power of p. If T is a maximal tree of X and $G_T = \pi(\mathcal{G}, T)$ the corresponding fundamental group over T with vertex groups $G_{v(i)}$, $v(i) \in V(X) = V(T)$ and edge groups $H_{\iota(e)}\theta_e = H_{\tau(e)}$, $e \in E(T)$, then in the realization $(\epsilon, G_T\epsilon)$ of G_T the image $(G_T)\epsilon = K$ is a finite abelian p-group. For each $e_j \in E(X) \setminus E(T)$ the isomorphism θ_{e_j}, where $(H_{\iota(e_j)})\theta_{e_j}$, induces an isomorphism $\varphi_j : (H_{\iota(e_j)})\epsilon \to (H_{\tau(e_j)})\epsilon$ in the realization of G_τ. By Proposition 10 there is a finite abelian p-group V with $K \leq V$ and $\theta_1, \theta_2, \dots, \theta_v \in \mathrm{Aut}V$ with $o(\theta_j) = p^{\lambda_j}$, $\theta_{j|H_j\epsilon} = \varphi_j$, where H_j are the cores of the loops in the resulting graph of groups by contracting the maximal tree T to a point. Let Y be the subgroup of the holomorph of V defined by the right regular representation $R(V)$ of V and the group $W = \langle \theta_1, \theta_2, \dots \theta_v \rangle$ $(Y = R(V)]W)$. The map $\bar{\epsilon} : G = \pi(\mathcal{G}, X) \to Y$ defined by $G_{v(i)}\bar{\epsilon} = G_{v(i)}\epsilon$ for $v(i) \in V(X)$ and $t_j\bar{\epsilon} = \theta_j$ for each generator which corresponds to the loop with core H_j, is well defined. The map $\bar{\epsilon}$ is a group homomorphism because the relations of the type $t_j^{-1}H_{i(e_j)}t_j = H_{\tau(e_j)}$ in $G = \pi(\mathcal{G}, X)$ are sent to $\theta_j^{-1}H_{\iota(e_j)}\epsilon \cdot \theta_j = (H_{\iota(e_j)}\epsilon)\theta_j = H_{\tau(e_j)}\epsilon$ in Y by $\bar{\epsilon}$. Also $Ker\ \bar{\epsilon} \cap G_{v(i)} = 1$, since $Ker\ \epsilon \cap G_{v(i)} = 1$.

We are now in a position to state.

Proposition 11. *Let $\mathcal{L}(\mathcal{G}, X), V, W, Y$ be as above. If W is a finite p-group, then the fundamental group $G = \pi(\mathcal{G}, X)$ is $\mathcal{RF}_p$.*

Proof. The group G is free by finite-p, since $Ker\ \bar{\epsilon}$ is a free group. □

The necessary condition stated in Corollary 3.1, where each automorphism θ_j corresponding to the circuit C_j, considered by itself, must have order a power of p, is not known to be sufficient, becasue we do not know if the group $W = \langle \theta_1, \theta_2, \dots, \theta_v \rangle$ is a p-group.

Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups with the properties

(i) The graph X is a tree.

(ii) There is a vertex $v(0)$ called the vertex of level zero.

(iii) The vertex $v(0)$ is joined with vertices of level one. Inductively a vertex $v(n)$ of level $n > 0$ is joined by an edge to exactly one vertex of level $n - 1$ (since X is a tree) and either it is a terminal vertex or the other vertices joined with it are of level $n + 1$.

(iv) All the vertex groups, except (possibly) the vertex group $G_{v(0)}$ of level zero, are finite abelian p-groups.

(v) If $G_{v(i)}$, $G_{v(i+1)}$ are two vertex groups of levels i and $i+1$ linked by the edge groups $H_{\iota(e)} \cong H_{\tau(e)}$, $(\iota(e) = v(i),\ \tau(e) = v(i+1))$, then $r_p(H_{\tau(e)}) = r_p(G_{v(i+1)})$.

The graph $\mathcal{L}(\mathcal{G}, X)$ with the properties (i)-(v) is said to be a *graph of strong cohesion.* So if $G_{v(0)}, G_{v(1)}, \ldots, G_{v(n)}, \ldots$ are the vertex groups along a path with starting point the vertex of level zero, then there exists an index n_0 such that $r_p(G_n) = r_p(G_{n_0})$ for every $n \geq n_0$.

Lemma 12. *Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of strong cohesion, where the exponents of $G_{v(i)}$, $v(i) \in V(X)$, $i \neq 0$ are bounded. The fundamental group $G = \pi(\mathcal{G}, X)$ has infinitely p-divisible elements if and only if the vertex group $G_{v(0)}$ has.*

Proof. Suppose that the equation $g = x^{p^n}$, $g \neq 1$, has solutions for infinitely many $n \in \mathbb{N}$. By Lemma 4, the element g and every root of $x^{p^n} = g$ belong to conjugates of vertex groups. Let p^m be an upper bound of the exponents of $G_{v(i)}$, $v(i) \in V(X)$ $i \neq 0$, then $x^{p^n} = 1$ for every $x \in G_{v(i)}$ for all $v(i) \in V(X)$ $i \neq 0$ and $n \geq m$, a contradiction, unless the roots of $x^{p^n} = g$ belong to $G_{v(0)}$ for almost all $n \in \mathbb{N}$. Therefore $G_{v(0)}$ has infinitely p-divisible elements. $\square$

Theorem 13. *Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of strong cohesion with the vertex group $G_{v(0)}$ of level zero a finite abelian p-group. For the fundamental group $G = \pi(\mathcal{G}, X)$ the following propositions are equivalent.*

(i) *The exponents of the vertex groups $G_{v(i)}, v(i) \in V(X)$ are bounded.*
(ii) *The realization $\epsilon : G \to R$ has finite image $G\epsilon$.*
(iii) *The group G is $\mathcal{RF}_p$.*
(iv) *The group G does not have infinitely p-divisible elements.*

Proof. (i) $\Rightarrow$ (ii) It is easy to see that the image $G\epsilon$ is finite, since the graph is of strong cohesion and the exponents of $G_{v(i)}, v(i) \in V(X)$ are bounded.

(ii) $\Rightarrow$ (iii) The group G is a free by finite p-group, since $Ker\ \epsilon$ is a free group.

(iii) $\Rightarrow$ (iv) Lemma 1.

(iv) $\Rightarrow$ (i) Suppose that the exponents of $G_{v(i)}$, $v(i) \in V(X)$ are not bounded. Then there is a sequence of $x_n \in G_{v(n)}$ in the vertex groups with unbounded orders. Because of the strong cohesion of $\mathcal{L}(\mathcal{G}, X)$ there is $m_n \in \mathbb{N}$ such that $1 \neq x_n^{p^{m_n}} \in G_{v(0)}$. The group $G_{v(0)}$ is finite so there are infinitely many of $x_n^{p^{m_n}}$ which coincide on a common element $x_0 \in G_{v(0)}$. Therefore $x_0 = x_n^{p^{m_n}}$, a contradiction, since the orders of x_n are unbounded and G has not infinitely p-divisible elements. □

Corollary 13.1. *If the tree X is finite, then the fundamental group $G = \pi(\mathcal{G}, X)$ is $\mathcal{RF}_p$.*

Corollary 13.2. *Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of finite abelain p-groups over the finite tree X. Without the assumption that $\mathcal{L}(\mathcal{G}, X)$ is of strong cohesion, the fundamental group $G = \pi(\mathcal{G}, X)$ is $\mathcal{RF}_p$.*

Proof. Because of the construction of the realization, since X is finite, we have that $G = \pi(\mathcal{G}, X)$ is a free by finite p-group. □

4. The residual p-finiteness of the fundamental group of a tree of free abelian groups.

Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups with vertex groups finitely generated free abelian groups over the tree X. Suppose that there is a vertex group $G_{v(0)}$ such that $r(G_{v(i)}) \leq r(G_{v(0)}) = k$, $v(i) \in V(X)$ where $r(G_{v(i)})$ is the (free) rank of $G_{v(i)}$. Let $R = \mathbb{Q} \times \cdots \times \mathbb{Q}$ be the direct product of k copies of $(\mathbb{Q}, +)$. Then the natural embedding of $G_{v(0)}$ in R, $\epsilon_0 : G_{v(0)} \to R$ can be extended to a homomorphism $\epsilon : G = \pi(\mathcal{G}, X) \to R$, by the definition of the fundamental group $G = \pi(\mathcal{G}, X)$. It is easy to see that $Ker\ \epsilon \cap G_{v(i)} = 1$, $v(i) \in V(X)$. So we have a realization of the fundamental group $G = \pi(\mathcal{G}, X)$.

In a similar way with definition preceding Lemma 12 we can define a graph of groups of strong cohesion in the case where the vertex groups are free abelian groups of finite rank. Simply we replace the statements (iv) and 9v) there by:

(iv') All the vertex groups, except (possibly) the vertex group $G_{v(0)}$ of level zero, are finitely generated free abelian groups.

(v') If $G_{v(i)}, G_{v(i+1)}$ are two vertex groups of levels i and $i+1$ linked by the edge groups $H_{\iota(e)} \cong H_{\tau(e)}, (\iota(e) = v(i),\ \tau(e) = v(i+1))$, then $r(H_{\tau(e)}) = r(G_{v(i+1)})$. Namely $H_{\tau(e)}$ is of finite index in $G_{v(i+1)}$.

Proposition 14. *Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups over the connected graph X with vertex groups G_v, $v \in V(X)$, finitely generated abelian groups.*

Suppose that the fundamental group $G = \pi(\mathcal{G}, X)$ is $\mathcal{RN}$. If G_v, G_u are two neighbouring vertex groups joined by the edge e $(\iota(e) = v, \tau(e) = u)$, then $\left|\overline{H}_{\iota(e)} : H_{\iota(e)}\right| = q^{v(e)}$ and $\left|\overline{H}_{\tau(e)} : H_{\tau(e)}\right| = q^{v(\overline{e})}$, for a prime q and $v(e)$, $v(\overline{e}) \geq 0$, where $\overline{H}_{\iota(e)}$ (resp. $\overline{H}_{\tau(e)}$) is the direct factor of $G_v(G_u)$ such that $H_{\iota(e)} \leq \overline{H}_{\iota(e)}$ $(H_{\tau(e)} \leq \overline{H}_{\tau(e)})$ and $r(\overline{H}_{\iota(e)}) = r(H_{\iota(e)})$ $(r(\overline{H}_{\iota(e)}) = r(H_{\tau(e)}))$. Moreover if G if $\mathcal{RF}_p$, then $q = p$ and $\left|\overline{H}_{\iota(e)} : H_{\iota(e)}\right| = p^{v(e)}$ for every $e \in E(X)$.

Proof. Let G_v, G_u be two neighboring vertex groups. Without loss of generality we can suppose that there is a $x \in \overline{H}_{\iota(e)} \setminus H_{\iota(e)}$. If there are two different primes p, q such that $p \mid \left|\overline{H}_{\iota(e)} : H_{\iota(e)}\right|$ and $q \mid \left|\overline{H}_{\tau(e)} : H_{\tau(e)}\right|$, then we can choose $x \in \bar{H}_{\iota(i)}$ and $y \in \overline{H}_{\tau(e)}$ such that $x^p \in H_{\iota(e)}$ and $y^q \in H_{\tau(e)}$ and since $x \notin H_{\iota(e)}$ we have $[x, y] \neq 1$ and $[x, y]^p \equiv [x^p, y] \equiv 1$ mod $\gamma_3(G)$, $[x, y]^q \equiv [x, y^q] \equiv 1$ mod $\gamma_3(G)$, whence $1 \neq [x, y] \in \gamma_3(G)$. Similarly $[x, y] \in \gamma_n(G)$ for every $n \in \mathbb{N}$. If $y \in G_u$, then $[x, y]N = [xN, yN] = [aN, yN]$, where $a \in H$. Hence $[x, y]N = [a, y]N = N$. Therefore $[x, y] \in N$ for every NG with $|G : N| = p^\lambda$, a contradiction, since G is $\mathcal{RN}$. so $p = q$.

If G is $\mathcal{RF}_p$ and there is a prime q such that $q \mid \left|\overline{H}_{\iota(e)} : H_{\iota(e)}\right|$, then for a $x \in G_v$ such that $x^q \in H_{\iota(e)}$, we have that $x \in \cap\{H_{\iota(e)}N \mid N \lhd G$ such that $|G : N| = p^\lambda\}$, a contradiction, since G is $\mathcal{RF}_p$ and as we point out above, we can find $x \in G_v$ such that $[x, y] \neq 1$. □

Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups of strong cohesion over the tree X. Suppose that the vertex group G_0 of level zero is a finitely generated free abelian group. Let $(\epsilon, G\epsilon)$ be the realization of the fundamental group $\pi(\mathcal{G}, S)$ in R $(R = \mathbb{Q} \times \cdots \times \mathbb{Q},$ k copies of $\mathbb{Q}$, $k = r(G_0))$. Let p be a prime and $f_n = \epsilon \circ \varphi_n$ $n \in \mathbb{N}$, where $\varphi_n : G\epsilon \to G\epsilon/(G\epsilon)^{p^n}$ is the natural epimorphism. If $K_n = Ker\ f_n$ $n \in \mathbb{N}$, then every element $gK_n \in G/K_n$ is of bounded order, since $(g^{p^n})f_n = (g^{p^n})\epsilon \circ \varphi_n = (g\epsilon)^{p^n}(G\epsilon)^{p^n} = (G\epsilon)^{p^n}$. Therefore each coset group $G_{v(i)}K_n/K_n$ is finite.

Proposition 14 gives a necessary condition for the $\mathcal{RF}_p$ of the fundamental group of a graph of groups with vertex groups *f.g.* abelian groups. Now we give a sufficient condition.

Proposition 15. *Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of groups of strong cohesion over the tree X with the vertex group G_0 of level zero a f.g. free abelian group.*

Moreover we suppose that for every $e \in E(X)$ there exists a prime p and $v(e) \geq 0$ such that $|\overline{H}_{\iota(e)}| : H_{\iota(e)} |= p^{v(e)}$. Then the following statements are equivalent.

(i) $\cap_n G^{p^n}\ Ker\ \epsilon = \ Ker\ \epsilon$

(ii) $\cap_n (G_{v(i)} \cap K_n) = 1,\ v(i) \in V(X)$.

(iii) *The realization $G\epsilon$ has no elements of infinite p-height.*

Proof. (i) $\Rightarrow$ (ii) By definition above $K_n = \ Ker\ f_n$, so $K_n = G^{p^n}\ Ker\ \epsilon$. But $\cap_n G^{p^n}\ Ker\ \epsilon = \ Ker\ \epsilon$ and $G_{v(i)} \cap \ Ker\ \epsilon = 1, v(i) \in V(X)$, whence $\cap_n (G_{v(i)} \cap K_n) = 1,\ v(i) \in V(X)$.

(ii) $\Rightarrow$ (iii) Let $g\epsilon \in G\epsilon$ be an element of infinite p-height. If $g = x_1 \cdots x_k,\ x_j \in G_{v(j)}$, then $g\epsilon = (y_n\epsilon)^{p^n}$ for infinitely many $n \in \mathbb{N}$. Since $x_j \in G_{v(j)}$, the graph is of strong cohesion and $G\epsilon$ is abelian, there exists $\lambda(g) = \lambda \in \mathbb{N}$ such that $x_j^{p^\lambda} \in G_0$ for $j = 1, \ldots, k$. Therefore $(g^{p^\lambda})\epsilon = (x_1^{p^\lambda} \cdots x_k^{p^\lambda})\epsilon = (y_n\epsilon)^{p^{n+\lambda}}$, and we can suppose that there is a $h \in G_0$ ($h = x_k^{p^\lambda}$) such that $h\epsilon$ has infinite p-height. So $h\epsilon \in \cap_n (G\epsilon)^{p^n}$, namely $h \in \cap_n K_n$ which implies $h \in \cap_n (G_0 \cap K_n) = 1$. So $x_1^{p^\lambda} \cdots x_k^{p^\lambda} = 1$ and $(y_n\epsilon)^{p+\lambda} = 1$, but $G\epsilon$ is torsion free, whence $y_n\epsilon = 1$ and $g\epsilon = 1$. Therefore the group $G\epsilon$ has not elements of infinite p-height. So $\cap_n (G\epsilon)^{p^n} = 1$.

(iii) $\Rightarrow$ (i) Since $\cap_n (G\epsilon)^{p^n} = 1$, we have $1 = \cap_n (G\epsilon)^{p^n} = \cap_n (G^{p^n})\epsilon = \cap_n (G^{p^n}\ Ker\ \epsilon)\epsilon \supseteq (\cap_n G^{p^n}\ Ker\ \epsilon)\epsilon$. Therefore $\cap_n G^{p^n}\ Ker\ \epsilon \subseteq \ Ker\ \epsilon$. □

Corollary 15.1. *Suppose that one of the (i) - (iii) of the Proposition 15 is valid, then the fundamental group $G = \pi(\mathcal{G}, X)$ is $\mathcal{RF}_p$.*

Proof. For every $n \in \mathbb{N}$ we can define a new graph of groups, say $\mathcal{L}(\mathcal{G}_n, X)$, over the same underlying tree X, where to each vertex $v(i) \in V(X)$ we assign the group $G_{v(i)}K_n/K_n$ and to each edge $e \in E(X)$ with $\iota(e) = v(i)$ we assign the "induced" subgroups $H_{\iota(e)}K_n/K_n \cong H_{\tau(e)}K_n/K_n$. If $r(H_{\tau(e)}) = r(G_{v(i)})$, where $\tau(e) = v(i)$, (the graph $\mathcal{L}(\mathcal{G}, X)$ is of strong cohesion), then there exists a sufficiently large $n \in \mathbb{N}$ such that $r_p(H_{T(e)}K_n/K_n) = r_p(G_{v(i)}K_n/K_n)$. So the graph $\mathcal{L}(\mathcal{G}_n, X)$ is of strong cohesion. Therefore the fundamental group $\overline{G}_n = \pi(\mathcal{G}_n, X)$ is $\mathcal{RF}_p$ from Theorem 13. The epimorphism $\theta_i : G_{v(i)} \to G_{v(i)}K_n/K_n,\ v(i) \in V(X)$, evidently, can be extended to a common homomorphism $\overline{\theta} : G = \pi(\mathcal{G}, X) \to \overline{G}_n = \pi(\mathcal{G}_n, X)$.

Since $\cap_n(G_{v(i)} \cap K_n) = 1,\ v(i) \in V(X)$, we have that for $g \in G,\ g \neq 1$, there exists $n \in \mathbb{N}$ such that $g\overline{\theta} \neq 1$, and finally G is $\mathcal{RF}_p$. □

Corollary 15.2. *Let $\mathcal{L}(\mathcal{G}, X)$ be a graph of finitely generated free abelian groups over the finite tree X. without the assumption that $\mathcal{L}(\mathcal{G}, X)$ is of strong cohesion, the fundamental group $G = \pi(\mathcal{G}, X)$ is $\mathcal{RF}_p$ if and only if $\left|\overline{H}_{i(e)} : H_{i(e)}\right| = p^{v(e)}$ for every $e \in E(X)$.*

Proof. It is clear that the realization $G\epsilon$ of G is finitely generated, since X is finite. So the result follows from the Proposition 14 and the proof of the Proposition 15. □

References

1. S. Andreadakis, E. Raptis, D. Varsos, *Extending isomorphisms to automorphisms*, Arch. Math. **53** (1989), 121-125.
2. H. Bass, *Covering theory for graphs of groups*, J.P.A.A. **89** (1993), 3-47.
3. K.W. Gruenberg, *Residual properties of infinite soluble groups*, Proc. London Math. Soc. **7** (1957), 29-62.
4. G. Higman, *Amalgams of p-Groups*, J. Algebra **1** (1964), 301-305.
5. A. L. Lichtman, *Necessary and sufficient conditions for the residual nilpotence of free products of groups*, J.P.A.A. **12** (1978), 49-64.
6. A.I. Malc'ev, *Generalized nilpotent Algebras and their adjoint groups*, Amer. Math. Soc. Translations (2) **69** (1968), 1-21.
7. E. Raptis, D. Varsos, *Residual properties of HNN-extensions with base group an abelian group*, J.P.A.A. **59** (1989), 285-290.
8. ———, *The residual nilpotence of HNN-extensions with base group a finite or a fg abelian group*, J.P.A.A. **76** (1991), 167-178.
9. ———, *On the subgroup separability of the fundamental group of a finite graph of groups*, (To appear in Demonstratio Mathematica).
10. J.P. Serre, *Trees*, Springer-Verlag.

Received November 15, 1993

Revised version received May 6, 1995

Department of Mathematics, University of Athens, Panepistemiopolis 15784, Athens, GREECE