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Tree Products of Conjugacy Separable Groups

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A group G is said to be conjugacy separable if for each pair of elements $x, y \in G$ such that x and y are not conjugate in G, there exists a finite homomorphic image $\bar{G}$ of G such that the images of x, y are not conjugate in $\bar{G}$. In this note, we show that the tree products of finitely many conjugacy separable and subgroup separable groups amalgamating central subgroups with trivial intersections are conjugacy separable. We then apply our results to polycyclic-by-finite groups and free-by-finite groups.

Key Words: generalised free product; tree product; conjugacy separable; polycyclic-by-finite groups; free-by-finite groups

1. INTRODUCTION

Baumslag [3] proved that the generalised free products of two polycyclic-by-finite groups amalgamating a central subgroup are residually finite. In this note we shall prove that the tree products of finitely many conjugacy separable and subgroup separable groups amalgamating central subgroups with trivial intersections are conjugacy separable. This generalises our result in [27]. As a consequence the tree products of finitely many polycyclic-by-finite groups or free-by-finite groups amalgamating central subgroups with trivial intersections are conjugacy separable. More precisely we shall prove the following:

THEOREM 2. *Let $G_1, G_2, \ldots, G_n$ be conjugacy separable and subgroup separable groups. Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the finitely generated subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$, where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Then G is conjugacy separable.*

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We recall that a group G is said to be conjugacy separable if for each pair of elements $x, y \in G$ such that x and y are not conjugate in G, there exists a finite homomorphic image $\bar{G}$ of G such that the images of x, y are not conjugate in $\bar{G}$. It is well known that free groups, finitely generated nilpotent groups, and surface groups are conjugacy separable (Stebe [21]; Blackburn [4]; Scott [20]). Dyer [5], Formanek [8], Remeslennikov [18], and Fine and Rosenberger [7] extended these results by showing that free-by-finite groups, polycyclic-by-finite groups, and Fuchsian groups (finite extension of surface groups) are conjugacy separable, respectively. The free products of conjugacy separable groups are shown to be conjugacy separable by Stebe [21]. The systematic study of the conjugacy separability of generalised free products began with Stebe [21]. Dyer [6] completed Stebe's work by showing that the generalised free products of two free groups or finitely generated nilpotent groups amalgamating a cyclic subgroup is conjugacy separable. By using the structure of certain generalised free products, Tang [24] and Allenby and Tang [2] then showed that certain one-relator groups with torsion are conjugacy separable. More recently Allenby [1] showed that certain one-relator products are conjugacy separable by using similar methods.

The study of the conjugacy separability of generalised free products has been continued by many mathematicians. Their results can be briefly summarised as follows: Let G be a generalised free product of two groups amalgamating a cyclic subgroup. Then G is conjugacy separable if the two factor groups are free groups or finitely generated nilpotent groups (Dyer [6]) or surface groups (Tang [26]) or finitely generated Fuchsian groups (Kim and Tang [11]) or polycyclic groups (Ribes, Segal and Zaleskii: see the paper by Kim and Tang [11]) or free-by-finite groups or finitely generated nilpotent groups (first proved by Tang [25] with some additional conditions and later in general by Ribes and Zalesskii [19]). More recently Kim and Tang [13] showed that the tree products of finitely many conjugacy separable residually finitely generated torsion-free nilpotent groups amalgamating cyclic subgroups are conjugacy separable. As a consequence the tree products of finitely many free groups, finitely generated torsion-free nilpotent groups, or surface groups amalgamating cyclic subgroups are conjugacy separable. In the same paper, Kim and Tang [13] asked whether the tree products of finitely many Fuchsian groups or the tree products of finitely many polycyclic groups amalgamating cyclic subgroups are conjugacy separable. Our theorem complements these results.

We shall proceed to prove our results as follows: in Section 2 we state the definitions and essential lemmas; in Section 3 we give a brief outline of the proof of Theorem 2 and state the lemmas (without proofs) used directly in the proof; in Section 4 we prove Theorem 2 and give some applications to polycyclic-by-finite groups and free-by-finite groups and finally in Section 5 we give the proofs of all the lemmas.

The notations used in this note are standard. In addition, the following will be used for any group G and $x, y \in G$:

$N \triangleleft_f G$ means N is a normal subgroup of finite index in G.

$Z(G)$ denotes the centre of G.

$x \sim_G y$ means x is conjugate to y in G.

$\{x\}^G$ denotes the conjugacy class of x in G.

K^x means $x^{-1}Kx$ for any subgroup K in G.

If G is a generalised free product and $x \in G$, then $\|x\|$ will denote the usual reduced length of x.

2. PRELIMINARIES

In this section, we give the main definitions and essential lemmas. We begin with a theorem of Magnus, Karrass, and Solitar [16].

LEMMA 1. *Let $E = G_1 *_H G_2$ and let $x \in E$ be of minimal length in its conjugacy class. Suppose that $y \in E$ is cyclically reduced and $y \sim_E x$.*

(a) *If $\|x\| = 0$, then $\|y\| \leq 1$ and if $y \in G_1$ say, there exists a sequence $h_1, h_2, \ldots, h_r$ of elements in H such that $y \sim_{G_1} h_1 \sim_{G_2} h_2 \sim_{G_1} \cdots \sim_{G_2} h_r = x$.*

(b) *If $\|x\| = 1$, then $\|y\| = 1$ and either $x, y \in G_1$ and $x \sim_{G_1} y$ or else $x, y \in G_2$ and $x \sim_{G_2} y$.*

(c) *If $\|x\| \geq 2$, then $\|y\| = \|x\|$ and $y \sim_H x'$ where x' is a cyclic permutation of x.*

Let $x = u_1 u_2 \cdots u_r$, $y = v_1 v_2 \cdots v_r$, where $r \geq 2$, be cyclically reduced elements of E as in Lemma 1. Consider the system of equations

$$\begin{aligned} u_{i+1} &= k_0^{-1} v_1 k_1 \\ u_{i+2} &= k_1^{-1} v_2 k_2 \\ &\vdots \\ u_{i+r} &= k_{r-1}^{-1} v_r k_r. \end{aligned} \qquad \text{(Ii)}$$

The system of equations (Ii) has a solution of H if and only if there exists a finite sequence $h_0, h_1, \ldots, h_r$ of elements in H which satisfy the system of equations (Ii) simultaneously and $h_r = h_0$. This is equivalent to $x' =$

$h_0^{-1}yh_0$, where $x' = u_i u_{i+1} \cdots u_{i-1}$ is a cyclic permutation of x. So $x \sim_E y$ if and only if the system of equations (Ii) has a solution in H for some i, $0 \le i < r$. (See Dyer [6].)

We recall the definition of a conjugacy separable group.

DEFINITION 1. A group G is said to be conjugacy separable (c.s.) if for each pair of elements x, $y \in G$ such that x and y are not conjugate in G, there exists a finite homomorphic image $\bar{G}$ of G such that the images of x, y are not conjugate in $\bar{G}$.

LEMMA 2. *Let $E = G_1 *_H G_2$, where G_1 and G_2 are finite groups. Then E is conjugacy separable.*

Proof. Theorem 1 in Dyer [6].

Next we give the definition of a subgroup separable group.

DEFINITION 2. A group G is called H-separable for the subgroup H if for each $x \in G \backslash H$, there exists $N \lhd_f G$ such that $x \notin HN$.

G is termed subgroup separable if G is H-separable for every finitely generated subgroup H. Similarly G is HK-separable for the subgroups H, K if for each $x \in G \backslash HK$, there exists $N \lhd_f G$ such that $g \notin (HK)N$.

LEMMA 3. *Let G be free-by-finite. Then G is HK-separable for all finitely generated subgroups H, K of G. In particular, G is subgroup separable.*

Proof. Lemma 3.2 in Tang [25].

We note that if $E = G_1 *_H G_2$, where G_1 and G_2 are finite groups, then E is free-by-finite and hence E is HK-separable for all finitely generated subgroups H, K of E.

DEFINITION 3. Let $G_1, G_2, \ldots, G_n$ be groups and let H_{ij} be a subgroup of G_i. Then $G = \langle G_1, G_2, \ldots, G_n;\ H_{ij} = H_{ji} \rangle$ shall denote a tree product of the $G_1, G_2, \ldots, G_n$ with the subgroups H_{ij} of G_i and H_{ji} of G_j amalgamated.

3. OUTLINE OF THE PROOF OF THEOREM 2

In [27], we proved the following:

THEOREM 1. *Let $E = G_1 *_H G_2$, where $G_1 \cap G_2 = H$ and $H \subseteq Z(G_1) \cap Z(G_2)$. Suppose that each G_i is conjugacy separable and subgroup separable. Then E is conjugacy separable.*

In this note, we shall extend Theorem 1 to Theorem 2. First we give a brief description of the methods we shall use. The proof of Theorem 2 is based on methods and techniques first developed by Stebe [22] and Dyer [6]. Later Allenby, Kim, McCarron, and Tang independently and jointly, further developed, expanded, and refined these and other new methods in their papers [1, 2, 11, 13–15, 24–26]. Among the many concepts used and introduced by Allenby, Kim, McCarron, and Tang, the following or similar properties will be used in the proof of Theorem 2:

Let G be a group and let H be a finitely generated subgroup of G.

(i) Let $R \lhd_f H$. Then there exists $N \lhd_f G$ such that $N \cap H = R$.

(ii) Let $x \in G \backslash H$. Then there exists $N \lhd_f G$ such that $x \notin HN$.

(iii) Let $\{x\}^G \cap H = \emptyset$, where $x \in G$. Then there exists $P \lhd_f G$ such that $\{\bar{x}\}^{\bar{G}} \cap \bar{H} = \emptyset$ in $\bar{G} = G/P$.

(iv) Let $x \notin HyH$, where $x, y \in G$. Then there exists $P \lhd_f G$ such that $\bar{x} \notin \bar{H}\bar{y}\bar{H}$ in $\bar{G} = G/P$.

Properties (i), (ii), (iii), and (iv) are derived from the definitions of potency (see Allenby and Tang [2]), subgroup separability, subgroup conjugacy separability (see Kim and Tang [12]), and double coset separability, respectively.

Our proof of Theorem 2 follows the method of Allenby, Kim, McCarron, and Tang which is based on the criterion of Magnus, Karrass, and Solitar (Lemma 1 above). We shall use the theorem of Dyer (Lemma 2 above) and the properties (i)–(iv) repeatedly. However these properties and theorems together are not sufficient to prove the conjugacy separability of a generalised free product in general. The properties of the factor groups and the type of subgroup amalgamated play an important role in determining the conjugacy separability of the generalised free product. This is where Allenby, Kim, McCarron, and Tang have made many contributions.

We now give a brief outline of the proof of Theorem 2. Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji} \rangle$ be a tree product of conjugacy separable and subgroup separable groups $G_1, G_2, \ldots, G_n$ amalgamating the finitely generated subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$, where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. To prove that G is conjugacy separable, we use induction on n. The case $n = 2$ follows from Theorem 1. Let $n \geq 3$. The tree product G has an extremal vertex, say G_n, which is joined to a unique vertex, say G_{n-1}. The subgroup of G generated by $G_1, G_2, \ldots, G_{n-1}$ is just their tree product. Let E denote this subgroup. Then we write $G = E *_H G_n$, where $H = H_{(n-1)n} = H_{n(n-1)}$. This implies that G is a generalised free product of the two groups E and G_n with the subgroup H amalgamated. Note that H is central only in G_n.

Next we apply the criterion of Magnus, Karrass, and Solitar (Lemma 1 above). To do that we need to show that E is conjugacy separable and E has the properties (i)–(iv). In addition, E must have the following property:

(v) Let $H^x \cap H = 1, H^y \cap H = 1$, where $x, y \in G$ and $R \triangleleft_f H$. Then there exists $P \triangleleft_f G$ such that $P \cap H \subseteq R$ and $H^x P \cap HP = P$, $H^y P \cap HP = P$.

By induction, E is conjugacy separable. By Lemmas 4–6, E satisfies the hypotheses of Lemmas 8, 10, 13, 26, and 32 and this implies that E has the properties (i)–(v). Now by assumption, G_n is conjugacy separable and subgroup separable. Again by Lemmas 4 and 6, G_n has the properties (i)–(iv). (The property (v) is not applicable to G_n.) So now we can proceed to use the criterion of Magnus, Karrass, and Solitar (Lemma 1 above). Let $x, y \in G$ be such that $x \not\sim_G y$. We can assume x, y to be of minimal lengths in their conjugacy classes. We then divide the proof into several cases according to the length of x and y. In each case, by using the above properties, we find an image $\bar{G}$ of G such that $\bar{G}$ is a generalised free product of two finite groups and such that $\bar{x} \not\sim_{\bar{G}} \bar{y}$, where $\bar{x}, \bar{y}$ are the images of x, y, respectively, in $\bar{G}$. By the theorem of Dyer (Lemma 2 above), $\bar{G}$ is conjugacy separable and we are done. This completes the brief outline of the proof of Theorem 2.

We shall give the complete proof of Theorem 2 in the next section. Before that, we state (without proofs) Lemmas 8, 10, 13, 26, and 32. As given in the outline above, these lemmas state that certain tree products have properties (i)–(v). Thus they play an important role in the proof of Theorem 2. The complete proofs of these and other lemmas will be given in Section 5.

LEMMA 8. *Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of G_i and H_{ji} of G_j. Let K be a subgroup of G_r and suppose that*

(a) *for each $S \triangleleft_f K$, there exists $M \triangleleft_f G_r$ such that $M \cap K = S$;*

(b) *for each $R_{ij} \triangleleft_f H_{ij}$, there exists $N_i \triangleleft_f G_i$ such that $N_i \cap H_{ij} = R_{ij}$.*

Then there exists $P \triangleleft_f G$ such that $P \cap K = S$.

LEMMA 10. *Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of G_i and H_{ji} of G_j. Let K be a subgroup of G_r and suppose that*

(a) *G_i is H_{ij}-separable and G_r is K-separable;*

(b) *for each $R_{ij} \triangleleft_f H_{ij}$, there exists $N_i \triangleleft_f G_i$ such that $N_i \cap H_{ij} = R_{ij}$.*

Then G is K-separable.

LEMMA 13. *Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$.*

Let K be a subgroup of $Z(G_r)$ such that $K \cap H_{ri} = 1$. Suppose that

(a) *G_i is H_{ij}-separable and G_r is K-separable;*

(b) *for each $R_{ri} \lhd_f H_{ri}$, there exists $M_{ri} \lhd_f G_r$ such that $M_{ri} \cap H_{ri} = R_{ri}$ and $KM_{ri} \cap H_{ri}M_{ri} = M_{ri}$;*

(c) *for each $R_{ij} \lhd_f H_{ij}$, there exists $N_i \lhd_f G_i$ such that $N_i \cap H_{ij} = R_{ij}$.*

Let $\{x\}^G \cap K = \emptyset$, where $x \in G$ is of minimal length in its conjugacy class. Then there exists $P \lhd_f G$ such that $\{\bar{x}\}^{\bar{G}} \cap \bar{K} = \emptyset$ in $\bar{G} = G/P$.

LEMMA 26. *Let $G_1, G_2, \ldots, G_n$ be residually finite groups. Let $G = \langle G_1, G_2, \ldots, G_n;\ H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$, where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Let K_1, K_2 be finitely generated subgroups of $Z(G_r)$, $Z(G_s)$, respectively, such that $K_1 \cap H_{ri} = 1 = K_2 \cap H_{si}$. Suppose that*

(a) *G_i is H_{ij}-separable, G_r is K_1-separable, and G_s is K_2-separable;*

(b) *for each $R_{ijk} \lhd_f H_{ij}$, there exists $M_{ijk} \lhd_f G_i$ such that $M_{ijk} \cap H_{ij} = R_{ijk}$ and $H_{ij}M_{ijk} \cap H_{ik}M_{ijk} = M_{ijk}$;*

(c) *for each $R_{ri} \lhd_f H_{ri}$, there exists $M_{ri} \lhd_f G_r$ such that $M_{ri} \cap H_{ri} = R_{ri}$ and $K_1M_{ri} \cap H_{ri}M_{ri} = M_{ri}$;*

(d) *for each $R_{si} \lhd_f H_{si}$, there exists $M_{si} \lhd_f G_s$ such that $M_{si} \cap H_{si} = R_{si}$ and $K_2M_{si} \cap H_{si}M_{si} = M_{si}$;*

(e) *for $u \notin J_1vJ_2$, where $u, v \in G_i$ and $J_1, J_2 \subseteq Z(G_i)$, there exists $L_i \lhd_f G_i$ such that $\bar{u} \notin \bar{J}_1\bar{v}\bar{J}_2$ in $\bar{G}_i = G_i/L_i$;*

(f) *for each $R_{ij} \lhd_f H_{ij}$, there exists $N_i \lhd_f G_i$ such that $N_i \cap H_{ij} = R_{ij}$.*

Let $x \notin K_1yK_2$, where $x, y \in G$. Then there exists $P \lhd_f G$ such that $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ in $\bar{G} = G/P$.

LEMMA 32. *Let $G = \langle G_1, G_2, \ldots, G_n;\ H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$. Let K be a subgroup of $Z(G_r)$ such that $H_{ri} \cap K = 1$. Suppose that*

(a) *G_i is H_{ij}-separable;*

(b) *for each $S \lhd_f K$, there exists $M \lhd_f G_r$ such that $M \cap K = S$ and $H_{ri}M \cap KM = M$;*

(c) *for each $R_{ij} \lhd_f H_{ij}$, there exists $M_i \lhd_f G_i$ such that $M_i \cap H_{ij} = R_{ij}$.*

Then $K^z \cap K = 1$ for all $z \in G\backslash G_r$ and for each $S \lhd_f K$ and $x, y \in G\backslash G_r$, there exists $P \lhd_f G$ such that $P \cap K \subseteq S$ and $K^xP \cap KP = P$, $K^yP \cap KP = P$.

4. TREE PRODUCTS OF CONJUGACY SEPARABLE GROUPS

In this section, we prove Theorem 2 and give some applications. First we prove some properties of subgroup separable groups.

LEMMA 4. *Let G be a subgroup separable group and let H be a finitely generated subgroup of $Z(G)$. Then for each $R \triangleleft_f H$, there exists $N \triangleleft_f G$ such that $N \cap H = R$.*

Proof. Let $R \triangleleft_f H$ be given. Since $R \subset Z(G)$, we can form $\bar{G} = G/R$. Since R is finitely generated, $\bar{G}$ is residually finite. Furthermore, the subgroup $\bar{H} = H/R$ is finite in $\bar{G}$. Hence there exists $\bar{N} \triangleleft_f \bar{G}$ such that $\bar{N} \cap \bar{H} = \bar{1}$. Let N be the preimage of $\bar{N}$ in G. Then $N \triangleleft_f G$ and $N \cap H = R$.

LEMMA 5. *Let G be a subgroup separable group and let K, H be finitely generated subgroups of $Z(G)$ such that $K \cap H = 1$. Then for each $R \triangleleft_f H$, there exists $N \triangleleft_f G$ such that $N \cap H = R$ and $KN \cap HN = N$.*

Proof. Let $R \triangleleft_f H$ be given. By Lemma 4, there exists $M \triangleleft_f G$ such that $M \cap H = R$. Let $S = M \cap K$. Then $S \triangleleft_f K$. Since $R, S \subset Z(G)$, we can form $\bar{G} = G/RS$. Since R, S are finitely generated, $\bar{G}$ is residually finite. Furthermore, the subgroups $\bar{K}, \bar{H}$ are finite in $\bar{G}$. Hence there exists $\bar{N} \triangleleft_f \bar{G}$ such that $\bar{N} \cap \bar{K}\bar{H} = \bar{1}$. Let N be the preimage of $\bar{N}$ in G. Then $N \triangleleft_f G$ and $KN \cap HN = N$. Also, $\bar{N} \cap \bar{H} = \bar{1}$. It follows that $N \cap H = R$. Hence N is the required normal subgroup.

LEMMA 6. *Let G be a subgroup separable group and let K, H be finitely generated subgroups of $Z(G)$. Then for $x \notin KyH$, where $x, y \in G$, there exists $N \triangleleft_f G$ such that $\bar{x} \notin \bar{K}\bar{y}\bar{H}$ in $\bar{G} = G/N$.*

Proof. Since $K, H \subseteq Z(G)$, then $x \notin KyH$ if and only if $xy^{-1} \notin KH$. Since G is subgroup separable and K, H are finitely generated, there exists $N \triangleleft_f G$ such that $xy^{-1} \notin (KH)N$. Let $\bar{G} = G/N$. Since $\bar{x}\bar{y}^{-1} \notin \overline{KH}$ and $\bar{K}, \bar{H} \subseteq Z(\bar{G})$, it follows that $\bar{x} \notin \bar{K}\bar{y}\bar{H}$.

THEOREM 2. *Let $G_1, G_2, \ldots, G_n$ be conjugacy separable and subgroup separable groups. Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the finitely generated subgroups H_{ij} of $Z(G_i)$, and H_{ji} of $Z(G_j)$ where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Then G is conjugacy separable.*

Proof. We use induction on n. The case $n = 2$ follows from Theorem 1. Let $n \geq 3$. The tree product G has an extremal vertex, say G_n, which is joined to a unique vertex, say G_{n-1}. The subgroup of G generated by $G_1, G_2, \ldots, G_{n-1}$ is just their tree product. Let E denote this subgroup. Then we write $G = E *_H G_n$, where $H = H_{(n-1)n} = H_{n(n-1)}$. By induction,

E is c.s. By Lemmas 4–6, E satisfies the hypotheses of Lemmas 8, 10, 13, 26, and 32. Let $x, y \in G$ be such that $x \nsim_G y$. We can assume x, y to be of minimal lengths in their conjugacy classes. We divide the proof into several cases.

Case 1. $\|x\| = \|y\| = 0$

Since $H_{n(n-1)} \subseteq Z(G_n)$, by Lemma 1(a), $x \sim_G y$ if and only if $x \sim_E y$ or $x = y$. So $x \nsim_G y$ if and only if $x \nsim_E y$. Since E is c.s. there exists $N_1 \lhd_f E$ such that $\bar{x} \nsim_{\bar{E}} \bar{y}$, where $\bar{E} = E/N_1$. By Lemma 4, there exists $N_2 \lhd_f G_n$ such that $N_2 \cap H_{n(n-1)} = N_1 \cap H_{(n-1)n}$. Now, we form $\bar{G} = \bar{E} *_H \bar{G}_n$, where $\bar{E} = E/N_1$, $\bar{G}_n = G_n/N_2$, and $\bar{H} = H_{(n-1)n}N_1/N_1 = H_{n(n-1)}N_2/N_2$. Clearly $\bar{G}$ is a homomorphic image of G. By the choice of N_1, $\bar{x} \nsim_{\bar{E}} \bar{y}$. Since $\bar{H}_{n(n-1)} \subseteq Z(\bar{G}_n)$, by Lemma 1(a) again, $\bar{x} \nsim_{\bar{G}} \bar{y}$. Since $\bar{E}$ and $\bar{G}_n$ are finite, $\bar{G}$ is c.s. by Lemma 2 and our result follows.

Case 2. $\|x\|, \|y\| \leq 1$

Subcase 1. $\|x\| = 0$ and $\|y\| = 1$. First suppose $y \in E \backslash H_{(n-1)n}$. Since y is of minimal length in its conjugacy class, then $\{y\}^E \cap H_{(n-1)n} = \varnothing$. By Lemma 13, there exists $N_1 \lhd_f E$ such that $\{\bar{y}\}^{\bar{E}} \cap \bar{H}_{(n-1)n} = \varnothing$ in $\bar{E} = E/N_1$. By Lemma 4, there exists $N_2 \lhd_f G_n$ such that $N_2 \cap H_{n(n-1)} = N_1 \cap H_{(n-1)n}$. As in Case 1, we form $\bar{G}$. Clearly $\{\bar{y}\}^{\bar{G}} \cap \bar{H}_{(n-1)n} = \varnothing$ and hence $\bar{x} \nsim_G \bar{y}$. Our result then follows as above.

Now suppose $y \in G_n \backslash H_{n(n-1)}$. By $H_{n(n-1)}$-separability of G_n, there exists $N_2 \lhd_f G_n$ such that $\bar{y} \notin \bar{H}_{n(n-1)}$ in $\bar{G}_n = G_n/N_2$. Since $\bar{H}_{n(n-1)} \subseteq Z(\bar{G}_n)$, we have $\{\bar{y}\}^{\bar{G}_n} \cap \bar{H}_{n(n-1)} = \varnothing$. By Lemma 8, there exists $N_1 \lhd_f E$ such that $N_1 \cap H_{(n-1)n} = N_2 \cap H_{n(n-1)}$. Again we form $\bar{G}$. Clearly $\{\bar{y}\}^{\bar{G}} \cap \bar{H}_{n(n-1)} = \varnothing$ and hence $\bar{x} \nsim_{\bar{G}} \bar{y}$. Our result now follows.

The case where $\|x\| = 1$ and $\|y\| = 0$ can be similarly proved.

Subcase 2. $\|x\| = \|y\| = 1$. First suppose x and y are in different factors of G. WLOG, let $x \in E \backslash H_{(n-1)n}$ and $y \in G_n \backslash H_{n(n-1)}$. Since x is of minimal length in its conjugacy class, then $\{x\}^E \cap H_{(n-1)n} = \varnothing$. By Lemma 13, there exists $M_1 \lhd_f E$ such that $\{x\}^{\bar{E}} \cap \bar{H}_{(n-1)n} = \varnothing$ in $\bar{E} = E/M_1$. By $H_{n(n-1)}$-separability of G_n, there exists $M_2 \lhd_f G_n$ such that $\bar{y} \notin \bar{H}_{n(n-1)}$ in $\bar{G}_n = G_n/M_2$. Since $\bar{H}_{n(n-1)} \subseteq Z(\bar{G}_n)$, we have $\{\bar{y}\}^{\bar{G}_n} \cap \bar{H}_{n(n-1)} = \varnothing$. By Lemmas 4 and 8, we can find $N_1 \lhd_f E$, $N_2 \lhd_f G_n$ such that $N_1 \subseteq M_1$, $N_2 \subseteq M_2$, and $N_1 \cap H_{(n-1)n} = N_2 \cap H_{n(n-1)}$. Again we form $\bar{G}$. By the choice of N_1 and N_2, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

Now suppose x, y are in the same factor of G, say E. Since x, y are of minimal lengths in their conjugacy classes, $\{x\}^E \cap H_{(n-1)n} = \varnothing$ and

$\{y\}^E \cap H_{(n-1)n} = \emptyset$. By the conjugacy separability of E and by Lemma 13, there exists $N_1 \triangleleft_f E$ such that $\bar{x} \nsim_{\bar{E}} \bar{y}$, $\{\bar{x}\}^{\bar{E}} \cap \bar{H}_{(n-1)n} = \emptyset$ and $\{\bar{y}\}^{\bar{E}} \cap \bar{H}_{(n-1)n} = \emptyset$ in $\bar{E} = E/N_1$. By Lemma 4, there exists $N_2 \triangleleft_f G_n$ such that $N_2 \cap H_{n(n-1)} = N_1 \cap H_{(n-1)n}$. Then we form $\bar{G}$. Clearly $\bar{x} \nsim_{\bar{G}} \bar{y}$ and our result follows. Similarly for the case $x, y \in G_n$.

Case 3. $\|x\| \neq \|y\|$ *and One of* $\|x\|$ *and* $\|y\|$ *Is at Least* 2

By Lemma 10, E is $H_{(n-1)n}$-separable and by Lemma 8, for each $R \triangleleft_f H_{(n-1)n}$, there exists $M_1 \triangleleft_f E$ such that $M_1 \cap H_{(n-1)n} = R$. By assumption, G_n is $H_{n(n-1)}$-separable and by Lemma 4, for each $R \triangleleft_f H_{n(n-1)}$, there exists $M_2 \triangleleft_f G_n$ such that $M_2 \cap H_{n(n-1)} = R$. So we can find $N_1 \triangleleft_f E$ and $N_2 \triangleleft_f G_n$ such that in $\bar{G} = \bar{E} *_{\bar{H}} \bar{G}_n$, $\|\bar{x}\| = \|x\| \neq \|y\| = \|\bar{y}\|$. Since $\|\bar{x}\|$ and $\|\bar{y}\|$ are each of minimal length in their respective conjugacy classes, then $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

Case 4. $\|x\| = \|y\| \geq 2$

Let $x = u_1 u_2 \cdots u_r$ and $y = v_1 v_2 \cdots v_r$ be cyclically reduced words in G, $r \geq 2$. Since $x \nsim_G y$, the system of equations (Ii) has no solution in H for all $0 \leq i < r$. Therefore we need to show that, for each i, there exists $N_i \triangleleft_f G$ such that in $\bar{G}_i = G/N_i$, the corresponding system of equations (Ii) has no solution in H. Letting N be the intersection of the normal subgroups N_i in G we have $\bar{x} \nsim_{\bar{G}} \bar{y}$ in $\bar{G} = G/N$ and the result follows. Hence it is sufficient to show the case $i = 0$ in (Ii).

Since E is $H_{(n-1)n}$-separable and G_n is $H_{n(n-1)}$-separable, there exist $M_1 \triangleleft_f E$, $M_2 \triangleleft_f G_n$ such that $u_i, v_i \notin H_{(n-1)n} M_1$ if $u_i, v_i \in E \setminus H_{(n-1)n}$ and $u_j, v_j \notin H_{n(n-1)} M_2$ if $u_j, v_j \in G_n \setminus H_{n(n-1)}$.

Suppose that there exists some t such that $u_t \notin H v_t H$, where $u_t, v_t \in E$ or $u_t, v_t \in G_n$. First suppose $u_t, v_t \in E$. By Lemma 26, there exists $T_1 \triangleleft_f E$ such that $\bar{u}_t \notin \bar{H}_{(n-1)n} \bar{v}_t \bar{H}_{(n-1)n}$ in $\bar{E} = E/T_1$. By Lemmas 4 and 8, there exist $N_1 \triangleleft_f E$, $N_2 \triangleleft_f G_n$ such that $N_1 \subseteq M_1 \cap T_1$, $N_2 \subseteq M_2$ and $N_1 \cap H_{(n-1)n} = N_2 \cap H_{n(n-1)}$. Again we form $\bar{G}$. Then $\bar{u}_t \notin \bar{H}_{(n-1)n} \bar{v}_t \bar{H}_{(n-1)n}$ in $\bar{G}$ and hence $\bar{x} \nsim_{\bar{G}} \bar{y}$. The result now follows.

Now suppose $u_t \notin H_{n(n-1)} v_t H_{n(n-1)}$, where $u_t, v_t \in G_n$. Since $H_{n(n-1)} \subseteq Z(G_n)$, we have $u_t v_t^{-1} \notin H_{n(n-1)}$. Then by $H_{n(n-1)}$-separability of G_n, there exists $T_2 \triangleleft_f G_n$ such that $\bar{u}_t \bar{v}_t^{-1} \notin \bar{H}_{n(n-1)}$ in $\bar{G}_n = G_n/T_2$. This implies that $\bar{u}_t \notin \bar{H}_{n(n-1)} \bar{v}_t \bar{H}_{n(n-1)}$ and we proceed as above to obtain our result.

Consequently, we may assume that the system of equations (Ii) has no solution in H but each individual equation of (Ii) is soluble in H. Thus there exists a finite sequence $x_1, y_1, \ldots, x_r, y_r$ of elements in H such that the following hold:

$$u_1 = x_1^{-1} v_1 y_1$$
$$u_2 = x_2^{-1} v_2 y_2$$
$$\vdots$$
$$u_r = x_r^{-1} v_r y_r.$$

Recall that $y = v_1 v_2 \cdots v_r$, where $v_i \in E \backslash H_{(n-1)n}$ or $v_i \in G_n \backslash H_{n(n-1)}$. First suppose that all the v_i from $E \backslash H_{(n-1)n}$ are actually in $G_{n-1} \backslash H_{(n-1)n}$. So each of these v_i commutes with every element of $H_{(n-1)n}$ since $H_{(n-1)n} \subseteq Z(G_{n-1})$. In this case by Lemma 1(c), $x \sim_G y$ if and only if $x = y$. As before, we can find $N_1 \lhd_f E$, $N_2 \lhd_f G_n$ such that in $\bar{G} = \bar{E} *_{\bar{H}} \bar{G}_n$, $\|\bar{x}\| = \|\bar{y}\|$ and $\bar{x} \neq \bar{y}$. Hence $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

So we may assume that for at least one i, the v_i from $E \backslash H_{(n-1)n}$ is not in $G_{n-1} \backslash H_{(n-1)n}$. Then by Lemma 32 (with $K = H_{(n-1)n}$), we have $H_{(n-1)n}^{v_i} \cap H_{(n-1)n} = 1$ and hence the equation $u_i = x_i^{-1} v_i y_i$ has unique solution x_i^{-1}, y_i. Fixing this i, we consider the next equation $u_{i+1} = x_{i+1}^{-1} v_{i+1} y_{i+1}$ and arrange, if possible, so that $x_{i+1} = y_i$. Continuing in this way, we see that this matching must eventually fail at some equation, say $u_j = x_j^{-1} v_j y_j$, where $x_j \neq y_{j-1}$. This equation may be the equation we started with. Furthermore this v_j is not in $G_{n-1} \backslash H_{(n-1)n}$, for otherwise, this v_j commutes with every element of $H_{(n-1)n}$ and we can match $x_j = y_{j-1}$. Again by Lemma 32 (with $K = H_{(n-1)n}$), we have $H_{(n-1)n}^{v_j} \cap H_{(n-1)n} = 1$ and the equation $u_j = x_j^{-1} v_j y_j$ has unique solution x_j^{-1}, y_j. Now by the residual finiteness of H, there exists $L \lhd_f H$ such that $x_j y_{j-1}^{-1} \notin L$. Let $R = M_1 \cap M_2 \cap L$. Then $R \lhd_f H$. By Lemma 32, there exists $T_1 \lhd_f E$ such that $T_1 \cap H = R_1 \subseteq R$ and $H_{(n-1)n}^{v_j} T_1 \cap H_{(n-1)n} T_1 = T_1$, $H_{(n-1)n}^{v_i} T_1 \cap H_{(n-1)n} T_1 = T_1$. Let $N_1 = M_1 \cap T_1$. Then $N_1 \lhd_f E$ such that $N_1 \cap H_{(n-1)n} = R_1$ and $H_{(n-1)n}^{v_j} N_1 \cap H_{(n-1)n} N_1 = N_1$, $H_{(n-1)n}^{v_i} N_1 \cap H_{(n-1)n} N_1 = N_1$. Now, by Lemma 4, we can find $T_2 \lhd_f G_n$ such that $T_2 \cap H_{n(n-1)} = R_1$. Let $N_2 = M_2 \cap T_2$. Then $N_2 \lhd_f G_n$ and $N_2 \cap H_{n(n-1)} = R_1$. As above, we form $\bar{G}$. Then $\bar{H}_{(n-1)n}^{\bar{v}_j} \cap \bar{H}_{(n-1)n} = \bar{1}$, $\bar{H}_{(n-1)n}^{\bar{v}_i} \cap \bar{H}_{(n-1)n} = \bar{1}$ in $\bar{G}$. This implies that both the equations $\bar{u}_j = \bar{x}_j^{-1} \bar{v}_j \bar{y}_j$ and $\bar{u}_i = \bar{x}_i^{-1} \bar{v}_i \bar{y}_i$ have unique solutions. Since $\bar{x}_j \neq \bar{y}_{j-1}$ in $\bar{G}$, the matching of $\bar{x}_j$ with $\bar{y}_{j-1}$ fail at the equation $\bar{u}_j = \bar{x}_j^{-1} \bar{v}_j \bar{y}_j$. Therefore $\bar{x} \nsim_{\bar{G}} \bar{y}$ and our result follows.

We now give some applications of Theorem 2 to polycyclic-by-finite groups and free-by-finite groups. We note that by Remeslennikov [18], Formanek [8], and Dyer [5], polycyclic-by-finite groups and free-by-finite groups are conjugacy separable. Furthermore, polycyclic groups and free groups are subgroup separable (Mal'cev [17]; Hall [9]). Since a finite extension of a subgroup separable group is again subgroup separable, polycyclic-by-finite groups and free-by-finite groups are subgroup separable. Hence from Theorem 2, we have the following two corollaries:

COROLLARY 1. *Let $G_1, G_2, \ldots, G_n$ be polycyclic-by-finite or free-by-finite groups. Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the finitely generated subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$, where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Then G is conjugacy separable.*

COROLLARY 2. *Let $G_1, G_2, \ldots, G_n$ be finitely generated abelian groups. Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the finitely generated subgroups H_{ij} of G_i and H_{ji} of G_j, where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Then G is conjugacy separable.*

5. PROOFS OF THE LEMMAS

In this section, we shall state and give the complete proofs of all the lemmas. For each of these results, we first prove the case when the tree product has two factors and then apply induction for the case when the tree product has $n \geq 2$ factors. This will lead to the proofs of Lemmas 8, 10, 13, 26, and 32.

LEMMA 7. *Let $E = G_1 *_H G_2m$, where $G_1 \cap G_2 = H$. Let K be a subgroup of G_1 and suppose that*

(a) *for each $S \lhd_f K$, there exists $M \lhd_f G_1$ such that $M \cap K = S$;*

(b) *for each $R \lhd_f H$, there exists $N \lhd_f G_2$ such that $N \cap H = R$.*

Then there exists $P \lhd_f E$ such that $P \cap K = S$.

Proof. Let $S \lhd_f K$ and $M \lhd_f G_1$ be such that $M \cap K = S$. Let $R = M \cap H$. Then $R \lhd_f H$ and hence by (b), there exists $N \lhd_f G_2$ such that $N \cap H = R$. Let $\bar{E} = \bar{G}_1 *_H \bar{G}_2$, where $\bar{G}_1 = G_1/M$, $\bar{G}_2 = G_2/N$, and $\bar{H} = HM/M = HN/N$. Clearly $\bar{E}$ is a homomorphic image of E. Since $\bar{E}$ is residually finite and $\bar{K} = KM/M$ is finite, there exists $\bar{P} \lhd_f \bar{E}$ such that $\bar{P} \cap \bar{K} = \bar{1}$. Let P be the preimage of $\bar{P}$ in E. Then $P \lhd_f E$ and $P \cap K = S$.

LEMMA 8. *Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of G_i and H_{ji} of G_j. Let K be a subgroup of G_r and suppose that*

(a) *for each $S \lhd_f K$, there exists $M \lhd_f G_r$ such that $M \cap K = S$;*

(b) *for each $R_{ij} \lhd_f H_{ij}$, there exists $N_i \lhd_f G_i$ such that $N_i \cap H_{ij} = R_{ij}$.*

Then there exists $P \lhd_f G$ such that $P \cap K = S$.

Proof. The proof follows by induction on n with the case $n = 2$ being Lemma 7.

LEMMA 9. *Let $E = G_1 *_H G_2$, where $G_1 \cap G_2 = H$. Let K be a subgroup of G_1 and suppose that*

(a) *G_1 is H-separable, K-separable and G_2 is H-separable;*

(b) *for each $R \lhd_f H$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $N_1 \cap H = R = N_2 \cap H$.*

Then E is K-separable.

Proof. Lemma 2.3 in Kim [10].

LEMMA 10. *Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of G_i and H_{ji} of G_j. Let K be a subgroup of G_r and suppose that*

(a) *G_i is H_{ij}-separable and G_r is K-separable;*

(b) *for each $R_{ij} \lhd_f H_{ij}$, there exists $N_i \lhd_f G_i$ such that $N_i \cap H_{ij} = R_{ij}$.*

Then G is K-separable.

Proof. We use induction on n. The case $n = 2$ follows from Lemma 9. Let $n \geq 3$. The tree product G has an extremal vertex, say G_n, which is joined to a unique vertex, say G_{n-1}. The subgroup of G generated by $G_1, G_2, \ldots, G_{n-1}$ is just their tree product. Let E denote this subgroup. Then we write $G = E *_H G_n$, where $H = H_{(n-1)n} = H_{n(n-1)}$. By induction, E is $H_{(n-1)n}$-separable and by Lemma 8, for each $R \lhd_f H_{(n-1)n}$, there exists $L \lhd_f E$ such that $L \cap H_{(n-1)n} = R$. First suppose that $K < E$. By induction, E is K-separable and hence G is K-separable by Lemma 9. Now suppose $K < G_n$. By our assumption, G_n is K-separable and hence G is K-separable by Lemma 9.

LEMMA 11. *Let $E = G_1 *_H G_2$, where $G_1 \cap G_2 = H$ and $H \subseteq Z(G_2)$. Let K be a subgroup of G_1 and suppose that*

(a) *G_1, G_2 are H-separable;*

(b) *for $\{y\}^{G_1} \cap K = \emptyset$, where $y \in G_1$, there exists $M \lhd_f G_1$ such that $\{\bar{y}\}^{\bar{G}_1} \cap \bar{K} = \emptyset$ in $\bar{G}_1 = G_1/M$;*

(c) *for each $R \lhd_f H$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $N_1 \cap H = R = N_2 \cap H$.*

Let $\{x\}^E \cap K = \emptyset$, where $x \in E$ is of minimal length in its conjugacy class. Then there exists $P \lhd_f E$ such that $\{\bar{x}\}^{\bar{E}} \cap \bar{K} = \emptyset$ in $\bar{E} = E/P$.

Proof. Let x be a reduced element of E.

Case 1. $x \in G_1$. Since $\{x\}^E \cap K = \emptyset$, we have $\{x\}^{G_1} \cap K = \emptyset$. By (b), there exists $N_1 \triangleleft_f G_1$ such that $\{\bar{x}\}^{\bar{G}_1} \cap \bar{K} = \emptyset$ in $\bar{G}_1 = G_1/N_1$. By (c), there exists $N_2 \triangleleft_f G_2$ such that $N_1 \cap H = N_2 \cap H$. We form $\bar{E} = \bar{G}_1 *_H \bar{G}_2$, where $\bar{G}_1 = G_1/N_1$, $\bar{G}_2 = G_2/N_2$, and $\bar{H} = HN_1/N_1 = HN_2/N_2$. Let $\bar{x}, \bar{K}$, denote the image of x, K, respectively, in $\bar{E}$. Since $\{\bar{x}\}^{\bar{G}_1} \cap \bar{K} = \emptyset$, $\bar{H} \subseteq Z(\bar{G}_2)$, and $\bar{K} \subseteq \bar{G}_1$, it follows that $\{\bar{x}\}^{\bar{E}} \cap \bar{K} = \emptyset$. Since $\bar{E}$ is c.s. by Lemma 2 and $\bar{K}$ is finite, there exists $\bar{P} \triangleleft_f \bar{E}$ such that $\{\hat{x}\}^{\hat{E}} \cap \hat{K} = \emptyset$ in $\hat{E} = \bar{E}/\bar{P}$. Let P be the preimage of $\bar{P}$ in E. Then $P \triangleleft_f E$ and $\{\tilde{x}\}^{\tilde{E}} \cap \tilde{K} = \emptyset$ in $\tilde{E} = E/P$.

Case 2. $x \in G_2 \backslash H$. By H-separability of G_2, there exists $N_2 \triangleleft_f G_2$ such that $x \notin HN_2$. By (c), there exists $N_1 \triangleleft_f G_1$ such that $N_1 \cap H = N_2 \cap H$. As in Case 1, we form $\bar{E}$. Since $\bar{x} \in \bar{G}_2 \backslash \bar{H}$, $\bar{H} \subseteq Z(\bar{G}_2)$, and $\bar{K} \subseteq \bar{G}_1$, it follows that $\{\bar{x}\}^{\bar{E}} \cap \bar{K} = \emptyset$. We can now proceed as in Case 1 to obtain our result.

Case 3. $x \notin G_1 \cup G_2$, that is, $\|x\| \geq 2$ in E. WLOG, let $x = a_1 b_1 \cdots a_n b_n$, where $a_i \in G_1 \backslash H$ and $b_i \in G_2 \backslash H$ for all i. By H-separability of G_1, G_2 and by (c), we can find $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that $a_i \notin HN_1$, $b_i \notin HN_2$ for all i and $N_1 \cap H = N_2 \cap H$. As above we form $\bar{E}$. Then $\bar{x}$ is reduced and $\|\bar{x}\| = \|x\|$ in $\bar{E}$. Clearly $\{\bar{x}\}^{\bar{E}} \cap \bar{K} = \emptyset$ and our result follows.

LEMMA 12. *Let $E = G_1 *_H G_2$, where $G_1 \cap G_2 = H$ and $H \subseteq Z(G_2)$. Let K be a subgroup of $Z(G_2)$ such that $K \cap H = 1$. Suppose that*

(a) *G_1 is H-separable and G_2 is H-separable, K-separable*;

(b) *for each $R \triangleleft_f H$, there exist $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that $N_1 \cap H = R = N_2 \cap H$ and $KN_2 \cap HN_2 = N_2$.*

Let $\{x\}^E \cap K = \emptyset$, where $x \in E$ is of minimal length in its conjugacy class. Then there exists $P \triangleleft_f E$ such that $\{\bar{x}\}^{\bar{E}} \cap \bar{K} = \emptyset$ in $\bar{E} = E/P$.

Proof. Let x be a reduced element of E.

Case 1. $x \in G_2 \backslash K$. By K-separability of G_2, there exists $M \triangleleft_f G_2$ such that $x \notin KM$. Let $R = M \cap H$. Then $R \triangleleft_f H$. By (b), there exist $T_1 \triangleleft_f G_1$, $T_2 \triangleleft_f G_2$ such that $T_1 \cap H = R = T_2 \cap H$, $KT_2 \cap HT_2 = T_2$. Let $N_1 = T_1$ and $N_2 = M \cap T_2$. Then $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that $N_2 \subseteq M$ and $N_1 \cap H = R = N_2 \cap H$.

Next we show that $KN_2 \cap HN_2 = N_2$. Clearly $N_2 \subseteq KN_2 \cap HN_2$. Let $u \in KN_2 \cap HN_2$. Then $u = kn_1 = hn_2$ for some $k \in K$, $h \in H$, and $n_1, n_2 \in N_2$. Since $N_2 \subseteq T_2$, we have $KN_2 \cap HN_2 \subseteq KT_2 \cap HT_2 = T_2$ and so $u \in T_2$. This implies that $h = un_2^{-1} \in T_2$. Therefore $h \in T_2 \cap H = R = N_2 \cap H$ and hence $h \in N_2$. It follows that $u = hn_2 \in N_2$ and so $KN_2 \cap HN_2 \subseteq N_2$. Hence $KN_2 \cap HN_2 = N_2$.

Now we form $\bar{E} = \bar{G}_1 *_H \bar{G}_2$, where $\bar{G}_1 = G_1/N_1$, $\bar{G}_2 = G_2/N_2$, and $\bar{H} = HN_1/N_1 = HN_2/N_2$. Let $\bar{x}$, $\bar{K}$ denote the images of x, K, respectively, in $\bar{E}$. Then $\bar{x} \notin \bar{K}$, $\bar{K} \cap \bar{H} = \bar{1}$ and $\bar{K} \subseteq Z(\bar{G}_2)$ in $\bar{E}$. It follows that $\{\bar{x}\}^{\bar{E}} \cap \bar{K} = \emptyset$. We can now proceed as in Case 1 of Lemma 11 to obtain our result.

Case 2. $x \in G_1 \backslash H$. By H-separability of G_1, there exists $N_1 \lhd_f G_1$ such that $x \notin HN_1$. By (b), we can find $N_2 \lhd_f G_2$ such that $N_1 \cap H = N_2 \cap H$, $KN_2 \cap HN_2 = N_2$. As above, we form $\bar{E}$. Then $\bar{x} \notin \bar{H}$, $\bar{K} \cap \bar{H} = \bar{1}$, and $\bar{K} \subseteq Z(\bar{G}_2)$ in $\bar{E}$. It follows that $\{\bar{x}\}^{\bar{E}} \cap \bar{K} = \emptyset$ and our result follows as above.

Case 3. $x \notin G_1 \cup G_2$, that is, $\|x\| \geq 2$ in E. As in Case 3 of Lemma 11, we form $\bar{E}$ in which $\bar{x}$ is reduced and $\|\bar{x}\| = \|x\|$ in $\bar{E}$. Clearly $\{\bar{x}\}^{\bar{E}} \cap \bar{K} = \emptyset$ and our result follows.

LEMMA 13. *Let $G = \langle G_1, G_2, \ldots, G_n;\ H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$. Let K be a subgroup of $Z(G_r)$ such that $K \cap H_{ri} = 1$. Suppose that*

(a) *G_i is H_{ij}-separable and G_r is K-separable*;

(b) *for each $R_{ri} \lhd_f H_{ri}$, there exists $M_{ri} \lhd_f G_r$ such that $M_{ri} \cap H_{ri} = R_{ri}$ and $KM_{ri} \cap H_{ri}M_{ri} = M_{ri}$*;

(c) *for each $R_{ij} \lhd_f H_{ij}$, there exists $N_i \lhd_f G_i$ such that $N_i \cap H_{ij} = R_{ij}$.*

Let $\{x\}^G \cap K = \emptyset$, where $x \in G$ is of minimal length in its conjugacy class. Then there exists $P \lhd_f G$ such that $\{\bar{x}\}^{\bar{G}} \cap \bar{K} = \emptyset$ in $\bar{G} = G/P$.

Proof. We use induction on n. The case $n = 2$ follows easily from Lemma 12.

Let $n \geq 3$. As in Lemma 10, we write $G = E *_H G_n$, where E is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$ and $H = H_{(n-1)n} = H_{n(n-1)}$. By Lemma 10, E is $H_{(n-1)n}$-separable and by Lemma 8, for each $R \lhd_f H_{(n-1)n}$, there exists $L \lhd_f E$ such that $L \cap H_{(n-1)n} = R$. Let $\{x\}^G \cap K = \emptyset$, where $x \in G$ is of minimal length in its conjugacy class.

Case 1. $K < E$. By induction, for $\{y\}^E \cap K = \emptyset$, where $y \in E$ is of minimal length in its conjugacy class, there exists $Q \lhd_f E$ such that $\{\bar{y}\}^{\bar{E}} \cap \bar{K} = \emptyset$ in $\bar{E} = E/Q$. By assumption, G_n is $H_{n(n-1)}$-separable and for each $R \lhd_f H_{n(n-1)}$, there exists $N \lhd_f G_n$ such that $N \cap H_{n(n-1)} = R$. Since $H_{n(n-1)} \subseteq Z(G_n)$, our result follows from Lemma 11.

Case 2. $K < G_n$. Our result follows from Lemma 12.

LEMMA 14. *Let $E = G_1 *_H G_2$, where $G_1 \cap G_2 = H$ and G_2 is residually finite. Let K_1, K_2 be subgroups of G_1 such that $K_1 \cap H = 1 = K_2 \cap H$.*

Suppose that

(a) *G_1 is H-separable, K_1H-separable, HK_2-separable and G_2 is H-separable*;

(b) *for each $R \lhd_f H$, there exists $M \lhd_f G_1$ such that $M \cap H \subseteq R$ and $K_1M \cap HM = M$, $K_2M \cap HM = M$*;

(c) *for $u \notin K_1vK_2$, where u, $v \in G_1$, there exists $N \lhd_f G_1$ such that $\bar{u} \notin \bar{K}_1\bar{v}\bar{K}_2$ in $\bar{G}_1 = G_1/N$*;

(d) *for each $R \lhd_f H$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $N_1 \cap H = R = N_2 \cap H$.*

Let $x \notin K_1yK_2$, where $x \in E$ and $y \in G_1 \cup G_2$. Then there exists $P \lhd_f E$ such that $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ in $\bar{E} = E/P$.

Proof. Let $x \notin K_1yK_2$ be a reduced element of E.

(i) $y \in G_1$.

Case 1. $x \in G_1$

By (c), there exists $N_1 \lhd_f G_1$ such that $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ in $\bar{G}_1 = G_1/N_1$. Then by (d), there exists $N_2 \lhd_f G_2$ such that $N_1 \cap H = N_2 \cap H$. Now we form $\bar{E} = \bar{G}_1 *_H \bar{G}_2$, where $\bar{G}_1 = G_1/N_1$, $\bar{G}_2 = G_2/N_2$, and $\bar{H} = HN_1/N_1 = HN_2/N_2$. Clearly $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ in $\bar{E}$. Since $\bar{E}$ is free-by-finite, then by Lemma 3, there exists $\bar{P} \lhd_f \bar{E}$ such that $\bar{x}\bar{y}^{-1} \notin (\bar{K}_1\bar{y}\bar{K}_2\bar{y}^{-1})\bar{P}$. Let P be the preimage of $\bar{P}$ in E. Then P is the required normal subgroup.

Case 2. $x \in G_2\backslash H$

By H-separability of G_2, there exists $N_2 \lhd_f G_2$ such that $x \notin HN_2$. Then by (d), we can find $N_1 \lhd_f G_1$ such that $N_1 \cap H = N_2 \cap H$. As in Case 1, we form $\bar{E}$. Since $\bar{x} \in \bar{G}_2\backslash\bar{H}$ and $\bar{K}_1\bar{y}\bar{K}_2 \subset \bar{G}_1$, it follows that $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$. Our result then follows as in Case 1.

Case 3. $x \notin G_1 \cup G_2$; *That Is*, $\|x\| \geq 2$ *in* E

As in Case 3 of Lemma 11, we form $\bar{E}$ in which $\bar{x}$ is reduced and $\|\bar{x}\| = \|x\|$ in $\bar{E}$. It follows that $\bar{x} \notin \bar{G}_1$ and hence $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$. Again we are done.

(ii) $y \in G_2\backslash H$.

Case 1. $x \in G_1$

Clearly $x \notin K_1yK_2$ if and only if $y \notin K_1xK_2$. As in Case 2 above, there exists $P \lhd_f E$ such that $\bar{y} \notin \bar{K}_1\bar{x}\bar{K}_2$ in $\bar{E} = E/P$. Therefore $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ and we are done.

Case 2. $x \in G_2 \backslash H$

Since $K_1 \cap H = 1 = K_2 \cap H$ and $K_1, K_2 \subset G_1$, then $x \in K_1 y K_2$ if and only if $x = y$. Hence we have $xy^{-1} \neq 1$. By the residual finiteness and H-separability of G_2, there exists $M \triangleleft_f G_2$ such that $xy^{-1} \notin M$ and x, $y \notin HM$. Let $R = M \cap H$. Then $R \triangleleft_f H$. By (b), there exists $N_1 \triangleleft_f G_1$ such that $N_1 \cap H \subseteq R$ and $K_1 N_1 \cap HN_1 = N_1, K_2 N_1 \cap HN_1 = N_1$. By (d), we can find $N_2 \triangleleft_f G_2$ such that $N_2 \subseteq M$ and $N_2 \cap H = N_1 \cap H$. Again we form $\bar{E}$. Then $\bar{x} \neq \bar{y}, \bar{x}, \bar{y} \in \bar{G}_2 \backslash \bar{H}$ and $\bar{K}_1 \cap \bar{H} = \bar{1} = \bar{K}_2 \cap \bar{H}$ in $\bar{E}$. It follows that $\bar{x} \notin \bar{K}_1 \bar{y} \bar{K}_2$ and the result follows.

Case 3. $x \notin G_1 \cup G_2$; *That Is*, $\|x\| \geq 2$ *in* E

Subcase 1. $\|x\| > 3$. As before, we form $\bar{E}$ in which $\bar{x}$ is reduced and $\|\bar{x}\| = \|x\|$ in $\bar{E}$. Clearly $\bar{x} \notin \bar{K}_1 \bar{y} \bar{K}_2$ and our result follows.

Subcase 2. $\|x\| = 3$. Let $x = a_1 b_1 a_2$, where $a_1, a_2 \in G_1 \backslash H$ and $b_1 \in G_2 \backslash H$. Since G_1, G_2 are H-separable, there exist $M_1 \triangleleft_f G_1$, $M_2 \triangleleft_f G_2$ such that a_1, $a_2 \notin HM_1$ and b_1, $y \notin HM_2$. We note that $x \in K_1 y K_2$ if and only if $x = k_1 y k_2$ for some $k_1 \in K_1$, $k_2 \in K_2$ and there exist h_1, $h_2 \in H$ such that $a_1 = k_1 h_1$, $b_1 = h_1^{-1} y h_2$ and $a_2 = h_2^{-1} k_2$. First suppose $a_1 \notin K_1 H$. Since G_1 is $K_1 H$-separable, there exists $M \triangleleft_f G_1$ such that $a_1 \notin (K_1 H) M$. Then by (d), there exist $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that $N_1 \subseteq M_1 \cap M$, $N_2 \subseteq M_2$, and $N_1 \cap H = N_2 \cap H$. As before, we form $\bar{E}$. Then $\bar{a}_1$, $\bar{a}_2 \in \bar{G}_1 \backslash \bar{H}$, $\bar{y}$, $\bar{b}_1 \in \bar{G}_2 \backslash \bar{H}$, and $\bar{a}_1 \notin \bar{K}_1 \bar{H}$ in $\bar{E}$. It follows that $\bar{x} \notin \bar{K}_1 \bar{y} \bar{K}_2$ and we are done as above. Similarly for the case $a_2 \notin HK_2$.

Now suppose $a_1 \in K_1 H$ and $a_2 \in HK_2$. Let $a_1 = k_1 h_1$ and $a_2 = h_2^{-1} k_2$, where $k_1 \in K_1$, $k_2 \in K_2$, and h_1, $h_2 \in H$. Since $K_1 \cap H = 1 = K_2 \cap H$, we see that k_j, h_j are uniquely determined. Since $x \notin K_1 y K_2$, then $b_1 \neq h_1^{-1} y h_2$. By residual finiteness of G_2, there exists $N \triangleleft_f G_2$ such that $b_1^{-1} h_1^{-1} y h_2 \notin N$. Let $R = M_1 \cap M_2 \cap N$. Then $R \triangleleft_f H$. By (b), there exists $N_1 \triangleleft_f G_1$ such that $N_1 \subseteq M_1$, $N_1 \cap H \subseteq R$, and $K_1 N_1 \cap HN_1 = N_1$, $K_2 N_1 \cap HN_1 = N_1$. By (d), there exists $N_2 \triangleleft_f G_2$ such that $N_2 \subseteq M_2 \cap N$ and $N_2 \cap H = N_1 \cap H$. As before, we form $\bar{E}$. Then $\bar{a}_1, \bar{a}_2 \in \bar{G}_1 \backslash \bar{H}$, $\bar{y}$, $\bar{b}_1 \in \bar{G}_2 \backslash \bar{H}$, and $\bar{b}_1 \neq \bar{h}_1^{-1} \bar{y} \bar{h}_2$ in $\bar{E}$. Furthermore, $\bar{K}_1 \cap \bar{H} = \bar{1} = \bar{K}_2 \cap \bar{H}$ and hence $\bar{k}_j$, $\bar{h}_j$ are uniquely determined. Since $\bar{b}_1 \neq \bar{h}_1^{-1} \bar{y} \bar{h}_2$, then $\bar{x} \notin \bar{K}_1 \bar{y} \bar{K}_2$ and again we are done.

Let $x = b_1 a_1 b_2$, where $a_1 \in G_1 \backslash H$ and b_1, $b_2 \in G_2 \backslash H$. By H-separability of G_1, G_2, and by (d), there exist $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that $a_1 \notin HN_1$, b_1, $b_2 \notin HN_2$, and $N_1 \cap H = N_2 \cap H$. As above, we form $\bar{E}$. Then $\bar{a}_1 \in \bar{G}_1 \backslash \bar{H}$ and $\bar{b}_1$, $\bar{b}_2 \in \bar{G}_2 \backslash \bar{H}$ in $\bar{E}$. Clearly $\bar{x} \notin \bar{K}_1 \bar{y} \bar{K}_2$ and the result follows.

Subcase 3. $\|x\| = 2$. Let $x = ab$, where $a \in G_1 \backslash H$ and $b \in G_2 \backslash H$. Since $K_2 \cap H = 1$, then $x \in K_1 y K_2$ if and only if $x = k_1 y$ for some $k_1 \in K_1$ and there exists $h \in H$ such that $a = k_1 h$ and $b = h^{-1} y$. Therefore, using a similar argument as in Subcase 2, we can find $P \lhd_f E$ such that $\bar{x} \notin \bar{K}_1 \bar{y} \bar{K}_2$ in $\bar{E} = E/P$ as required. The case $x = ba$, where $a \in G_1 \backslash H$ and $b \in G_2 \backslash H$ can be similarly proved.

LEMMA 15. *Let $E = G_1 *_H G_2$, where $G_1 \cap G_2 = H$ and H is residually finite. Let K_1, K_2 be subgroups of G_1, G_2, respectively, such that $K_1 \cap H = 1 = K_2 \cap H$. Suppose that*

(a) *G_1 is H-separable, K_1-separable, K_1H-separable and G_2 is H-separable, K_2-separable, HK_2-separable;*

(b) *for each $R \lhd_f H$, there exist $M_1 \lhd_f G_1$, $M_2 \lhd_f G_2$ such that $M_1 \cap H = M_2 \cap H \subseteq R$ and $K_1 M_1 \cap HM_1 = M_1$, $K_2 M_2 \cap HM_2 = M_2$;*

(c) *for each $R \lhd_f H$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $N_1 \cap H = R = N_2 \cap H$.*

Let $x \notin K_1 y K_2$, where $x \in E$ and $y \in G_1 \cup G_2$. Then there exists $P \lhd_f E$ such that $\bar{x} \notin \bar{K}_1 \bar{y} \bar{K}_2$ in $\bar{E} = E/P$.

Proof. Let $x \notin K_1 y K_2$ be a reduced element of E. We shall only consider the case $y \in G_1$. The case where $y \in G_2$ is similar.

Case 1. $x \in G_1$

Since $K_2 \cap H = 1$, it follows that $x \in K_1 y K_2$ if and only if $xy^{-1} \in K_1$. Hence we have $xy^{-1} \notin K_1$. By K_1-separability of G_1, there exists $N \lhd_f G_1$ such that $xy^{-1} \notin K_1 N$. By (b), there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $N_1 \subseteq N$, $N_1 \cap H = N_2 \cap H$ and $K_1 N_1 \cap HN_1 = N_1$, $K_2 N_2 \cap HN_2 = N_2$. Now we form $\bar{E} = \bar{G}_1 *_{\bar{H}} \bar{G}_2$, where $\bar{G}_1 = G_1/N_1$, $\bar{G}_2 = G_2/N_2$, and $\bar{H} = HN_1/N_1 = HN_2/N_2$. Then $\bar{x}\bar{y}^{-1} \notin \bar{K}_1$ and $\bar{K}_1 \cap \bar{H} = \bar{1} = \bar{K}_2 \cap \bar{H}$ in $\bar{E}$. This implies that $\bar{x} \notin \bar{K}_1 \bar{y} \bar{K}_2$ and the result follows as in Lemma 14.

Case 2. $x \in G_2 \backslash H$

We note that $x \in K_1 y K_2$ if and only if $x = k_1 y k_2$ for some $k_1 \in K_1$, $k_2 \in K_2$ and there exists $h \in H$ such that $x = hk_2$ and $y = k_1^{-1} h$. First suppose $x \notin HK_2$. Since G_2 is HK_2-separable, there exists $M \lhd_f G_2$ such that $x \notin (HK_2)M$. By (c), we can find $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $N_2 \subseteq M$ and $N_1 \cap H = N_2 \cap H$. As in Case 1, we form $\bar{E}$. Then $\bar{x} \notin \bar{H}\bar{K}_2$ in $\bar{E}$. This implies that $\bar{x} \notin \bar{K}_1 \bar{y} \bar{K}_2$ and we are done. The case where $y \notin K_1 H$ can be similarly proved.

Now suppose $x \in HK_2$ and $y \in K_1 H$. Let $x = h_1 k_2$, and $y = k_1^{-1} h_2$, where $k_1 \in K_1$, $k_2 \in K_2$, and h_1, $h_2 \in H$. Since $K_1 \cap H = 1 = K_2 \cap H$, we see that k_j, h_j are uniquely determined. Clearly $x \notin K_1 y K_2$ implies that $h_1 \neq h_2$. By residual finiteness of H, there exists $S \lhd_f H$ such that $h_1 h_2^{-1} \notin$

S. By H-separability of G_2, there exists $T \triangleleft_f G_2$ such that $x \notin HT$. Let $R = S \cap T$. Then $R \triangleleft_f H$. Hence by (b), there exist $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that $N_2 \subseteq T$, $N_1 \cap H = N_2 \cap H \subseteq R$ and $K_1N_1 \cap HN_1 = N_1$, $K_2N_2 \cap HN_2 = N_2$. Again we form $\bar{E}$. Then $\bar{x} \in \bar{G}_2 \backslash \bar{H}$, $\bar{h}_1 \neq \bar{h}_2$, and $\bar{K}_1 \cap \bar{H} = \bar{1} = \bar{K}_2 \cap \bar{H}$ in $\bar{E}$. This implies that $\bar{k}_j$, $\bar{h}_j$ are uniquely determined. Since $\bar{h}_1 \neq \bar{h}_2$, we have $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ and the result follows.

Case 3. *$x \notin G_1 \cup G_2$; That Is, $\|x\| \geq 2$ in E*

Subcase 1. $\|x\| > 2$. As before, we form $\bar{E}$ in which $\bar{x}$ is reduced and $\|\bar{x}\| = \|x\|$ in $\bar{E}$. Clearly $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ and our result follows.

Subcase 2. $\|x\| = 2$. Let $x = ab$, where $a \in G_1 \backslash H$ and $b \in G_2 \backslash H$.

(i) $y \in H$. Then $x \in K_1yK_2$ if and only if $x = k_1yk_2$ for some $k_1 \in K_1, k_2 \in K_2$ and there exist $h_1, h_2 \in H$ such that $a = k_1h_1, b = h_2k_2$, where $y = h_1h_2$. Following a very similar argument as in Case 2, we can find $P \triangleleft_f E$ such that $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ in $\bar{E} = E/P$ as required.

(ii) $y \notin H$. Then $x \in K_1yK_2$ if and only if $x = k_1yk_2$ for some $k_1 \in K_1$, $k_2 \in K_2$ and there exists $h \in H$ such that $a = k_1yh$ and $b = h^{-1}k_2$. Following a very similar argument as in Case 2, we can find $P \triangleleft_f E$ such that $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ in $\bar{E} = E/P$ as required.

Let $x = ba$, where $a \in G_1 \backslash H$ and $b \in G_2 \backslash H$. By H-separability of G_1, G_2 and by (c), there exist $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that $a \notin HN_1$, $b \notin HN_2$, and $N_1 \cap H = N_2 \cap H$. As before, we form $\bar{E}$. Then $\bar{a} \in \bar{G}_1 \backslash \bar{H}$ and $\bar{b} \in \bar{G}_2 \backslash \bar{H}$ in $\bar{E}$. Clearly $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ and we are done.

LEMMA 16. *Let $E = G_1 *_H G_2$, where $G_1 \cap G_2 = H$ and G_2 is residually finite. Let K_1, K_2 be subgroups of G_1 such that $K_1^y \cap H = 1 = K_2^y \cap H$ for all $y \in G_1$. Suppose that*

(a) *G_1, G_2 are H-separable;*

(b) *for each $R \triangleleft_f H$ and $u, v \in G_1$, there exists $M \triangleleft_f G_1$ such that $M \cap H \subseteq R$ and $K_1^uM \cap HM = M$, $K_2^vM \cap HM = M$;*

(c) *for $u \notin K_1vK_2$, where $u, v \in G_1$, there exists $N \triangleleft_f G_1$ such that $\bar{u} \notin \bar{K}_1\bar{v}\bar{K}_2$ in $\bar{G}_1 = G_1/N$;*

(d) *for $u \notin K_1vH$, where $u, v \in G_1$, there exists $S \triangleleft_f G_1$ such that $\bar{u} \notin \bar{K}_1\bar{v}\bar{H}$ in $\bar{G}_1 = G_1/S$;*

(e) *for $u \notin HvK_2$, where $u, v \in G_1$, there exists $T \triangleleft_f G_1$ such that $\bar{u} \notin \bar{H}\bar{v}\bar{K}_2$ in $\bar{G}_1 = G_1/T$;*

(f) *for each $R \triangleleft_f H$, there exist $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that $N_1 \cap H = R = N_2 \cap H$.*

Let $x \notin K_1yK_2$, where x, $y \in E$. Then there exists $P \triangleleft_f E$ such that $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$, in $\bar{E} = E/P$.

Proof. The case $y \in G_1 \cup G_2$ follows from Lemma 14. Let $y \notin G_1 \cup G_2$ and $x \notin K_1yK_2$ be a reduced element of E.

Case 1. $x \in G_1 \cup G_2$

Clearly $x \notin K_1yK_2$ if and only if $y \notin K_1xK_2$. As in Lemma 14, we can find $P \triangleleft_f E$ such that $\bar{y} \notin \bar{K}_1\bar{x}\bar{K}_2$ in $\bar{E} = E/P$. Therefore $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ in $\bar{E} = E/P$ and we are done.

Case 2. $x \notin G_1 \cup G_2$; *That Is*, $\|x\| \geq 2$ *in* E

Subcase 1. $\|x\| \neq \|y\| \pm 2$, $\|y\| \pm 1$ and $\|y\|$. As before, we form $\bar{E}$ in which $\|\bar{x}\| = \|x\|$ and $\|\bar{y}\| = \|y\|$ in $\bar{E}$. So $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ and our result follows.

Subcase 2. $\|x\| = \|y\| \pm 2, \|y\| \pm 1$ or $\|y\|$. We shall only consider the case $x = a_1b_1a_2\cdots a_nb_na_{n+1}$, $y = u_1v_1u_2\cdots u_nv_nu_{n+1}$ where a_i, $u_i \in G_1\backslash H$, b_i, $v_i \in G_2\backslash H$ for all i. The other cases are similar. Since G_1, G_2 are H-separable, there exist $M_1 \triangleleft_f G_1$, $M_2 \triangleleft_f G_2$ such that a_i, $u_i \notin HM_1$, b_i, $v_i \notin HM_2$ for all i. We note that $x \in K_1yK_2$ if and only if $x = k_1yk_2 = k_1u_1v_1u_2\cdots u_nv_nu_{n+1}k_2$ for some $k_1 \in K_1$, $k_2 \in K_2$ and there exist $h_1, l_1, \ldots, h_n, l_n \in H$ such that

$$
\begin{aligned}
a_1 &= k_1u_1h_1\\
b_1 &= h_1^{-1}v_1l_1\\
a_2 &= l_1^{-1}u_2h_2\\
&\;\vdots\\
a_n &= l_{n-1}^{-1}u_nh_n\\
b_n &= h_n^{-1}v_nl_n\\
a_{n+1} &= l_n^{-1}u_{n+1}k_2.
\end{aligned}
\tag{1}
$$

Since $K_1^{u_1} \cap H = 1 = K_2^{u_{n+1}} \cap H$ by our assumption, we see that k_j, h_j, and l_j are uniquely determined. Now, $x \notin K_1yK_2$ implies that at least one equation of (1) is not satisfied. Suppose that $b_i = h_i^{-1}v_il_i$ is the first equation not satisfied, that is, $v_i^{-1}h_ib_i \notin H$. By H-separability of G_2, there exists $M \triangleleft_f G_2$ such that $v_i^{-1}h_ib_i \notin HM$. Hence by (b) and (f), there exist $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that $N_1 \subseteq M_1$, $N_2 \subseteq M_2 \cap M$, $N_1 \cap H = N_2 \cap H$, and $K_1^{u_1}N_1 \cap HN_1 = N_1$, $K_2^{u_{n+1}}N_2 \cap HN_2 = N_2$. As before, we form $\bar{E}$. Then $\bar{a}_i$, $\bar{u}_i \in \bar{G}_1\backslash\bar{H}$, $\bar{b}_i$, $\bar{v}_i \in \bar{G}_2\backslash\bar{H}$ for all i, and $\bar{v}_i^{-1}\bar{h}_i\bar{b}_i \notin \bar{H}$ in $\bar{E}$. Furthermore, $\bar{K}_1^{u_1} \cap \bar{H} = \bar{1} = \bar{K}_2^{u_{n+1}} \cap \bar{H}$ in $\bar{E}$.

Suppose $\bar{x} \in \bar{K}_1\bar{y}\bar{K}_2$. Then

$$\bar{x} = \bar{a}_1\bar{b}_1\bar{a}_2\cdots\bar{a}_n\bar{b}_n\bar{a}_{n+1} = \bar{t}_1\bar{u}_1\bar{v}_1\bar{u}_2\cdots\bar{u}_n\bar{v}_n\bar{u}_{n+1}\bar{t}_2$$

for some $\bar{t}_1 \in \bar{K}_1$, $\bar{t}_2 \in \bar{K}_2$ and there exist $\bar{r}_1, \bar{w}_1, \ldots, \bar{r}_n, \bar{w}_n \in \bar{H}$ such that

$$\begin{aligned}
\bar{a}_1 &= \bar{t}_1\bar{u}_1\bar{r}_1\\
\bar{b}_1 &= \bar{r}_1^{-1}\bar{v}_1\bar{w}_1\\
\bar{a}_2 &= \bar{w}_1^{-1}\bar{u}_2\bar{r}_2\\
&\vdots\\
\bar{a}_n &= \bar{w}_{n-1}^{-1}\bar{u}_n\bar{r}_n\\
\bar{b}_n &= \bar{r}_n^{-1}\bar{v}_n\bar{w}_n\\
\bar{a}_{n+1} &= \bar{w}_n^{-1}\bar{u}_{n+1}\bar{t}_2.
\end{aligned} \tag{2}$$

Since $\bar{K}_1^{\bar{u}_1} \cap \bar{H} = \bar{1} = \bar{K}_2^{\bar{u}_{n+1}} \cap \bar{H}$, we see that $\bar{t}_j$, $\bar{r}_j$, and $\bar{w}_j$ are uniquely determined. Therefore $\bar{k}_1 = \bar{t}_1$, $\bar{h}_1 = \bar{r}_1$, $\bar{l}_1 = \bar{w}_1, \ldots, \bar{h}_{i-1} = \bar{r}_{i-1}$, $\bar{l}_{i-1} = \bar{w}_{i-1}$, and $\bar{h}_i = \bar{r}_i$. This implies that $\bar{v}_i^{-1}\bar{h}_i\bar{b}_i = \bar{v}_i^{-1}\bar{r}_i\bar{b}_i = \bar{w}_i \in \bar{H}$, a contradiction. Therefore $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ and we are done.

LEMMA 17. *Let $E = G_1 *_H G_2$, where $G_1 \cap G_2 = H$ and H is residually finite. Let K_1, K_2 be subgroups of G_1, G_2, respectively, such that $K_1^{y_1} \cap H = 1 = K_2^{y_2} \cap H$ for all $y_1 \in G_1$ and $y_2 \in G_2$. Suppose that*

(a) *G_1 is H-separable, K_1-separable and G_2 is H-separable, K_2-separable;*

(b) *for each $R \lhd_f H$ and $u \in G_1, v \in G_2$, there exist $M_1 \lhd_f G_1, M_2 \lhd_f G_2$ such that $M_1 \cap H = M_2 \cap H \subseteq R$ and $K_1^u M_1 \cap HM_1 = M_1$, $K_2^v M_2 \cap HM_2 = M_2$;*

(c) *for $u \notin K_1 vH$, where $u, v \in G_1$, there exists $S \lhd_f G_1$ such that $\bar{u} \notin \bar{K}_1\bar{v}\bar{H}$ in $\bar{G}_1 = G_1/S$;*

(d) *for $u \notin HvK_2$, where u, $v \in G_2$, there exists $T \lhd_f G_2$ such that $\bar{u} \notin \bar{H}\bar{v}\bar{K}_2$ in $\bar{G}_1 = G_1/T$;*

(e) *for each $R \lhd_f H$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $N_1 \cap H = R = N_2 \cap H$.*

Let $x \notin K_1 y K_2$, where $x, y \in E$. Then there exists $P \lhd_f E$ such that $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ in $\bar{E} = E/P$.

Proof. The case $y \in G_1 \cup G_2$ follows from Lemma 15. Let $y \notin G_1 \cup G_2$ and $x \notin K_1 y K_2$ be a reduced element of E. Then following a very similar argument as in Lemma 16, we can find $P \lhd_f E$ such that $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ in $\bar{E} = E/P$ as required.

To extend Lemmas 16 and 17 to a tree product (Lemma 26), we shall need the following extra lemmas (Lemmas 18–25):

LEMMA 18. *Let* $E = G_1 *_H G_2$, *where* $G_1 \cap G_2 = H$ *and* $H \subseteq Z(G_2)$. *Let* K, K_1, K_2 *be subgroups of* G_1 *such that* $K_1^y \cap K = 1 = K_2^y \cap K$ *for all* $y \in G_1$. *Suppose that*

(a) G_1, G_2 *are* H*-separable*;

(b) *for each* $S \lhd_f K$ *and* u, $v \in G_1$, *there exists* $M \lhd_f G_1$ *such that* $M \cap K \subseteq S$ *and* $K_1^u M \cap KM = M$, $K_2^v M \cap KM = M$;

(c) *for each* $R \lhd_f H$, *there exist* $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ *such that* $N_1 \cap H = R = N_2 \cap H$.

Then $K_1^z \cap K = 1$ *and* $K_2^z \cap K = 1$ *for all* $z \in E$. *Furthermore, for each* $S \lhd_f K$ *and* $x, y \in E$, *there exists* $P \lhd_f E$ *such that* $P \cap K \subseteq S$ *and* $K_1^x P \cap KP = P$, $K_2^y P \cap KP = P$.

Proof. Since $K_1^y \cap K = 1 = K_2^y \cap K$ for all $y \in G_1$ and $H \subseteq Z(G_2)$, clearly $K_1^z \cap K = 1$ and $K_2^z \cap K = 1$ for all $z \in E$. Let $x, y \in E$ and $S \lhd_f K$ be given. We divide the proof into several cases.

Case 1. $x, y \in G_1$. By (b), for $S \lhd_f K$ and x, $y \in G_1$, there exists $N_1 \lhd_f G_1$ such that $N_1 \cap K \subseteq S$ and $K_1^x N_1 \cap KN_1 = N_1$, $K_2^y N_1 \cap KN_1 = N_1$. Then by (c), there exists $N_2 \lhd_f G_2$ such that $N_2 \cap H = N_1 \cap H$. Now we form $\bar{E} = \bar{G}_1 *_H \bar{G}_2$, where $\bar{G}_1 = G_1/N_1$, $\bar{G}_2 = G_2/N_2$, and $\bar{H} = HN_1/N_1 = HN_2/N_2$. Clearly $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. Since $\bar{E}$ is residually finite and $\bar{K}$, $\bar{K}_1$, $\bar{K}_2$ are finite, there exists $\bar{P} \lhd_f \bar{E}$ such that $\bar{P} \cap \bar{K}_1^{\bar{x}} \bar{K} = \bar{1}$ and $\bar{P} \cap \bar{K}_2^{\bar{y}} \bar{K} = \bar{1}$. This implies that $\bar{P} \cap \bar{K} = \bar{1}$ and $\bar{K}_1^{\bar{x}} \bar{P} \cap \bar{K}\bar{P} = \bar{P}$, $\bar{K}_2^{\bar{y}} \bar{P} \cap \bar{K}\bar{P} = \bar{P}$. Let P be the preimage of $\bar{P}$ in E. Then P is the required normal subgroup.

Case 2. x, $y \in G_2 \backslash H$. By H-separability of G_2 and by (c), there exist $M_1 \lhd_f G_1$, $M_2 \lhd_f G_2$ such that x, $y \notin HM_2$ and $M_1 \cap H = M_2 \cap H$. Let $S_1 = M_1 \cap S$. Then $S_1 \lhd_f K$. Hence by (b), for $S_1 \lhd_f K$, there exists $T_1 \lhd_f G_1$ such that $T_1 \cap K = S_2 \subseteq S_1$ and $K_1 T_1 \cap KT_1 = T_1$, $K_2 T_1 \cap KT_1 = T_1$. Let $N_1 = M_1 \cap T_1$. Then $N_1 \lhd_f G_1$, $N_1 \cap K = S_2$ and $K_1 N_1 \cap KN_1 = N_1$, $K_2 N_1 \cap KN_1 = N_1$. By (c), we can find $N_2 \lhd_f G_2$ such that $N_2 \subseteq M_2$ and $N_1 \cap H = N_2 \cap H$. Again we form $\bar{E}$. Then $\bar{x}$, $\bar{y} \in \bar{G}_2 \backslash \bar{H}$ and $\bar{K}_1 \cap \bar{K} = \bar{1}$, $\bar{K}_2 \cap \bar{K} = \bar{1}$ in $\bar{E}$. Since $\bar{H} \subseteq Z(\bar{G}_2)$, it follows that $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$. The result then follows as in Case 1.

Case 3. x, $y \notin G_1 \cup G_2$. WLOG, let $x = a_1 b_1 \cdots a_n b_n$ and $y = c_1 d_1 \cdots c_m d_m$, where a_i, $c_i \in G_1 \backslash H$ and b_i, $d_i \in G_2 \backslash H$ for all i. By H-separability of G_1, G_2 and by (c), there exist $M_1 \lhd_f G_1$, $M_2 \lhd_f G_2$ such that a_i, $c_i \notin HM_1$, b_i, $d_i \notin HM_2$ for all i, and $M_1 \cap H = M_2 \cap H$. Let

$S_1 = M_1 \cap S$. Then $S_1 \lhd_f K$. Now, let $x_1 = a_1 a_2 \cdots a_n$ and $y_1 = c_1 c_2 \cdots c_m$. Then $x_1, y_1 \in G_1$. Hence by (b), for $S_1 \lhd_f K$ and $x_1, y_1 \in G_1$, there exists $N_1 \lhd_f G_1$ such that $N_1 \subseteq M_1$, $N_1 \cap K \subseteq S_1$ and $K_1^{x_1} N_1 \cap K N_1 = N_1, K_2^{y_1} N_1 \cap K N_1 = N_1$. By (c) we can find $N_2 \lhd_f G_2$ such that $N_2 \subseteq M_2$ and $N_1 \cap H = N_2 \cap H$. Again we form $\bar{E}$. Then $\bar{a}_i$, $\bar{c}_i \in \bar{G}_1 \backslash \bar{H}$, $\bar{b}_i$, $\bar{d}_i \in \bar{G}_2 \backslash \bar{H}$ for all i and $\bar{K}_1^{\bar{x}_1} \cap \bar{K} = \bar{1}$, $\bar{K}_2^{\bar{y}_1} \cap \bar{K} = \bar{1}$ in $\bar{E}$.

Next we show that $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. Let $\bar{u}$ be any element in $\bar{K}_1^{\bar{x}}$. Then $\bar{u} = \bar{k}^{\bar{x}} = \bar{b}_n^{-1} \bar{a}_n^{-1} \cdots \bar{b}_1^{-1} \bar{a}_1^{-1} \bar{k} \bar{a}_1 \bar{b}_1 \cdots \bar{a}_n \bar{b}_n$ for some $\bar{k} \in \bar{K}_1$. Now let $\bar{u}_i = \bar{a}_i^{-1} \bar{b}_{i-1}^{-1} \cdots \bar{b}_1^{-1} \bar{a}_1^{-1} \bar{k} \bar{a}_1 \bar{b}_1 \cdots \bar{b}_{i-1} \bar{a}_i$, where $1 \le i \le n$. Then $\bar{u} = \bar{k}^{\bar{x}} = \bar{b}_n^{-1} \bar{a}_n^{-1} \cdots \bar{b}_i^{-1} \bar{u}_i \bar{b}_i \cdots \bar{a}_n \bar{b}_n$ for each $1 \le i \le n$.

First suppose $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{i-1} \in \bar{H}$ but $\bar{u}_i \notin \bar{H}$ for some $1 \le i \le n$. Since $\bar{k}^{\bar{x}} = \bar{b}_n^{-1} \bar{a}_n^{-1} \cdots \bar{b}_i^{-1} \bar{u}_i \bar{b}_i \cdots \bar{a}_n \bar{b}_n$, it follows that $\bar{k}^{\bar{x}}$ has length greater than 1. Hence $\bar{u} = \bar{k}^{\bar{x}} \notin \bar{K}$.

Now suppose $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n \in \bar{H}$. This implies that $\bar{b}_i$ commutes with $\bar{u}_i$ for each i since $\bar{u}_i \in \bar{H} \subseteq Z(\bar{G}_2)$. Hence $\bar{k}^{\bar{x}} = \bar{b}_n^{-1} \bar{u}_n \bar{b}_n = \bar{u}_n = \bar{a}_n^{-1} \bar{b}_{n-1}^{-1} \bar{u}_{n-1} \bar{b}_{n-1} \bar{a}_n = \bar{a}_n^{-1} \bar{u}_{n-1} \bar{a}_n = \bar{a}_n^{-1} \bar{a}_{n-1}^{-1} \bar{b}_{n-2}^{-1} \bar{u}_{n-2} \bar{b}_{n-2} \bar{a}_{n-1} \bar{a}_n = \cdots = \bar{a}_n^{-1} \bar{a}_{n-1}^{-1} \cdots \bar{a}_1^{-1} \bar{k} \bar{a}_1 \cdots \bar{a}_{n-1} \bar{a}_n = \bar{x}_1^{-1} k \bar{x}_1$. Therefore $\bar{k}^{\bar{x}} = \bar{k}^{\bar{x}_1} \in \bar{K}_1^{\bar{x}_1}$. Since $\bar{K}_1^{\bar{x}_1} \cap \bar{K} = \bar{1}$, it follows that $\bar{u} = \bar{k}^{\bar{x}} \notin \bar{K}$. Hence $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$. Similarly we can show that $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$. Then our result follows as in Case 1.

Case 4. $x \in G_1$, $y \in G_2 \backslash H$. By H-separability of G_2 and by (b) and (c), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $y \notin HN_2$, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$, and $K_1^x N_1 \cap K N_1 = N_1$, $K_2 N_1 \cap K N_1 = N_1$. Again we form $\bar{E}$. Then $\bar{y} \in \bar{G}_2 \backslash \bar{H}$ and $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{K}_2 \cap \bar{K} = \bar{1}$ in $\bar{E}$. As in Case 2, we have $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$ and we are done as above.

Case 5. $x \in G_1$, $y \notin G_1 \cup G_2$. Let y and y_1 be as in Case 3. By H-separability of G_1, G_2 and by (b) and (c), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $c_i \notin HN_1$, $d_i \notin HN_2$ for all i, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$, and $K_1^x N_1 \cap K N_1 = N_1$, $K_2^{y_1} N_1 \cap K N_1 = N_1$. Again we form $\bar{E}$. Then $\bar{c}_i \in \bar{G}_1 \backslash \bar{H}$, $\bar{d}_i \in \bar{G}_2 \backslash \bar{H}$ for all i and $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{K}_2^{\bar{y}_1} \cap \bar{K} = \bar{1}$ in $\bar{E}$. As in Case 3, we have $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$ and our result follows as before.

Case 6. $x \in G_2 \backslash H$, $y \notin G_1 \cup G_2$. Let y and y_1 be as in Case 3. By H-separability of G_1, G_2 and by (b) and (c), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $c_i \notin HN_1$, x, $d_i \notin HN_2$ for all i, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$ and $K_1 N_1 \cap K N_1 = N_1$, $K_2^{y_1} N_1 \cap K N_1 = N_1$. Again we form $\bar{E}$. Then $\bar{c}_i \in \bar{G}_1 \backslash \bar{H}$, $\bar{x}$, $\bar{d}_i \in \bar{G}_2 \backslash \bar{H}$ for all i and $\bar{K}_1 \cap \bar{K} = \bar{1}$, $\bar{K}_2^{\bar{y}_1} \cap \bar{K} = \bar{1}$ in $\bar{E}$. As in Case 2 and Case 3, we have $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$, respectively. Our result now follows as before.

The following cases are similar to Cases 4, 5, and 6, respectively.

Case 7. $x \in G_2 \backslash H$, $y \in G_1$.

Case 8. $x \notin G_1 \cup G_2$, $y \in G_1$.

Case 9. $x \notin G_1 \cup G_2$, $y \in G_2 \backslash H$.

LEMMA 19. *Let* $E = G_1 *_H G_2$, *where* $G_1 \cap G_2 = H$ *and* $H \subseteq Z(G_2)$. *Let* K, K_1, K_2 *be subgroups of* G_2 *such that* $K_1, K_2 \subseteq Z(G_2)$ *and* $K_1 \cap K = K_2 \cap K = H \cap K = 1$. *Suppose that*

(a) G_1, G_2 *are* H-*separable*;

(b) *for each* $S \triangleleft_f K$, *there exists* $M \triangleleft_f G_2$ *such that* $M \cap K = S$ *and* $K_1 M \cap KM = M$;

(c) *for each* $S \triangleleft_f K$, *there exists* $N \triangleleft_f G_2$ *such that* $N \cap K = S$ *and* $K_2 N \cap KN = N$;

(d) *for each* $S \triangleleft_f K$, *there exists* $L \triangleleft_f G_2$ *such that* $L \cap K = S$ *and* $HL \cap KL = L$;

(e) *for each* $R \triangleleft_f H$, *there exist* $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ *such that* $N_1 \cap H = R = N_2 \cap H$.

Then $K_1^z \cap K = 1$ *and* $K_2^z \cap K = 1$ *for all* $z \in E$. *Furthermore, for each* $S \triangleleft_f K$ *and* $x, y \in E$, *there exists* $P \triangleleft_f E$ *such that* $P \cap K \subseteq S$ *and* $K_1^x P \cap KP = P$, $K_2^y P \cap KP = P$.

Proof. Since $J \cap K = 1$ and $J \subseteq Z(G_2)$ for $J = H, K_1, K_2$, clearly $K_1^z \cap K = 1$ and $K_2^z \cap K = 1$ for all $z \in E$. Let $x, y \in E$ and $S \triangleleft_f K$ be given. We divide the proof into several cases.

Case 1. x, $y \in G_1 \backslash H$. By H-separability of G_1 and by (d) and (e), for $S \triangleleft_f K$, there exist $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that $x, y \notin HN_1$, $N_2 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$ and $HN_2 \cap KN_2 = N_2$. Now we form $\bar{E} = \bar{G}_1 *_H \bar{G}_2$, where $\bar{G}_1 = G_1/N_1$, $\bar{G}_2 = G_2/N_2$ and $\bar{H} = HN_1/N_1 = HN_2/N_2$. Then $\bar{x}$, $\bar{y} \in \bar{G}_1 \backslash \bar{H}$ and $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}^{-1}}$, $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}_1 \cap \bar{K}^{\bar{x}^{-1}} = \bar{1}$, $\bar{K}_2 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Consequently, $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. We can now proceed as in Case 1 of Lemma 18 to obtain our result.

Case 2. x, $y \in G_2$. Since $K_1, K_2 \subseteq Z(G_2)$, then $K_1^x = K_1$ and $K_2^y = K_2$. By (b) and (c), for $S \triangleleft_f K$, there exists $N_2 \triangleleft_f G_2$ such that $N_2 \cap K = S$ and $K_1 N_2 \cap KN_2 = N_2$, $K_2 N_2 \cap KN_2 = N_2$. By (e), there exists $N_1 \triangleleft_f G_1$ such that $N_1 \cap H = N_2 \cap H$. As above, we form $\bar{E}$. Clearly $\bar{K}_1 \cap \bar{K} = \bar{1}$ and $\bar{K}_2 \cap \bar{K} = \bar{1}$ in $\bar{E}$. Since $\bar{K}_1, \bar{K}_2 \subseteq Z(\bar{G}_2)$, then $\bar{K}_1^{\bar{x}} = \bar{K}_1$, $\bar{K}_2^{\bar{y}} = \bar{K}_2$ and hence $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. The result now follows as in Case 1.

Case 3. $x, y \notin G_1 \cup G_2$. WLOG, let $x = a_1 b_1 \cdots a_n b_n$ and $y = c_1 d_1 \cdots c_m d_m$, where a_i, $c_i \in G_1 \backslash H$ and b_i, $d_i \in G_2 \backslash H$ for all i. By

H-separability of G_1, G_2 and by (d) and (e), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that a_i, $c_i \notin HN_1$ and b_i, $d_i \notin HN_2$ for all i, $N_2 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$, and $HN_2 \cap KN_2 = N_2$. As before, we form $\bar{E}$. Then $\bar{a}_i$, $\bar{c}_i \in \bar{G}_1 \backslash \bar{H}$, $\bar{b}_i$, $\bar{d}_i \in \bar{G}_2 \backslash \bar{H}$ for all i, and $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}^{-1}}$, $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}_1 \cap \bar{K}^{\bar{x}^{-1}} = \bar{1}$, $\bar{K}_2 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Consequently, $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. Our result then follows as above.

The remaining cases can be similarly proved as in Lemma 18.

LEMMA 20. *Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$. Let K, K_1, K_2 be subgroups of G_r such that K_1, $K_2 \subseteq Z(G_r)$ and $K_1 \cap K = K_2 \cap K = H_{ri} \cap K = 1$. Suppose that*

(a) *G_i is H_{ij}-separable;*

(b) *for each $S \lhd_f K$, there exists $M \lhd_f G_r$ such that $M \cap K = S$ and $K_1 M \cap KM = M$;*

(c) *for each $S \lhd_f K$, there exists $N \lhd_f G_r$ such that $N \cap K = S$ and $K_2 N \cap KN = N$;*

(d) *for each $S \lhd_f K$, there exists $L_{ri} \lhd_f G_r$ such that $L_{ri} \cap K = S$ and $H_{ri} L_{ri} \cap KL_{ri} = L_{ri}$;*

(e) *for each $R_{ij} \lhd_f H_{ij}$, there exists $N_i \lhd_f G_i$ such that $N_i \cap H_{ij} = R_{ij}$.*

Then $K_1^z \cap K = 1$ and $K_2^z \cap K = 1$ for all $z \in G$. Furthermore, for each $S \lhd_f K$ and x, $y \in G$, there exists $P \lhd_f G$ such that $P \cap K \subseteq S$ and $K_1^x P \cap KP = P$, $K_2^y P \cap KP = P$.

Proof. We use induction on n. The case $n = 2$ follows easily from Lemma 19.

Let $n \geq 3$. As before, we write $G = E *_H G_n$, where E is the tree product generated by G_1, $G_2, \ldots, G_{n-1}$ and $H = H_{(n-1)n} = H_{n(n-1)}$. By Lemma 10, E is $H_{(n-1)n}$-separable and by Lemma 8, for each $R \lhd_f H_{(n-1)n}$, there exists $L \lhd_f E$ such that $L \cap H_{(n-1)n} = R$.

Case 1. $K, K_1, K_2 < E$. By induction, $K_1^z \cap K = 1, K_2^z \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and $u, v \in E$, there exists $Q \lhd_f E$ such that $Q \cap K \subseteq S$ and $K_1^u Q \cap KQ = Q$, $K_2^v Q \cap KQ = Q$. Since $H_{n(n-1)} \subseteq Z(G_n)$, the result follows from Lemma 18.

Case 2. K, K_1, $K_2 < G_n$. The result follows from Lemma 19.

LEMMA 21. *Let $E = G_1 *_H G_2$, where $G_1 \cap G_2 = H$. Let K be a subgroup of G_1 such that $H^y \cap K = 1$ for all $y \in G_1$ and let K_1, K_2 be subgroups of G_2.*

Suppose that

(a) G_1, G_2 *are* H*-separable*;

(b) *for each* $S \triangleleft_f K$ *and* $u, v \in G_1$, *there exists* $M \triangleleft_f G_1$ *such that* $M \cap K \subseteq S$ *and* $H^u M \cap KM = M, H^v M \cap KM = M$;

(c) *for each* $R \triangleleft_f H$, *there exist* $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ *such that* $N_1 \cap H = R = N_2 \cap H$.

Then $K_1^z \cap K = 1$ *and* $K_2^z \cap K = 1$ *for all* $z \in E$. *Furthermore, for each* $S \triangleleft_f K$ *and* $x, y \in E$, *there exists* $P \triangleleft_f E$ *such that* $P \cap K \subseteq S$, $K_1^x P \cap KP = P$, *and* $K_2^y P \cap KP = P$.

Proof. Since $H^y \cap K = 1$ for all $y \in G_1$ and $K \subset G_1, K_1, K_2 \subset G_2$, then $K_1^z \cap K = 1$ and $K_2^z \cap K = 1$ for all $z \in E$. Let $x, y \in E$ and $S \triangleleft_f K$ be given. We divide the proof into several cases.

Case 1. x, $y \in G_1$. By (b), for $S \triangleleft_f K$ and $x, y \in G_1$, there exists $N_1 \triangleleft_f G_1$ such that $N_1 \cap K \subseteq S$ and $H^x N_1 \cap KN_1 = N_1$, $H^y N_1 \cap KN_1 = N_1$. By (c), there exists $N_2 \triangleleft_f G_2$ such that $N_2 \cap H = N_1 \cap H$. We form $\bar{E} = \bar{G}_1 *_H \bar{G}_2$, where $\bar{G}_1 = G_1/N_1$, $\bar{G}_2 = G_2/N_2$, and $\bar{H} = HN_1/N_1 = HN_2/N_2$. Clearly $\bar{H}^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{H}^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. This implies $\bar{H} \cap \bar{K}^{\bar{x}^{-1}} = \bar{1}$ and $\bar{H} \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Since $\bar{K}_1$, $\bar{K}_2 \subset \bar{G}_2$, it follows that $\bar{K}_1 \cap \bar{K}^{\bar{x}^{-1}} = \bar{1}$, $\bar{K}_2 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$ and hence $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$. We can now proceed as in Case 1 of Lemma 18 to obtain our result.

Case 2. x, $y \in G_2 \backslash H$. By H-separability of G_2 and by (b) and (c), for $S \triangleleft_f K$, there exist $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that x, $y \notin HN_2$, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$, and $HN_1 \cap KN_1 = N_1$. As above, we form $\bar{E}$. Then $\bar{x}$, $\bar{y} \in \bar{G}_2 \backslash \bar{H}$ and $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}^{-1}}$, $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}_1 \cap \bar{K}^{\bar{x}^{-1}} = \bar{1}$, $\bar{K}_2 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Consequently, $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. The result then follows as in Case 1.

Case 3. x, $y \notin G_1 \cup G_2$. First suppose $x = a_1 b_1 \cdots a_n b_n$ and $y = c_1 d_1 \cdots c_m d_m$, where a_i, $c_i \in G_1 \backslash H$ and b_i, $d_i \in G_2 \backslash H$ for all i. By H-separability of G_1, G_2 and by (b) and (c), for $S \triangleleft_f K$, there exist $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that a_i, $c_i \notin HN_1$ and b_i, $d_i \notin HN_2$ for all i, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$, and $HN_1 \cap KN_1 = N_1$. Again we form $\bar{E}$. Then $\bar{a}_i$, $\bar{c}_i \in \bar{G}_1 \backslash \bar{H}$, $\bar{b}_i$, $\bar{d}_i \in \bar{G}_2 \backslash \bar{H}$ for all i, and $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}^{-1}}$, $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}_1 \cap \bar{K}^{\bar{x}^{-1}} = \bar{1}$, $\bar{K}_2 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Consequently, $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. Our result now follows as before.

Now suppose $x = a_1 b_1 \cdots a_n$ and $y = c_1 d_1 \cdots c_m$, where a_i, $c_i \in G_1 \backslash H$ and b_i, $d_i \in G_2 \backslash H$ for all i. As above, we can form $\bar{E}$ such that $\bar{a}_i$, $\bar{c}_i \in \bar{G}_1 \backslash \bar{H}$, $\bar{b}_i$, $\bar{d}_i \in \bar{G}_2 \backslash \bar{H}$ for all i and $\bar{H}^{\bar{a}_n} \cap \bar{K} = \bar{1}$, $\bar{H}^{\bar{c}_m} \cap \bar{K} = \bar{1}$ in $\bar{E}$. Clearly $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. Our result now follows as before.

The remaining cases can be similarly proved as in Lemma 18.

Lemma 21 can be extended to a tree product as follows:

LEMMA 22. *Let* $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ *be a tree product of* $G_1, G_2, \ldots, G_n$ *amalgamating the subgroups* H_{ij} *of* $Z(G_i)$ *and* H_{ji} *of* $Z(G_j)$. *Let* K *be a subgroup of* G_r *such that* $H_{ri} \cap K = 1$ *and let* K_1, K_2 *be subgroups of* $Z(G_s)$, *where* $r \neq s$. *Suppose that*

(a) G_i *is* H_{ij}*-separable*;

(b) *for each* $S \lhd_f K$, *there exists* $M_{ri} \lhd_f G_r$ *such that* $M_{ri} \cap K = S$ *and* $H_{ri}M_{ri} \cap KM_{ri} = M_{ri}$;

(c) *for each* $R_{ij} \lhd_f H_{ij}$, *there exists* $N_i \lhd_f G_i$ *such that* $N_i \cap H_{ij} = R_{ij}$.

Then $K_1^z \cap K = 1$ *and* $K_2^z \cap K = 1$ *for all* $z \in G$. *Furthermore, for each* $S \lhd_f K$ *and* $x, y \in G$, *there exists* $P \lhd_f G$ *such that* $P \cap K \subseteq S$ *and* $K_1^x P \cap KP = P$, $K_2^y P \cap KP = P$.

Proof. We use induction on n. Let $n = 2$. WLOG, we may assume that $K < G_1$. Since $H_{12} \subseteq Z(G_1)$, then $H_{12}^z = H_{12}$ for all $z \in G_1$. By assumption, $H_{12} \cap K = 1$ and by (b), for each $S \lhd_f K$, there exists $M \lhd_f G_1$ such that $M \cap K = S$ and $H_{12}M \cap KM = M$. The result then follows from Lemma 21.

Let $n \geq 3$. As before, we write $G = E *_H G_n$, where E is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$ and $H = H_{(n-1)n} = H_{n(n-1)}$. By Lemma 10, E is $H_{(n-1)n}$-separable and by Lemma 8, for each $R \lhd_f H_{(n-1)n}$, there exists $L \lhd_f E$ such that $L \cap H_{(n-1)n} = R$.

Case 1. $K, K_1, K_2 < E$. By induction, $K_1^z \cap K = 1$, $K_2^z \cap K = 1$, for all $z \in E$, and for each $S \lhd_f K$ and $u, v \in E$, there exists $N \lhd_f E$ such that $N \cap K \subseteq S$ and $K_1^u N \cap KN = N$, $K_2^v N \cap KN = N$. Since $H_{n(n-1)} \subseteq Z(G_n)$, our result follows from Lemma 18.

Case 2. $K < E$, $K_1, K_2 < G_n$. First suppose $H_{(n-1)n}$ and K are in different factors of E. Then by induction (with $K_1 = K_2 = H_{(n-1)n}$), we have $H_{(n-1)n}^z \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and $u, v \in E$, there exists $Q \lhd_f E$ such that $Q \cap K \subseteq S$ and $H_{(n-1)n}^u Q \cap KQ = Q$, $H_{(n-1)n}^v Q \cap KQ = Q$. Hence we are done by Lemma 21.

Now suppose $H_{(n-1)n}$ and K are in the same factor of E. Then we argue as above but in this case, we use Lemma 20 (with $K_1 = K_2 = H_{(n-1)n}$) instead of induction.

Case 3. $K_1, K_2 < E$, $K < G_n$. Since $H_{n(n-1)} \subseteq Z(G_n)$, then $H_{n(n-1)}^z = H_{n(n-1)}$ for all $z \in G_n$. By assumption, $H_{n(n-1)} \cap K = 1$ and by (b), for each $S \lhd_f K$, there exists $M \lhd_f G_n$ such that $M \cap K = S$ and $H_{n(n-1)}M \cap KM = M$. Again we are done by Lemma 21.

LEMMA 23. *Let* $E = G_1 *_H G_2$, *where* $G_1 \cap G_2 = H$. *Let* K, K_1 *be subgroups of* G_1 *such that* $K_1^y \cap K = 1 = H^y \cap K$ *for all* $y \in G_1$ *and let* K_2 *be a subgroup of* G_2. *Suppose that*

(a) G_1, G_2 *are* H*-separable*;

(b) *for each* $S \lhd_f K$ *and* $u, v \in G_1$, *there exists* $M \lhd_f G_1$ *such that* $M \cap K \subseteq S$ *and* $K_1^u M \cap KM = M$, $H^v M \cap KM = M$;

(c) *for each* $S \lhd_f K$ *and* $u, v \in G_1$, *there exists* $N \lhd_f G_1$ *such that* $N \cap K \subseteq S$ *and* $H^u N \cap KN = N$, $H^v N \cap KN = N$;

(d) *for each* $R \lhd_f H$, *there exist* $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ *such that* $N_1 \cap H = R = N_2 \cap H$.

Then $K_1^z \cap K = 1$ *and* $K_2^z \cap K = 1$ *for all* $z \in E$. *Furthermore, for each* $S \lhd_f K$ *and* $x, y \in E$, *there exists* $P \lhd_f E$ *such that* $P \cap K \subseteq S$ *and* $K_1^x P \cap KP = P$, $K_2^y P \cap KP = P$.

Proof. Since $K_1^y \cap K = 1 = H^y \cap K$ for all $y \in G_1$ and $K_2 \subset G_2$, it is easy to see that $K_1^z \cap K = 1$ and $K_2^z \cap K = 1$ for all $z \in E$. Let $x, y \in E$ and $S \lhd_f K$ be given. We divide the proof into several cases.

Case 1 x, $y \in G_1$. By (b), for $S \lhd_f K$ and x, $y \in G_1$, there exists $N_1 \lhd_f G_1$ such that $N_1 \cap K \subseteq S$ and $K_1^x N_1 \cap KN_1 = N_1$, $H^y N_1 \cap KN_1 = N_1$. By (d), there exists $N_2 \lhd_f G_2$ such that $N_2 \cap H = N_1 \cap H$. Now we form $\bar{E} = \bar{G}_1 *_{\bar{H}} \bar{G}_2$, where $\bar{G}_1 = G_1/N_1$, $\bar{G}_2 = G_2/N_2$, and $\bar{H} = HN_1/N_1 = HN_2/N_2$. Clearly $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{H}^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. This implies $\bar{H} \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Since $\bar{K}_2 \subset \bar{G}_2$, it follows that $\bar{K}_2 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$ and hence $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$. We can now proceed as in Case 1 of Lemma 18 to obtain our result.

Case 2. x, $y \in G_2 \backslash H$. By H-separability of G_2 and by (c) and (d), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $x, y \notin HN_2$, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$, and $HN_1 \cap KN_1 = N_1$. Again we form $\bar{E}$. Then $\bar{x}$, $\bar{y} \in \bar{G}_2 \backslash \bar{H}$ and $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}^{-1}}$, $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}_1 \cap \bar{K}^{\bar{x}^{-1}} = \bar{1}$, $\bar{K}_2 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Consequently, $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. Our result then follows as in Case 1.

Case 3. x, $y \notin G_1 \cup G_2$. WLOG, let $x = a_1 b_1 \cdots a_n b_n$ and $y = c_1 d_1 \cdots c_m d_m$, where a_i, $c_i \in G_1 \backslash H$ and b_i, $d_i \in G_2 \backslash H$ for all i. By H-separability of G_1, G_2 and by (c) and (d), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that a_i, $c_i \notin HN_1$ and $b_i, d_i \notin HN_2$ for all i, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$, and $HN_1 \cap KN_1 = N_1$. As above, we form $\bar{E}$. Then $\bar{a}_i$, $\bar{c}_i \in \bar{G}_1 \backslash \bar{H}$, $\bar{b}_i$, $\bar{d}_i \in \bar{G}_2 \backslash \bar{H}$ for all i, and $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}^{-1}}$, $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}_1 \cap \bar{K}^{\bar{x}^{-1}} = \bar{1}$, $\bar{K}_2 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Consequently, $\bar{K}_1^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_2^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. The result now follows as above.

The remaining cases can be similarly proved as in Lemma 18.

Next we extend Lemma 23 to a tree product.

LEMMA 24. *Let* $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ *be a tree product of* $G_1, G_2, \ldots, G_n$ *amalgamating the subgroups* H_{ij} *of* $Z(G_i)$ *and* H_{ji} *of* $Z(G_j)$. *Let* K, K_1 *be subgroups of* G_r *such that* $K_1 \subseteq Z(G_r)$ *and* $K_1 \cap K = 1 = H_{ri} \cap K$. *Let* K_2 *be a subgroup of* G_s *such that* $K_2 \subseteq Z(G_s)$, *where* $s \neq r$. *Suppose that*

(a) G_i *is* H_{ij}*-separable*;

(b) *for each* $S \lhd_f K$, *there exists* $M \lhd_f G_r$ *such that* $M \cap K = S$ *and* $K_1M \cap KM = M$;

(c) *for each* $S \lhd_f K$, *there exists* $N_{ri} \lhd_f G_r$ *such that* $N_{ri} \cap K = S$ *and* $H_{ri}N_{ri} \cap KN_{ri} = N_{ri}$;

(d) *for each* $R_{ij} \lhd_f H_{ij}$, *there exists* $N_i \lhd_f G_i$ *such that* $N_i \cap H_{ij} = R_{ij}$.

Then $K_1^z \cap K = 1$ *and* $K_2^z \cap K = 1$ *for all* $z \in G$. *Furthermore, for each* $S \lhd_f K$ *and* $x, y \in G$, *there exists* $P \lhd_f G$ *such that* $P \cap K \subseteq S$ *and* $K_1^xP \cap KP = P$, $K_2^yP \cap KP = P$.

Proof. We use induction on n. Let $n = 2$. WLOG, we may assume that $K < G_1$. Since $H_{12}, K_1 \subseteq Z(G_1)$, then $H_{12}^z = H_{12}$ and $K_1^z = K_1$ for all $z \in G_1$. By assumption, $H_{12} \cap K = 1$ and $K_1 \cap K = 1$. By (c), for each $S \lhd_f K$, there exists $N \lhd_f G_1$ such that $N \cap K = S$ and $H_{12}N \cap KN = N$. Furthermore, by (b) and (c), for each $S \lhd_f K$, there exists $T \lhd_f G_1$ such that $T \cap K = S$ and $K_1T \cap KT = T$, $H_{12}T \cap KT = T$. The result then follows from Lemma 23.

Let $n \geq 3$. As before, we write $G = E *_H G_n$, where E is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$ and $H = H_{(n-1)n} = H_{n(n-1)}$. By Lemma 10, E is $H_{(n-1)n}$-separable and by Lemma 8, for each $R \lhd_f H_{(n-1)n}$, there exists $L \lhd_f E$ such that $L \cap H_{(n-1)n} = R$.

Case 1. $K, K_1, K_2 < E$. By induction, $K_1^z \cap K = 1$, $K_2^z \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and $u, v \in E$, there exists $Q \lhd_f E$ such that $Q \cap K \subseteq S$ and $K_1^uQ \cap KQ = Q$, $K_2^vQ \cap KQ = Q$. Since $H_{n(n-1)} \subseteq Z(G_n)$, our result follows from Lemma 18.

Case 2. $K, K_1 < E$, $K_2 < G_n$. First suppose $H_{(n-1)n}$ and K are in different factors of E. Then by induction (with $K_2 = H_{(n-1)n}$), we have $K_1^z \cap K = 1, H_{(n-1)n}^z \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and u, $v \in E$, there exists $Q_1 \lhd_f E$ such that $Q_1 \cap K \subseteq S$ and $K_1^uQ_1 \cap KQ_1 = Q_1$, $H_{(n-1)n}^vQ_1 \cap KQ_1 = Q_1$. By Lemma 22 (with $K_1 = K_2 = H_{(n-1)n}$), we have $H_{(n-1)n}^z \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and $u, v \in E$, there exists $Q_2 \lhd_f E$ such that $Q_2 \cap K \subseteq S$ and $H_{(n-1)n}^uQ_2 \cap KQ_2 = Q_2, H_{(n-1)n}^vQ_2 \cap KQ_2 = Q_2$. The result then follows from Lemma 23.

Now suppose $H_{(n-1)n}$ and K are in the same factor of E. Then we argue as above but in this case, we use Lemma 20 twice (with $K_2 = H_{(n-1)n}$ and with $K_1 = K_2 = H_{(n-1)n}$) instead of induction and Lemma 22, respectively.

Case 3. $K_2 < E$, K, $K_1 < G_n$. Since $H_{n(n-1)}$, $K_1 \subseteq Z(G_n)$, then $H^z_{n(n-1)} = H_{n(n-1)}$ and $K^z_1 = K_1$ for all $z \in G_n$. By assumption, $H_{n(n-1)} \cap K = 1$ and $K_1 \cap K = 1$. By (b), for each $S \lhd_f K$, there exists $N \lhd_f G_n$ such that $N \cap K = S$ and $H_{n(n-1)}N \cap KN = N$. Furthermore, by (b) and (c), for each $S \lhd_f K$, we can find $T \lhd_f G_n$ such that $T \cap K = S$ and $K_1T \cap KT = T$, $H_{n(n-1)}T \cap KT = T$. The result now follows from Lemma 23.

LEMMA 25. *Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$. Let K be a subgroup of G_r such that $H_{ri} \cap K = 1$ and let K_1, K_2 be subgroups of $Z(G_s)$, $Z(G_t)$, respectively, where $s \neq r \neq t$, $s \neq t$. Suppose that*

(a) *G_i is H_{ij}-separable;*

(b) *for each $S \lhd_f K$, there exists $M_{ri} \lhd_f G_r$ such that $M_{ri} \cap K = S$ and $H_{ri}M_{ri} \cap KM_{ri} = M_{ri}$;*

(c) *for each $R_{ij} \lhd_f H_{ij}$, there exists $N_i \lhd_f G_i$ such that $N_i \cap H_{ij} = R_{ij}$.*

Then $K^z_1 \cap K = 1$ and $K^z_2 \cap K = 1$ for all $z \in G$. Furthermore, for each $S \lhd_f K$ and $x, y \in G$, there exists $P \lhd_f G$ such that $P \cap K \subseteq S$ and $K^x_1 P \cap KP = P$, $K^y_2 P \cap KP = P$.

Proof. We use induction on n. As before, we write $G = E *_H G_n$, where E is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$ and $H = H_{(n-1)n} = H_{n(n-1)}$. By Lemma 10, E is $H_{(n-1)n}$-separable and by Lemma 8, for each $R \lhd_f H_{(n-1)n}$, there exists $L \lhd_f E$ such that $L \cap H_{(n-1)n} = R$.

First, let $n = 3$. WLOG, let $G = \langle G_1, G_2, G_3; H_{12} = H_{21}, H_{23} = H_{32}\rangle$ and $E = \langle G_1, G_2; H_{12} = H_{21}\rangle$. We shall consider the following cases:

Case 1. $K, K_1 < E$, $K_2 < G_3$

First suppose $K < G_1$. Since $H_{12} \subseteq Z(G_1)$, then $H^z_{12} = H_{12}$ for all $z \in G_1$. By assumption, $H_{12} \cap K = 1$ and by (b), for each $S \lhd_f K$, there exists $M_{12} \lhd_f G_1$ such that $M_{12} \cap K = S$ and $H_{12}M_{12} \cap KM_{12} = M_{12}$. Then by Lemma 21 (with $K_2 = H_{23}$), we have $K^z_1 \cap K = 1$, $H^z_{23} \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and $u, v \in E$, there exists $M \lhd_f E$ such that $M \cap K \subseteq S$ and $K^u_1 M \cap KM = M$, $H^v_{23}M \cap KM = M$. Again, by Lemma 21 (with $K_1 = K_2 = H_{23}$), we have $H^z_{23} \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and $u, v \in E$, there exists $N \lhd_f E$ such that $N \cap K \subseteq S$ and $H^u_{23}N \cap KN = N$, $H^v_{23}N \cap KN = N$. The result now follows from Lemma 23.

Now suppose $K < G_2$. Since $H_{21} \subseteq Z(G_2)$, then $H^z_{21} = H_{21}$ for all $z \in G_2$. By assumption, $H_{21} \cap K = 1$ and by (b), for each $S \lhd_f K$, there exists

$M_{21} \lhd_f G_2$ such that $M_{21} \cap K = S$ and $H_{21}M_{21} \cap KM_{21} = M_{21}$. Then we argue as above but we use Lemma 23 (with $K_1 = H_{23}$, $K_2 = K_1$) and Lemma 19 (with $K_1 = K_2 = H_{23}$) instead of using Lemma 21 twice. The result again follows from Lemma 23.

Case 2. $K, K_2 < E, K_1 < G_3$

This case is similar to Case 1.

Case 3. $K_1, K_2 < E, K < G_3$

Since $H_{32} \subseteq Z(G_3)$, then $H_{32}^z = H_{32}$ for all $z \in G_3$. By assumption, $H_{32} \cap K = 1$ and by (b), for each $S \lhd_f K$, there exists $M_{32} \lhd_f G_3$ such that $M_{32} \cap K = S$ and $H_{32}M_{32} \cap KM_{32} = M_{32}$. The result now follows from Lemma 21.

Let $n \geq 4$. We shall consider the following cases:

Case 1. $K, K_1, K_2 < E$

By induction, $K_1^z \cap K = 1$, $K_2^z \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and $u, v \in E$, there exists $Q \lhd_f E$ such that $Q \cap K \subseteq S$ and $K_1^u Q \cap KQ = Q$, $K_2^v Q \cap KQ = Q$. Since $H_{n(n-1)} \subseteq Z(G_n)$, the result follows from Lemma 18.

Case 2. $K, K_1 < E, K_2 < G_n$

We shall consider the following subcases:

Subcase 1. Suppose $H_{(n-1)n}$, K, and K_1 are in different factors of E. Then by induction (with $K_2 = H_{(n-1)n}$), we have $K_1^z \cap K = 1$, $H_{(n-1)n}^z \cap K = 1$ for all $z \in E$, and for each $S \lhd_f K$ and $u, v \in E$, there exists $M \lhd_f E$ such that $M \cap K \subseteq S$ and $K_1^u M \cap KM = M$, $H_{(n-1)n}^v M \cap KM = M$. Furthermore, by Lemma 22 (with $K_1 = K_2 = H_{(n-1)n}$), we have $H_{(n-1)n}^z \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and $u, v \in E$, there exists $N \lhd_f E$ such that $N \cap K \subseteq S$ and $H_{(n-1)n}^u N \cap KN = N$, $H_{(n-1)n}^v N \cap KN = N$. Our result now follows from Lemma 23.

Subcase 2. Suppose $H_{(n-1)n}$ and K_1 are in the same factor of E. We argue as in Subcase 1 but in this case, we use Lemma 22 (with $K_2 = H_{(n-1)n}$) instead of induction.

Subcase 3. Suppose $H_{(n-1)n}$ and K are in the same factor of E. We argue as in Subcase 1 but in this case, we use Lemma 24 (with $K_1 = H_{(n-1)n}$, $K_2 = K_1$) and Lemma 20 (with $K_1 = K_2 = H_{(n-1)n}$) instead of induction and Lemma 22, respectively.

Case 3. K, $K_2 < E$, $K_1 < G_n$

This case is similar to Case 2.

Case 4. K_1, $K_2 < E$, $K < G_n$

Since $H_{n(n-1)} \subseteq Z(G_n)$, then $H^z_{n(n-1)} = H_{n(n-1)}$ for all $z \in G_n$. By assumption, $H_{n(n-1)} \cap K = 1$ and by (b), for each $S \lhd_f K$, there exists $M \lhd_f G_n$ such that $M \cap K = S$ and $H_{n(n-1)}M \cap KM = M$. So we are done by Lemma 21.

Now we can extend Lemmas 16 and 17 to a tree product (Lemma 26).

LEMMA 26. *Let $G_1, G_2, \ldots, G_n$ be residually finite groups. Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$, where $H_{ij} \cap H_{ik} = 1$ for $j \neq k$. Let K_1, K_2 be finitely generated subgroups of $Z(G_r)$, $Z(G_s)$, respectively, such that $K_1 \cap H_{ri} = 1 = K_2 \cap H_{si}$. Suppose that*

(a) *G_i is H_{ij}-separable, G_r is K_1-separable, and G_s is K_2-separable;*

(b) *for each $R_{ijk} \lhd_f H_{ij}$, there exists $M_{ijk} \lhd_f G_i$ such that $M_{ijk} \cap H_{ij} = R_{ijk}$ and $H_{ij}M_{ijk} \cap H_{ik}M_{ijk} = M_{ijk}$;*

(c) *for each $R_{ri} \lhd_f H_{ri}$, there exists $M_{ri} \lhd_f G_r$ such that $M_{ri} \cap H_{ri} = R_{ri}$ and $K_1M_{ri} \cap H_{ri}M_{ri} = M_{ri}$;*

(d) *for each $R_{si} \lhd_f H_{si}$, there exists $M_{si} \lhd_f G_s$ such that $M_{si} \cap H_{si} = R_{si}$ and $K_2M_{si} \cap H_{si}M_{si} = M_{si}$;*

(e) *for $u \notin J_1vJ_2$ where $u, v \in G_i$ and $J_1, J_2 \subseteq Z(G_i)$, there exists $L_i \lhd_f G_i$ such that $\bar{u} \notin \bar{J}_1\bar{v}\bar{J}_2$ in $\bar{G}_i = G_i/L_i$;*

(f) *for each $R_{ij} \lhd_f H_{ij}$, there exists $N_i \lhd_f G_i$ such that $N_i \cap H_{ij} = R_{ij}$.*

Let $x \notin K_1yK_2$, where $x, y \in G$. Then there exists $P \lhd_f G$ such that $\bar{x} \notin \bar{K}_1\bar{y}\bar{K}_2$ in $\bar{G} = G/P$.

Proof. We use induction on n. The case $n = 2$ follows from Lemmas 16 and 17.

Let $n \geq 3$. As before, we write $G = E *_H G_n$, where E is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$ and $H = H_{(n-1)n} = H_{n(n-1)}$. By Lemma 10, E is $H_{(n-1)n}$-separable and by Lemma 8, for each $R \lhd_f H_{(n-1)n}$, there exists $N_1 \lhd_f E$ such that $N_1 \cap H_{(n-1)n} = R$.

Case 1. K_1, $K_2 < E$. By Lemmas 20, 22, 24, or 25 (with $K = H_{(n-1)n}$), we have $K_1^y \cap H_{(n-1)n} = 1$, $K_2^y \cap H_{(n-1)n} = 1$ for all $y \in E$, and for each $R \lhd_f H_{(n-1)n}$ and $u, v \in E$, there exists $M \lhd_f E$ such that $M \cap H_{(n-1)n} \subseteq R$ and $K_1^uM \cap H_{(n-1)n}M = M$, $K_2^vM \cap H_{(n-1)n}M = M$. By induction,

(i) for $u \notin K_1 v K_2$, where $u, v \in E$, there exists $L_1 \lhd_f E$ such that $\bar{u} \notin \bar{K}_1 \bar{v} \bar{K}_2$ in $\bar{E} = E/L_1$;

(ii) for $u \notin K_1 v H_{(n-1)n}$, where u, $v \in E$, there exists $L_2 \lhd_f E$ such that $\bar{u} \notin \bar{K}_1 \bar{v} \bar{H}_{(n-1)n}$ in $\bar{E} = E/L_2$;

(iii) for $u \notin H_{(n-1)n} v K_2$, where $u, v \in E$, there exists $L_3 \lhd_f E$ such that $\bar{u} \notin \bar{H}_{(n-1)n} \bar{v} \bar{K}_2$ in $\bar{E} = E/L_3$.

Our result now follows from Lemma 16.

Case 2. $K_1 < E$, $K_2 < G_n$. By Lemma 10, E is K_1-separable and by assumption, G_n is K_2-separable. By Lemmas 20 or 22 (with $K = H_{(n-1)n}$, $K_2 = K_1$), we have $K_1^y \cap H_{(n-1)n} = 1$ for all $y \in E$ and for each $R \lhd_f H_{(n-1)n}$ and $u \in E$, there exists $M_1 \lhd_f E$ such that $M_1 \cap H_{(n-1)n} \subseteq R$ and $K_1^u M_1 \cap H_{(n-1)n} M_1 = M_1$. Since $K_2 \subseteq Z(G_n)$, then $K_2^z = K_2$ for all $z \in G_n$. By assumption, $K_2 \cap H_{n(n-1)} = 1$ and there exists $M_2 \lhd_f G_n$ such that $M_2 \cap H_{n(n-1)} = M_1 \cap H_{(n-1)n}$ and $K_2 M_2 \cap H_{n(n-1)} M_2 = M_2$. Now, by induction, for $u \notin K_1 v H_{(n-1)n}$, where u, $v \in E$, there exists $L_1 \lhd_f E$ such that $\bar{u} \notin \bar{K}_1 \bar{v} \bar{H}_{(n-1)n}$ in $\bar{E} = E/L_1$. By (e), for $u \notin H_{n(n-1)} v K_2$, where u, $v \in G_n$, there exists $L_2 \lhd_f G_n$ such that $\bar{u} \notin \bar{H}_{n(n-1)} \bar{v} \bar{K}_2$ in $\bar{G}_n = G_n/L_2$. Our result now follows from Lemma 17.

Case 3. $K_1 < G_n$, $K_2 < E$. This case is similar to Case 2.

Case 4. K_1, $K_2 < G_n$. Since K_1, $K_2 \subseteq Z(G_n)$, then $K_1^z = K_1$ and $K_2^z = K_2$ for all $z \in G_n$. By assumption, $K_1 \cap H_{n(n-1)} = 1$ and $K_2 \cap H_{n(n-1)} = 1$. By (c) and (d), for each $R \lhd_f H_{n(n-1)}$, there exists $T \lhd_f G_n$ such that $T \cap H_{n(n-1)} = R$ and $K_1 T \cap H_{n(n-1)} T = T$, $K_2 T \cap H_{n(n-1)} T = T$. Furthermore, by (e),

(i) for $u \notin K_1 v K_2$, where $u, v \in G_n$, there exists $L_1 \lhd_f G_n$ such that $\bar{u} \notin \bar{K}_1 \bar{v} \bar{K}_2$ in $\bar{G}_n = G_n/L_1$;

(ii) for $u \notin K_1 v H_{n(n-1)}$, where u, $v \in G_n$, there exists $L_2 \lhd_f G_n$ such that $\bar{u} \notin \bar{K}_1 \bar{v} \bar{H}_{n(n-1)}$ in $\bar{G}_n = G_n/L_2$;

(iii) for $u \notin H_{n(n-1)} v K_2$, where $u, v \in G_n$, there exists $L_3 \lhd_f G_n$ such that $\bar{u} \notin \bar{H}_{n(n-1)} \bar{v} \bar{K}_2$ in $\bar{G}_n = G_n/L_3$.

The result now follows from Lemma 16.

LEMMA 27. *Let $E = G_1 *_H G_2$, where $G_1 \cap G_2 = H$. Let K be a subgroup of G_1 such that $H^y \cap K = 1$ for all $y \in G_1$. Suppose that*

(a) *G_1, G_2 are H-separable*;

(b) *for each $S \lhd_f K$ and $u, v \in G_1$, there exists $M \lhd_f G_1$ such that $M \cap K \subseteq S$ and $H^u M \cap KM = M$, $H^v M \cap KM = M$*;

(c) *for each* $R \lhd_f H$, *there exist* $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ *such that* $N_1 \cap H = R = N_2 \cap H$.

Then $K^z \cap K = 1$ *for all* $z \in E \backslash G_1$ *and for each* $S \lhd_f K$ *and* $x, y \in E\backslash G_1$, *there exists* $P \lhd_f E$ *such that* $P \cap K \subseteq S$ *and* $K^x P \cap KP = P$, $K^y P \cap KP = P$.

Proof. Since $H^y \cap K = 1$ for all $y \in G_1$, clearly $K^z \cap K = 1$ for all $z \in E\backslash G_1$. Let $x, y \in E\backslash G_1$ and $S \lhd_f K$ be given. We shall consider the following cases:

Case 1. $x, y \in G_2\backslash H$. By H-separability of G_2 and by (b) and (c), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $x, y \notin HN_2$, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$, and $HN_1 \cap KN_1 = N_1$. Now we form $\bar{E} = \bar{G}_1 *_H \bar{G}_2$, where $\bar{G}_1 = G_1/N_1$, $\bar{G}_2 = G_2/N_2$, and $\bar{H} = HN_1/N_1 = HN_2/N_2$. Then $\bar{x}$, $\bar{y} \in \bar{G}_2\backslash\bar{H}$ and $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}}$, $\bar{K}^{\bar{y}}$ has length greater than 1 and hence $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{K}^{\bar{y}} \cap \bar{K} = \bar{1}$. Since $\bar{E}$ is free-by-finite and $\bar{K}$ is finite, there exists $\bar{P} \lhd_f \bar{E}$ such that $\bar{P} \cap \bar{K}^{\bar{x}}\bar{K} = \bar{1}$ and $\bar{P} \cap \bar{K}^{\bar{y}}\bar{K} = \bar{1}$. This implies that $\bar{P} \cap \bar{K} = \bar{1}$ and $\bar{K}^{\bar{x}}\bar{P} \cap \bar{K}\bar{P} = \bar{P}$, $\bar{K}^{\bar{y}}\bar{P} \cap \bar{K}\bar{P} = \bar{P}$. Let P be the preimage of $\bar{P}$ in E. Then $P \lhd_f E$, $P \cap K = S_1$ and $K^x P \cap KP = P$, $K^y P \cap KP = P$. Our result now follows.

Case 2. $x, y \notin G_1 \cup G_2$. WLOG, let $x = a_1 b_1 \cdots a_n b_n$ and $y = c_1 d_1 \cdots c_m d_m$, where $a_i, c_i \in G_1\backslash H$ and $b_i, d_i \in G_2\backslash H$ for all i. By H-separability of G_1, G_2 and by (b) and (c), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $a_i, c_i \notin HN_1$ and $b_i, d_i \notin HN_2$ for all i, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$ and $H^{a_1^{-1}} N_1 \cap KN_1 = N_1$, $H^{c_1^{-1}} N_1 \cap KN_1 = N_1$. Again we form $\bar{E}$. Then $\bar{a}_i, \bar{c}_i \in \bar{G}_1\backslash\bar{H}$, $\bar{b}_i, \bar{d}_i \in \bar{G}_2\backslash\bar{H}$ for all i and $\bar{H}^{\bar{a}_1^{-1}} \cap \bar{K} = \bar{1}$, $\bar{H}^{\bar{c}_1^{-1}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}}$, $\bar{K}^{\bar{y}}$ has length greater than 1 and hence $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{K}^{\bar{y}} \cap \bar{K} = \bar{1}$. Our result now follows as in Case 1.

The remaining cases can be similarly proved.

To extend Lemma 27 to a tree product (Lemma 32), we need the following extra lemmas (Lemmas 28–31):

LEMMA 28. *Let* $E = G_1 *_H G_2$, *where* $G_1 \cap G_2 = H$. *Let* K, K_1 *be subgroups of* G_1 *such that* $K_1^y \cap K = 1 = H^y \cap K$ *for all* $y \in G_1$. *Suppose that*

(a) G_1, G_2 *are* H-*separable*;

(b) *for each* $S \lhd_f K$ *and* $u, v \in G_1$, *there exists* $M \lhd_f G_1$ *such that* $M \cap K \subseteq S$ *and* $H^u M \cap KM = M$, $K_1^v M \cap KM = M$;

(c) *for each* $S \lhd_f K$ *and* $u, v \in G_1$, *there exists* $N \lhd_f G_1$ *such that* $N \cap K \subseteq S$ *and* $H^u N \cap KN = N$, $H^v N \cap KN = N$;

(d) *for each* $R \lhd_f H$, *there exist* $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ *such that* $N_1 \cap H = R = N_2 \cap H$.

Then $K^t \cap K = 1$ for all $t \in E \backslash G_1$ and $K_1^z \cap K = 1$ for all $z \in E$. Furthermore, for each $S \lhd_f K$ and $x \in E \backslash G_1$, $y \in E$, there exists $P \lhd_f E$ such that $P \cap K \subseteq S$ and $K^x P \cap KP = P$, $K_1^y P \cap KP = P$.

Proof. Since $H^y \cap K = 1 = K_1^y \cap K$ for all $y \in G_1$, clearly $K^t \cap K = 1$ for all $t \in E \backslash G_1$ and $K_1^z \cap K = 1$ for all $z \in E$. Let $x \in E \backslash G_1$, $y \in E$, and $S \lhd_f K$ be given. We divide the proof into several cases.

Case 1. x, $y \in G_2 \backslash H$. By H-separability of G_2 and by (c) and (d), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that x, $y \notin HN_2$, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$, and $HN_1 \cap KN_1 = N_1$. Now we form $\bar{E} = \bar{G}_1 *_H \bar{G}_2$, where $\bar{G}_1 = G_1/N_1$, $\bar{G}_2 = G_2/N_2$, and $\bar{H} = HN_1/N_1 = HN_2/N_2$. Then $\bar{x}$, $\bar{y} \in \bar{G}_2 \backslash \bar{H}$ and $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}}$, $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{K}_1 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Consequently, $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$. Since $\bar{E}$ is residually finite and $\bar{K}$, $\bar{K}_1$ are finite, there exists $\bar{P} \lhd_f \bar{E}$ such that $\bar{P} \cap \bar{K}^{\bar{x}} \bar{K} = \bar{1}$ and $\bar{P} \cap \bar{K}_1^{\bar{y}} \bar{K} = \bar{1}$. This implies that $\bar{P} \cap \bar{K} = \bar{1}$ and $\bar{K}^{\bar{x}} \bar{P} \cap \bar{K} \bar{P} = \bar{P}$, $\bar{K}_1^{\bar{y}} \bar{P} \cap \bar{K} \bar{P} = \bar{P}$. Let P be the preimage of $\bar{P}$ in E. Then P is the required normal subgroup.

Case 2. x, $y \notin G_1 \cup G_2$. WLOG, let $x = a_1 b_1 \cdots a_n b_n$ and $y = c_1 d_1 \cdots c_m d_m$, where $a_i, c_i \in G_1 \backslash H$ and b_i, $d_i \in G_2 \backslash H$ for all i. By H-separability of G_1, G_2 and by (c) and (d), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that a_i, $c_i \notin HN_1$ and b_i, $d_i \notin HN_2$ for all i, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$ and $H^{a_1^{-1}} N_1 \cap KN_1 = N_1$, $HN_1 \cap KN_1 = N_1$. As before we form $\bar{E}$. Then $\bar{a}_i$, $\bar{c}_i \in \bar{G}_1 \backslash \bar{H}$, $\bar{b}_i$, $\bar{d}_i \in \bar{G}_2 \backslash \bar{H}$ for all i and $\bar{H}^{\bar{a}_1^{-1}} \cap \bar{K} = \bar{1}$, $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}}$, $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{K}_1 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Therefore $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$ and we proceed as in Case 1 to obtain our result.

Case 3. $x \in G_2 \backslash H$, $y \in G_1$. By H-separability of G_2 and by (b) and (d), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $x \notin HN_2$, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$ and $HN_1 \cap KN_1 = N_1$, $K_1^y N_1 \cap KN_1 = N_1$. Again we form $\bar{E}$. Then $\bar{x} \in \bar{G}_2 \backslash \bar{H}$ and $\bar{H} \cap \bar{K} = \bar{1}$, $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. As in Case 1, we have $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$ and our result follows as above.

The remaining cases can be similarly proved.

LEMMA 29. *Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$. Let K, K_1 be subgroups of G_r such that $K_1 \subseteq Z(G_r)$ and $K_1 \cap K = 1 = H_{ri} \cap K$. Suppose that*

(a) *G_i is H_{ij}-separable*;

(b) *for each $S \lhd_f K$, there exists $M \lhd_f G_r$ such that $M \cap K = S$ and $K_1 M \cap KM = M$*;

(c) *for each* $S \lhd_f K$, *there exists* $N_{ri} \lhd_f G_r$ *such that* $N_{ri} \cap K = S$ *and* $H_{ri}N_{ri} \cap KN_{ri} = N_{ri}$;

(d) *for each* $R_{ij} \lhd_f H_{ij}$, *there exists* $N_i \lhd_f G_i$ *such that* $N_i \cap H_{ij} = R_{ij}$.

Then $K^t \cap K = 1$ *for all* $t \in G\backslash G_r$ *and* $K_1^z \cap K = 1$ *for all* $z \in G$. *Furthermore, for each* $S \lhd_f K$ *and* $x \in G\backslash G_r$, $y \in G$, *there exists* $P \lhd_f G$ *such that* $P \cap K \subseteq S$ *and* $K^x P \cap KP = P$, $K_1^y P \cap KP = P$.

Proof. We use induction on n. Let $n = 2$. WLOG, we may assume that K, $K_1 < G_1$. Since H_{12}, $K_1 \subseteq Z(G_1)$, then $H_{12}^z = H_{12}$ and $K_1^z = K_1$ for all $z \in G_1$. By assumption, $K_1 \cap K = 1$, $H_{12} \cap K = 1$ and by (c), for each $S \lhd_f K$, there exists $N \lhd_f G_1$ such that $N \cap K = S$ and $H_{12}N \cap KN = N$. Furthermore, by (b) and (c), for each $S \lhd_f K$, we can find $T \lhd_f G_1$ such that $T \cap K = S$ and $H_{12}T \cap KT = T$, $K_1T \cap KT = T$. Our result now follows from Lemma 28.

Let $n \geq 3$. As before, we let $G = E *_H G_n$, where E is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$ and $H = H_{(n-1)n} = H_{n(n-1)}$. Let $x \in G\backslash G_r$, $y \in G$ and $S \lhd_f K$ be given. We divide the proof into several cases.

Case 1 K, $K_1 < E$

First suppose $H_{(n-1)n}$ and K are in different factors of E. By induction, $K^u \cap K = 1$ for all $u \in E\backslash G_r$ and $K_1^v \cap K = 1$ for all $v \in E$. By Lemma 22 (with $K_1 = K_2 = H_{(n-1)n}$), $H_{(n-1)n}^w \cap K = 1$ for all $w \in E$. It follows that $K^t \cap K = 1$ for all $t \in G\backslash G_r$ and $K_1^z \cap K = 1$ for all $z \in G$.

Subcase 1. $x \notin E\backslash G_r$, $y \in G$. By Lemma 10, E is H-separable and by Lemma 8, for each $R \lhd_f H_{(n-1)n}$, there exists $L \lhd_f E$ such that $L \cap H_{(n-1)n} = R$. By Lemma 24 (with $K_2 = H_{(n-1)n}$), we have $H_{(n-1)n}^z \cap K = 1$, $K_1^z \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and $u, v \in E$, there exists $M \lhd_f E$ such that $M \cap K \subseteq S$ and $H_{(n-1)n}^u M \cap KM = M$, $K_1^v M \cap KM = M$. By Lemma 22 (with $K_1 = K_2 = H_{(n-1)n}$), we have $H_{(n-1)n}^z \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and u, $v \in E$, there exists $N \lhd_f E$ such that $N \cap K \subseteq S$ and $H_{(n-1)n}^u N \cap KN = N$, $H_{(n-1)n}^v N \cap KN = N$. Hence we are done by Lemma 28.

Subcase 2. $x \in E\backslash G_r$, $y \in E$. By induction, for $S \lhd_f K$ and $x \in E\backslash G_r$, $y \in E$, there exists $N_1 \lhd_f E$ such that $N_1 \cap K \subseteq S$ and $K^x N_1 \cap KN_1 = N_1$, $K_1^y N_1 \cap KN_1 = N_1$. Then by (d), there exists $N_2 \lhd_f G_n$ such that $N_2 \cap H_{n(n-1)} = N_1 \cap H_{(n-1)n}$. We form $\bar{G} = \bar{E} *_{\bar{H}} \bar{G}_n$, where $\bar{E} = E/N_1$, $\bar{G}_n = G_n/N_2$, and $\bar{H} = H_{(n-1)n}N_1/N_1 = H_{n(n-1)}N_2/N_2$. Clearly $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{G}$. We can now proceed as in Case 1 of Lemma 28 to obtain our result.

Subcase 3. $x \in E \backslash G_r$, $y \in G_n \backslash H_{n(n-1)}$. By $H_{n(n-1)}$-separability of G_n and by Lemma 8, there exist $M_1 \lhd_f E, M_2 \lhd_f G_n$ such that $y \notin H_{n(n-1)} M_2$ and $M_1 \cap H_{(n-1)n} = M_2 \cap H_{n(n-1)}$. Let $S_1 = M_1 \cap S$. Then $S_1 \lhd_f K$. By induction, for $S_1 \lhd_f K$ and $x \in E \backslash G_r$, there exists $N_1 \lhd_f E$ such that $N_1 \subseteq M_1, N_1 \cap K \subseteq S_1$ and $K^x N_1 \cap K N_1 = N_1$, $K_1 N_1 \cap K N_1 = N_1$. By (d), there exists $N_2 \lhd_f G_n$ such that $N_2 \subseteq M_2$ and $N_2 \cap H_{n(n-1)} = N_1 \cap H_{(n-1)n}$. Again we form $\bar{G}$. Then $\bar{y} \in \bar{G}_n \backslash \bar{H}_{n(n-1)}$ and $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{K}_1 \cap \bar{K} = \bar{1}$ in $\bar{G}$. As in Case 2 of Lemma 18, we have $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$ and we are done.

Subcase 4. $x \in E \backslash G_r$, $y \notin E \cup G_n$. WLOG, let $y = c_1 d_1 \cdots c_m d_m$, where $c_i \in E \backslash H_{(n-1)n}$ and $d_i \in G_n \backslash H_{n(n-1)}$ for all i. Since E is $H_{(n-1)n}$-separable and G_n is $H_{n(n-1)}$-separable, there exist $M_1 \lhd_f E$, $M_2 \lhd_f G_n$ such that $c_i \notin H_{(n-1)n} M_1$ and $d_i \notin H_{n(n-1)} M_2$ for all i. By Lemma 8 and (d), we can assume that $M_1 \cap H_{(n-1)n} = M_2 \cap H_{n(n-1)}$. Let $S_1 = M_1 \cap S$. Then $S_1 \lhd_f K$. Let $y_1 = c_1 c_2 \cdots c_m$. Then $y_1 \in E$. By induction, for $S_1 \lhd_f K$ and $x \in E \backslash G_r$, $y_1 \in E$, there exists $N_1 \lhd_f E$ such that $N_1 \subseteq M_1$, $N_1 \cap K \subseteq S_1$ and $K^x N_1 \cap K N_1 = N_1$, $K_1^{y_1} N_1 \cap K N_1 = N_1$. By (d), there exists $N_2 \lhd_f G_n$ such that $N_2 \subseteq M_2$ and $N_2 \cap H_{n(n-1)} = N_1 \cap H_{(n-1)n}$. Again we form $\bar{G}$. Then $\bar{c}_i \in \bar{E} \backslash \bar{H}_{(n-1)n}$, $\bar{d}_i \in \bar{G}_n \backslash \bar{H}_{n(n-1)}$ for all i and $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{K}_1^{\bar{y}_1} \cap \bar{K} = \bar{1}$ in $\bar{G}$. As in Case 3 of Lemma 18, we have $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$ and again we are done.

Now suppose $H_{(n-1)n}$ and K are in the same factor. Then we argue as above but in this case, we use Lemma 20 twice (with $K_1 = K_2 = H_{(n-1)n}$ and with $K_2 = H_{(n-1)n}$) instead of Lemma 22 and Lemma 24, respectively.

Case 2. K, $K_1 < G_n$

Since $H_{n(n-1)}$, $K_1 \subseteq Z(G_n)$, then $H^z_{n(n-1)} = H_{n(n-1)}$ and $K_1^z = K_1$ for all $z \in G_n$. By assumption, $K_1 \cap K = 1$, $H_{n(n-1)} \cap K = 1$ and by (c), for each $S \lhd_f K$, there exists $N \lhd_f G_n$ such that $N \cap K = S$ and $H_{n(n-1)} N \cap KN = N$. Furthermore, by (b) and (c), for each $S \lhd_f K$, we can find $T \lhd_f G_n$ such that $T \cap K = S$ and $H_{n(n-1)} T \cap KT = T$, $K_1 T \cap KT = T$. Hence the result follows from Lemma 28.

LEMMA 30. *Let $E = G_1 *_H G_2$, where $G_1 \cap G_2 = H$. Let K, K_1 be subgroups of G_1, G_2, respectively, such that $H^y \cap K = 1$ for all $y \in G_1$. Suppose that*

(a) *G_1, G_2 are H-separable*;

(b) *for each $S \lhd_f K$ and $u, v \in G_1$, there exists $M \lhd_f G_1$ such that $M \cap K \subseteq S$ and $H^u M \cap KM = M$, $H^v M \cap KM = M$*;

(c) *for each $R \lhd_f H$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $N_1 \cap H = R = N_2 \cap H$.*

Then $K^t \cap K = 1$ *for all* $t \in E \backslash G_1$ *and* $K_1^z \cap K = 1$ *for all* $z \in E$. *Furthermore, for each* $S \lhd_f K$ *and* $x \in E \backslash G_1$, $y \in E$, *there exists* $P \lhd_f E$ *such that* $P \cap K \subseteq S$ *and* $K^x P \cap KP = P$, $K_1^y P \cap KP = P$.

Proof. Since $H^y \cap K = 1$ for all $y \in G_1$, we have $K^t \cap K = 1$ for all $t \in E \backslash G_1$. Since $K_1 \subset G_2$ and $H^y \cap K = 1$, we have $K_1^z \cap K = 1$ for all $z \in E$. Let $x \in E \backslash G_1$, $y \in E$, and $S \lhd_f K$ be given. We shall consider the following cases:

Case 1. $x, y \in G_2 \backslash H$. By H-separability of G_2 and by (b) and (c), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that x, $y \notin HN_2$, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$, and $HN_1 \cap KN_1 = N_1$. As before, we form $\bar{E} = \bar{G}_1 *_H \bar{G}_2$, where $\bar{G}_1 = G_1/N_1$, $\bar{G}_2 = G_2/N_2$, and $\bar{H} = HN_1/N_1 = HN_2/N_2$. Then $\bar{x}$, $\bar{y} \in \bar{G}_2 \backslash \bar{H}$ and $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}}$, $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{K}_1 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Consequently, $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$. We can now proceed as in Case 1 of Lemma 28 to obtain our result.

Case 2. x, $y \notin G_1 \cup G_2$. WLOG, let $x = a_1 b_1 \cdots a_n b_n$ and $y = c_1 d_1 \cdots c_m d_m$, where a_i, $c_i \in G_1 \backslash H$ and b_i, $d_i \in G_2 \backslash H$ for all i. By H-separability of G_1, G_2 and by (b) and (c), for $S \lhd_f K$, there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that a_i, $c_i \notin HN_1$ and b_i, $d_i \notin HN_2$ for all i, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$ and $H^{a_1^{-1}} N_1 \cap KN_1 = N_1$, $HN_1 \cap KN_1 = N_1$. As before, we form $\bar{E}$. Then $\bar{a}_i$, $\bar{c}_i \in \bar{G}_1 \backslash \bar{H}$, $\bar{b}_i$, $\bar{d}_i \in \bar{G}_2 \backslash \bar{H}$ for all i and $\bar{H}^{\bar{a}^{-1}} \cap \bar{K} = \bar{1}$, $\bar{H} \cap \bar{K} = \bar{1}$ in $\bar{E}$. It follows that each non-trivial element of $\bar{K}^{\bar{x}}$, $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{K}_1 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Consequently, $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$ and our result follows as in Case 1.

Case 3. $x \in G_2 \backslash H$, $y \in G_1$. By H-separability of G_2 and by (b) and (c), there exist $N_1 \lhd_f G_1$, $N_2 \lhd_f G_2$ such that $x \notin HN_2$, $N_1 \cap K \subseteq S$, $N_1 \cap H = N_2 \cap H$ and $HN_1 \cap KN_1 = N_1$, $H^y N_1 \cap KN_1 = N_1$. As before, we form $\bar{E}$. Then $\bar{x} \in \bar{G}_2 \backslash \bar{H}$ and $\bar{H} \cap \bar{K} = \bar{1}$, $\bar{H}^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{E}$. As in Case 1, we have $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$. Since $\bar{H}^{\bar{y}} \cap \bar{K} = \bar{1}$ and $\bar{K}_1 \subset \bar{G}_2$, we have $\bar{K}_1 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$ and hence $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$. The result now follows as before.

The remaining cases can be similarly proved.

LEMMA 31. *Let* $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji} \rangle$ *be a tree product of* $G_1, G_2, \ldots, G_n$ *amalgamating the subgroups* H_{ij} *of* $Z(G_i)$ *and* H_{ji} *of* $Z(G_j)$. *Let* K, K_1 *be subgroups of* G_r, G_s, *respectively, such that* $K_1 \subseteq Z(G_s)$ *and* $H_{ri} \cap K = 1$, *where* $r \neq s$. *Suppose that*

(a) G_i *is* H_{ij}-*separable*;

(b) *for each* $S \lhd_f K$, *there exists* $M_{ri} \lhd_f G_r$ *such that* $M_{ri} \cap K = S$ *and* $H_{ri} M_{ri} \cap KM_{ri} = M_{ri}$;

(c) *for each* $R_{ij} \lhd_f H_{ij}$, *there exists* $M_i \lhd_f G_i$ *such that* $M_i \cap H_{ij} = R_{ij}$.

Then $K^t \cap K = 1$ *for all* $t \in G\backslash G_r$ *and* $K_1^z \cap K = 1$ *for all* $z \in G$. *Furthermore, for each* $S \lhd_f K$ *and* $x \in G\backslash G_r$, $y \in G$, *there exists* $P \lhd_f G$ *such that* $P \cap K \subseteq S$ *and* $K^x P \cap KP = P$, $K_1^y P \cap KP = P$.

Proof. We use induction on n. Let $n = 2$. WLOG, we may assume that $K < G_1$ and $K_1 < G_2$. Since $H_{12} \subseteq Z(G_1)$, then $H_{12}^z = H_{12}$ for all $z \in G_1$. By assumption, $H_{12} \cap K = 1$ and by (b), for each $S \lhd_f K$, there exists $M \lhd_f G_1$ such that $M \cap K = S$ and $H_{12}M \cap KM = M$. Hence we are done by Lemma 30.

Let $n \geq 3$. As before, we let $G = E_H G_n$, where E is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$ and $H = H_{(n-1)n} = H_{n(n-1)}$. Let $x \in G\backslash G_r$, $y \in G$ and $S \lhd_f K$ be given. We shall consider the following cases:

Case 1. $K, K_1 < E$

We argue as in Case 1 of Lemma 29 with the following changes: If $H_{(n-1)n}$ and K are in different factors of E, we use Lemma 22 (with $K_2 = H_{(n-1)n}$) instead of Lemma 24. If $H_{(n-1)n}$ and K are in the same factor of E, we use Lemma 20 (with $K_1 = K_2 = H_{(n-1)n}$) and Lemma 24 (with $K_1 = H_{(n-1)n}$ and $K_2 = K_1$).

Case 2. $K < E$, $K_1 < G_n$

First suppose $H_{(n-1)n}$ and K are in different factors of E. By induction (with $K_1 = H_{(n-1)n}$), $K^u \cap K = 1$ for all $u \in E\backslash G_r$ and $H_{(n-1)n}^v \cap K = 1$ for all $v \in E$. Hence $K^t \cap K = 1$ for all $t \in G\backslash G_r$. Since $K_1 \subset G_n$ and $H_{(n-1)n}^v \cap K = 1$ for all $v \in E$, we have $K_1^z \cap K = 1$ for all $z \in G$.

Subcase 1. $x \notin E\backslash G_r$, $y \in G$. By Lemma 10, E is $H_{(n-1)n}$-separable and by Lemma 8, for each $R \lhd_f H_{(n-1)n}$, there exists $L \lhd_f E$ such that $L \cap H_{(n-1)n} = R$. By Lemma 22 (with $K_1 = K_2 = H_{(n-1)n}$), we have $H_{(n-1)n}^z \cap K = 1$ for all $z \in E$ and for each $S \lhd_f K$ and $u, v \in E$, there exists $Q \lhd_f E$ such that $Q \cap K \subseteq S$ and $H_{(n-1)n}^u Q \cap KQ = Q$, $H_{(n-1)n}^v Q \cap KQ = Q$. Our result now follows from Lemma 30.

Subcase 2. $x \in E\backslash G_r$, $y \in E$. By induction (with $K_1 = H_{(n-1)n}$), for $S \lhd_f K$ and $x \in E\backslash G_r$, $y \in E$, there exists $N_1 \lhd_f E$ such that $N_1 \cap K \subseteq S$ and $K^x N_1 \cap KN_1 = N_1$, $H_{(n-1)n}^y N_1 \cap KN_1 = N_1$. By (c), there exists $N_2 \lhd_f G_n$ such that $N_2 \cap H_{(n-1)n} = N_1 \cap H_{n(n-1)}$. Now we form $\bar{G} = \bar{E} *_{\bar{H}} \bar{G}_n$, where $\bar{E} = E/N_1$, $\bar{G}_n = G_n/N_2$, and $\bar{H} = H_{(n-1)n}N_1/N_1 = H_{n(n-1)}N_2/N_2$. Clearly $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{H}_{(n-1)n}^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{G}$. Since $\bar{H}_{(n-1)n}^{\bar{y}} \cap \bar{K} = \bar{1}$ and $\bar{K}_1 \subset \bar{G}_n$, we have $\bar{K}_1 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$ and hence $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$. We can now proceed as in Case 1 of Lemma 28 to obtain our result.

Subcase 3. $x \in E \backslash G_r$, $y \in G_n \backslash H_{n(n-1)}$. By $H_{n(n-1)}$-separability of G_n and by Lemma 8, there exist $M_1 \lhd_f E$, $M_2 \lhd_f G_n$ such that $y \notin H_{n(n-1)} M_2$, and $M_1 \cap H_{(n-1)n} = M_2 \cap H_{n(n-1)}$. Let $S_1 = M_1 \cap S$. Then $S_1 \lhd_f K$. Hence by induction (with $K_1 = H_{(n-1)n}$), for $S_1 \lhd_f K$ and $x \in E \backslash G_r$, there exists $N_1 \lhd_f E$ such that $N_1 \subseteq M_1$, $N_1 \cap K \subseteq S_1$ and $K^x N_1 \cap K N_1 = N_1$, $H_{(n-1)n} N_1 \cap K N_1 = N_1$. By (c), there exists $N_2 \lhd_f G_n$ such that $N_2 \subseteq M_2$ and $N_2 \cap H_{(n-1)n} = N_1 \cap H_{n(n-1)}$. Now we form $\bar{G}$ as above. Then $\bar{y} \in \bar{G}_n \backslash \bar{H}_{n(n-1)}$ and $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{H}_{(n-1)n} \cap \bar{K} = \bar{1}$ in $\bar{G}$. It follows that each non-trivial element of $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}_1 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Consequently, $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$. The result now follows as in Subcase 2.

Subcase 4. $x \in E \backslash G_r$, $y \notin E \cup G_n$. WLOG, let $y = c_1 d_1 \cdots c_m d_m$, where $c_i \in E \backslash H_{(n-1)n}$ and $d_i \in G_n \backslash H_{n(n-1)}$ for all i. Since E is $H_{(n-1)n}$-separable and G_n is $H_{n(n-1)}$-separable, there exist $M_1 \lhd_f G_1$, $M_2 \lhd_f G_2$ such that $c_i \notin H_{(n-1)n} M_1$ and $d_i \notin H_{n(n-1)}$ for all i. By Lemma 8 and (c), we may assume that $M_1 \cap H_{(n-1)n} = M_2 \cap H_{n(n-1)}$. Let $S_1 = M_1 \cap S$. Then $S_1 \lhd_f K$. Hence by induction (with $K_1 = H_{(n-1)n}$), for $S_1 \lhd_f K$ and $x \in E \backslash G_r$, there exists $N_1 \lhd_f E$ such that $N_1 \subseteq M_1$, $N_1 \cap K \subseteq S_1$ and $K^x N_1 \cap K N_1 = N_1$, $H_{(n-1)n} N_1 \cap K N_1 = N_1$. By (c), there exists $N_2 \lhd_f G_n$ such that $N_2 \subseteq M_2$ and $N_2 \cap H_{n(n-1)} = N_1 \cap H_{(n-1)n}$. As above, we form $\bar{G}$. Then $\bar{c}_i \in \bar{E} \backslash \bar{H}_{(n-1)n}$, $\bar{d}_i \in \bar{G}_n \backslash \bar{H}_{n(n-1)}$ for all i, and $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{H}_{(n-1)n} \cap \bar{K} = \bar{1}$ in $\bar{G}$. It follows that each non-trivial element of $\bar{K}^{\bar{y}^{-1}}$ has length greater than 1 and hence $\bar{K}_1 \cap \bar{K}^{\bar{y}^{-1}} = \bar{1}$. Consequently, $\bar{K}_1^{\bar{y}} \cap \bar{K} = \bar{1}$ and the result follows as before.

Now suppose $H_{(n-1)n}$ and K are in the same factor of E. Then we argue as above but in this case, we use Lemma 20 (with $K_1 = K_2 = H_{(n-1)n}$) and Lemma 29 (with $K_1 = H_{(n-1)n}$) instead of Lemma 22 and induction, respectively.

Case 3. $K_1 < E$, $K < G_n$

Since $H_{n(n-1)} \subseteq Z(G_n)$, then $H^z_{(n-1)n} = H_{(n-1)n}$ for all $z \in G_n$. By assumption, $H_{(n-1)n} \cap K = 1$ and by (b), for each $S \lhd_f K$, there exists $M \lhd_f G_n$ such that $M \cap K = S$ and $H_{(n-1)n} M \cap KM = M$. Our result now follows from Lemma 30.

Now we can extend Lemma 27 to a tree product (Lemma 32) as follows:

LEMMA 32. *Let* $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji} \rangle$ *be a tree product of* $G_1, G_2, \ldots, G_n$ *amalgamating the subgroups* H_{ij} *of* $Z(G_i)$ *and* H_{ji} *of* $Z(G_j)$. *Let* K *be a subgroup of* $Z(G_r)$ *such that* $H_{ri} \cap K = 1$. *Suppose that*

(a) G_i *is* H_{ij}*-separable*;

(b) *for each* $S \lhd_f K$, *there exists* $M \lhd_f G_r$ *such that* $M \cap K = S$ *and* $H_{ri} M \cap KM = M$;

(c) *for each* $R_{ij} \lhd_f H_{ij}$, *there exists* $M_i \lhd_f G_i$ *such that* $M_i \cap H_{ij} = R_{ij}$.

Then $K^z \cap K = 1$ *for all* $z \in G \backslash G_r$ *and for each* $S \lhd_f K$ *and* $x, y \in G \backslash G_r$, *there exists* $P \lhd_f G$ *such that* $P \cap K \subseteq S$ *and* $K^x P \cap KP = P$, $K^y P \cap KP = P$.

Proof. We use induction on n. Let $n = 2$. WLOG, let $K < G_1$. Since $H_{12} \subseteq Z(G_1)$, then $H_{12}^z = H_{12}$ for all $z \in G_1$. By assumption, $H_{12} \cap K = 1$ and by (b), for each $S \lhd_f K$, there exists $M \lhd_f G_1$ such that $M \cap K = S$ and $H_{12}M \cap KM = M$. Hence we are done by Lemma 27.

Let $n \geq 3$. As before, we let $G = E *_H G_n$, where E is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$ and $H = H_{(n-1)n} = H_{n(n-1)}$. Let $x, y \in G \backslash G_r$ and $S \lhd_f K$ be given. We divide the proof into several cases.

Case 1. $K < E$

First suppose $H_{(n-1)n}$ and K are in different factors of E. By induction, $K^u \cap K = 1$ for all $u \in E \backslash G_r$ and by Lemma 22 (with $K_1 = K_2 = H_{(n-1)n}$), we have $H_{(n-1)n}^v \cap K = 1$ for all $v \in E$. This implies that $K^z \cap K = 1$ for all $z \in G \backslash G_r$.

Subcase 1. x, $y \notin E \backslash G_r$. By Lemma 10, E is $H_{(n-1)n}$-separable and by Lemma 8, for each $R \lhd_f H_{(n-1)n}$, there exists $L \lhd_f E$ such that $L \cap H_{(n-1)n} = R$. By Lemma 22 (with $K_1 = K_2 = H_{(n-1)n}$), for each $S \lhd_f K$ and u, $v \in E$, there exists $M \lhd_f E$ such that $M \cap K \subseteq S$ and $H_{(n-1)n}^u M \cap KM = M$, $H_{(n-1)n}^v M \cap KM = M$. The result now follows from Lemma 27.

Subcase 2. x, $y \in E \backslash G_r$. By induction, for $S \lhd_f K$ and x, $y \in E \backslash G_r$, there exists $N_1 \lhd_f E$ such that $N_1 \cap K \subseteq S$ and $K^x N_1 \cap KN_1 = N_1$, $K^y N_1 \cap KN_1 = N_1$. By (c), there exists $N_2 \lhd_f G_n$ such that $N_2 \cap H_{n(n-1)} = N_1 \cap H_{(n-1)n}$. Now we form $\bar{G} = \bar{E} *_{\bar{H}} \bar{G}_n$, where $\bar{E} = E/N_1$, $\bar{G}_n = G_n/N_2$, and $\bar{H} = H_{(n-1)n}N_1/N_1 = H_{(n-1)n}N_2/N_2$. Clearly $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$ and $\bar{K}^{\bar{y}} \cap \bar{K} = \bar{1}$ in $\bar{G}$. We can now proceed as in Case 1 of Lemma 27 to obtain our result.

Subcase 3. $x \in E \backslash G_r$, $y \in G_n \backslash H_{n(n-1)}$. By $H_{n(n-1)}$-separability of G_n and by Lemma 8, there exist $M_1 \lhd_f G_1$, $M_2 \lhd_f G_n$ such that $y \notin H_{n(n-1)}M_2$ and $M_1 \cap H_{(n-1)n} = M_2 \cap H_{n(n-1)}$. Let $S_1 = M_1 \cap S$. Then $S_1 \lhd_f K$. Hence by Lemma 31 (with $K_1 = H_{(n-1)n}$), for $S_1 \lhd_f K$ and $x \in E \backslash G_r$, there exists $N_1 \lhd_f E$ such that $N_1 \subseteq M_1$, $N_1 \cap K \subseteq S_1$ and $K^x N_1 \cap KN_1 = N_1$, $H_{(n-1)n}N_1 \cap KN_1 = N_1$. By (c), there exists $N_2 \lhd_f G_2$ such that $N_2 \subseteq M_2$ and $N_2 \cap H_{n(n-1)} = N_1 \cap H_{(n-1)n}$. Now we form $\bar{G}$ as in Subcase 2. Then $\bar{y} \in \bar{G}_n \backslash \bar{H}_{n(n-1)}$ and $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{H}_{(n-1)n} \cap \bar{K} = \bar{1}$ in $\bar{G}$. As in Case 1 of Lemma 27, we have $\bar{K}^{\bar{y}} \cap \bar{K} = \bar{1}$ and our result follows.

Subcase 4. $x \in E \backslash G_r$, $y \notin E \cup G_n$. WLOG, let $y = c_1 d_1 \cdots c_m d_m$ where $c_i \in E \backslash H_{(n-1)n}$ and $d_i \in G_n \backslash H_{n(n-1)}$ for all i. Since E is $H_{(n-1)n}$-separable and G_n is $H_{n(n-1)}$-separable, there exist $M_1 \lhd_f E$, $M_2 \lhd_f G_n$ such that $c_i \notin H_{(n-1)n}M_1$ and $d_i \notin H_{n(n-1)}M_2$ for all i. By Lemma 8 and (c), we can assume that $M_1 \cap H_{(n-1)n} = M_2 \cap H_{n(n-1)}$. Let $S_1 = M_1 \cap S$. Then $S_1 \lhd_f K$. Hence by Lemma 31 (with $K_1 = H_{(n-1)n}$), for $S_1 \lhd_f K$ and $x \in E \backslash G_r$, $c_1^{-1} \in E$, there exists $N_1 \lhd_f E$ such that $N_1 \subseteq M_1$, $N_1 \cap K \subseteq S_1$ and $K^x N_1 \cap KN_1 = N_1$, $H_{(n-1)n}^{c_1^{-1}} N_1 \cap KN_1 = N_1$. By (c), there exists $N_2 \lhd_f G_n$ such that $N_2 \subseteq M_2$ and $N_2 \cap H_{n(n-1)} = N_1 \cap H_{(n-1)n}$. Again we form $\bar{G}$. Then $\bar{c}_i \in \bar{E} \backslash \bar{H}_{(n-1)n}$, $\bar{d}_i \in \bar{G}_n \backslash \bar{H}_{n(n-1)}$ for all i and $\bar{K}^{\bar{x}} \cap \bar{K} = \bar{1}$, $\bar{H}_{(n-1)n}^{\bar{c}^{-1}} \cap \bar{K} = \bar{1}$ in $\bar{G}$. As in Case 2 of Lemma 27, we have $\bar{K}^{\bar{y}} \cap \bar{K} = \bar{1}$ and our result follows.

The following cases are similar to Subcases 3 and 4, respectively.

Subcase 5. $x \in G_n \backslash H_{n(n-1)}$, $y \in E \backslash G_r$.

Subcase 6. $x \notin E \cup G_r$, $y \in E \backslash G_r$.

Now suppose $H_{(n-1)n}$ and K_1 are in the same factor of E. Then we argue as above but in this case, we use Lemma 20 (with $K_1 = K_2 = H_{(n-1)n}$) and Lemma 29 (with $K_1 = H_{(n-1)n}$) instead of Lemma 22 and Lemma 31, respectively.

Case 2. $K < G_n$

Since $H_{n(n-1)} \subseteq Z(G_n)$, then $H_{n(n-1)}^z = H_{n(n-1)}$ for all $z \in G_n$. By assumption, $H_{n(n-1)} \cap K = 1$ and for each $S \lhd_f K$, there exists $M \lhd_f G_n$ such that $M \cap K = S$ and $H_{n(n-1)} M \cap KM = M$. Hence we are done by Lemma 27.

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