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Conjugacy Separability of Certain HNN Extensions of Conjugacy-Separable Groups

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Abstract. A group G is said to be conjugacy-separable if, for each pair of elements $x, y \in G$ such that x and y are not conjugate in G, there exists a finite homomorphic image $\bar{G}$ of G such that the images of x and y are not conjugate in $\bar{G}$. In this paper, we show that certain HNN extensions of conjugacy-separable groups are conjugacy-separable. We then apply our results to HNN extensions of polycyclic-by-finite groups.

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1 Introduction

A group G is said to be conjugacy-separable if, for each pair of elements $x, y \in G$ such that x and y are not conjugate in G, there exists a finite homomorphic image $\bar{G}$ of G such that the images of x and y are not conjugate in $\bar{G}$. In this paper, we show that certain HNN extensions of conjugacy-separable groups are conjugacy-separable.

More precisely, we let $G = \langle t, K \mid t^{-1}At = B, \varphi \rangle$ be an HNN extension, where K is a conjugacy-separable and subgroup-separable group, A and B are the associated subgroups contained in the center of K, and $\varphi : A \to B$ is the associated isomorphism from A to B. We will show that G is conjugacy-separable when $A \cap B = 1$, or $A = B$, or $A \cap B$ has finite index in A and in B with $(A \cap B)\varphi = A \cap B$. In addition, we will give a characterization for G to be conjugacy-separable when A and B are infinite cyclic. We then apply our results to show that certain HNN extensions of polycyclic-by-finite groups are conjugacy-separable.

The notation used here is standard. In addition, for any group G, $N \lhd_f G$ means that N is a normal subgroup of finite index in G, x^G denotes the conjugacy class of x in G, $x \sim_G y$ means that x is conjugate to y in G, and $Z(G)$ denotes the center of G.

Now let $G = \langle t, K \,|\, t^{-1}At = B, \varphi \rangle$. Then every element $x \in G$ is conjugate to a cyclically reduced element. If $x \in G$ is cyclically reduced, we express x in the form $x = t^{e_1}x_1t^{e_2}x_2 \cdots t^{e_n}x_n$, where $x_i \in K$ and $e_i = \pm 1$ for $1 \leq i \leq n$.

If $x, y \in G$, then $x \sim_{K,t} y$ means that $x, y \in K$ and either $x \sim_K y$, or $x \in A$ and $t^{-1}xt = y$, or $x \in B$ and $txt^{-1} = y$.

Finally, $\|x\|$ denotes the usual reduced length of x.

2 Preliminaries

In this section, we state the necessary definitions and lemmas.

Lemma 2.1. [1] *Let $G = \langle t, K \,|\, t^{-1}At = B, \varphi \rangle$ be an HNN extension and let x, y be cyclically reduced in G. Suppose $x \sim_G y$. Then $\|x\| = \|y\|$ and one of the following holds*:

(i) $\|x\| = \|y\| = 0$ *and there is a finite sequence $z_1, z_2, \ldots, z_m \in A \cup B$ such that $x \sim_K z_1 \sim_{K,t} z_2 \sim_{K,t} \cdots \sim_{K,t} z_m \sim_K y$.*

(ii) $\|x\| = \|y\| \geq 1$ *and $x \sim_{A \cup B} y^*$, where y^* is a cyclic permutation of y.*

Definition 2.2. Let $G = \langle t, K \,|\, t^{-1}At = B, \varphi \rangle$ be an HNN extension and let x, y be cyclically reduced in G with $\|x\| = \|y\| = n \geq 1$. Suppose $x = t^{e_1}x_1 \cdots t^{e_n}x_n$ and $y = t^{\epsilon_1}y_1 \cdots t^{\epsilon_n}y_n$, where $x_i, y_i \in K$ and $e_i, \epsilon_i = \pm 1$ for $1 \leq i \leq n$. Consider the following set of equations:

$$\begin{cases} x_1 = u_1^{-1}y_1v_1, \\ x_2 = u_2^{-1}y_2v_2, \\ \qquad \cdots \\ x_n = u_n^{-1}y_nv_n. \end{cases} \tag{1}$$

A pair of elements ρ_i, σ_i of K is called an *admissible solution* of the ith equation if $x_i = \rho_i^{-1}y_i\sigma_i$, where $\rho_i, \sigma_i \in A \cup B$.

A set of admissible solutions $\sigma_0, \rho_1, \sigma_1, \ldots, \rho_n, \sigma_n \in A \cup B$ to (1) is said to be *complete* if $t^{-e_i}\sigma_{i-1}t^{e_i} = \rho_i$ for each i, where $\sigma_0 = \sigma_n$.

For the case $A \cap B = 1$, $\rho_i \in A$ or B according as $e_i = -1$ or $e_i = 1$, respectively, and $\sigma_i \in A$ or B according as $e_{i+1} = 1$ or $e_{i+1} = -1$ (here, $e_{n+1} = e_1$). Therefore, by [2], $x \sim_G y$ if and only if $e_i = \epsilon_i$ for all i and there exists a complete set of solutions to (1). This is equivalent to $x = \sigma_0^{-1}y\sigma_0$, where $\sigma_0 \in A \cup B$.

Definition 2.3. A group G is said to be *conjugacy-separable* if, for each pair of elements $x, y \in G$ such that x and y are not conjugate in G, there exists a finite homomorphic image $\bar{G}$ of G such that the images of x and y are not conjugate in $\bar{G}$.

Lemma 2.4. [2] *Let $G = \langle t, K \,|\, t^{-1}At = B, \varphi \rangle$ be an HNN extension, where K is finite. Then G is conjugacy-separable.*

Definition 2.5. A group G is called *H-separable* for the subgroup H if, for each $x \in G \backslash H$, there exists $N \lhd_f G$ such that $x \notin HN$. G is called *subgroup-separable* if G is H-separable for every finitely generated subgroup H.

3 The Main Results

Let $G = \langle t, K \,|\, t^{-1}At = B, \varphi \rangle$, where K is a conjugacy-separable and subgroup-separable group and A and B are finitely generated subgroups of $Z(K)$. First, we prove that G is conjugacy-separable when $A \cap B = 1$.

Lemma 3.1. *Let K be a subgroup-separable group and A and B finitely generated subgroups of $Z(K)$ such that $A \cap B = 1$. Then for $R \lhd_f A$ and $S \lhd_f B$, there exists $N \lhd_f K$ such that $N \cap A = R$, $N \cap B = S$, and $AN \cap BN = N$.*

Proof. Since $R, S \subseteq Z(K)$, we can form $\bar{K} = K/RS$. Since K is subgroup-separable and R, S are finitely generated, $\bar{K}$ is residually finite. Furthermore, $\bar{A}$ and $\bar{B}$ are finite subgroups of $\bar{K}$. Hence, there exists $\bar{N} \lhd_f \bar{K}$ such that $\bar{N} \cap \bar{A}\bar{B} = \bar{1}$. This implies that $\bar{N} \cap \bar{A} = \bar{1}$, $\bar{N} \cap \bar{B} = \bar{1}$, and $\bar{A}\bar{N} \cap \bar{B}\bar{N} = \bar{N}$. Let N be the preimage of $\bar{N}$ in K. Then $N \lhd_f K$, $N \cap A = R$, $N \cap B = S$, and $AN \cap BN = N$. □

Theorem 3.2. *Let $G = \langle t, K \,|\, t^{-1}At = B, \varphi \rangle$ be an HNN extension, where K is a conjugacy-separable and subgroup-separable group. Suppose A and B are finitely generated subgroups of $Z(K)$ and $A \cap B = 1$. Then G is conjugacy-separable.*

Proof. We can assume A and B are infinite. Indeed, if A and B are finite, then G is conjugacy-separable by Theorem 13 in [2]. Let $x, y \in G$ be such that $x \nsim_G y$. Without loss of generality, we may assume x and y are cyclically reduced in G.

First, suppose $\|x\| \neq \|y\|$. Let $x = t^{e_1}x_1 \cdots t^{e_n}x_n$ and $y = t^{\epsilon_1}y_1 \cdots t^{\epsilon_m}y_m$, where $x_i, y_i \in K$, $e_i, \epsilon_i = \pm 1$, and $n \neq m$. Let a_i denote those x_i, y_i in $K \backslash A$ and b_i denote those x_i, y_i in $K \backslash B$. Since K is subgroup-separable, there exists $M \lhd_f K$ such that $a_i \notin AM$ and $b_i \notin BM$. Let $R_1 = M \cap A$ and $S_1 = M \cap B$. Let $R = R_1 \cap S_1\varphi^{-1}$ and $S = S_1 \cap R_1\varphi$. Then $R \lhd_f A$, $S \lhd_f B$, and $R\varphi = S$. By Lemma 3.1, there exists $T \lhd_f K$ such that $T \cap A = R$, $T \cap B = S$, and $AT \cap BT = T$. Let $N = M \cap T$. Then $N \lhd_f K$ is such that $N \cap A = R$, $N \cap B = S$, and $AN \cap BN = N$. In addition, $(N \cap A)\varphi = R\varphi = S = N \cap B$. We form $\bar{G} = \langle t, \bar{K} \,|\, t^{-1}\bar{A}t = \bar{B}, \bar{\varphi} \rangle$, where $\bar{K} = K/N$, $\bar{A} = AN/N$, $\bar{B} = BN/N$, and $\bar{\varphi}$ is the isomorphism from $\bar{A}$ onto

$\bar{B}$ induced by φ. Clearly, $\bar{G}$ is a homomorphic image of G. Let $\bar{g}$ denote the image of any element g of G in $\bar{G}$. Then $\bar{x}$ and $\bar{y}$ are cyclically reduced and $\|\bar{x}\| = \|x\| \neq \|y\| = \|\bar{y}\|$ in $\bar{G}$. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ by Lemma 2.1. Since $\bar{K}$ is finite, $\bar{G}$ is conjugacy-separable by Lemma 2.4 and the result follows. Therefore, we can assume $\|x\| = \|y\|$.

Case 1. $\|x\| = \|y\| = 0$.

(i) Suppose $x \notin A \cup B$ or $y \notin A \cup B$. Without loss of generality, let $x \notin A \cup B$. Since $A, B \subseteq Z(K)$, we see that $x \nsim_G y$ if and only if $x \nsim_K y$. Since K is subgroup-separable and conjugacy-separable, there exists $M \lhd_f K$ such that $\tilde{x} \notin \tilde{A} \cup \tilde{B}$ and $\tilde{x} \nsim_{\tilde{K}} \tilde{y}$ in $\tilde{K} = K/M$. By Lemma 3.1, there exists $N \lhd_f K$ such that $N \subseteq M$, $(N \cap A)\varphi = N \cap B$, and $AN \cap BN = N$. Now we form $\bar{G} = \langle t, \bar{K} \,|\, t^{-1}\bar{A}t = \bar{B}, \bar{\varphi} \rangle$ as above. Then $\bar{x} \notin \bar{A} \cup \bar{B}$ and $\bar{x} \nsim_{\bar{K}} \bar{y}$ in $\bar{G}$. Since $\bar{A}, \bar{B} \subseteq Z(\bar{K})$, it follows that $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

(ii) Suppose $x, y \in A \cup B$. First, suppose x and y are in the same associated subgroup, say, A. Since $A, B \subseteq Z(K)$ and $A \cap B = 1$, we see that $x \nsim_G y$ if and only if $x \neq y$. Since K is residually finite, there exists $M \lhd_f K$ such that $xy^{-1} \notin M$. By Lemma 3.1, we can find $N \lhd_f K$ such that $N \subseteq M$, $(N \cap A)\varphi = N \cap B$, and $AN \cap BN = N$. As before, we form $\bar{G} = \langle t, \bar{K} \,|\, t^{-1}\bar{A}t = \bar{B}, \bar{\varphi} \rangle$. Then $\bar{x} \neq \bar{y}$ and $\bar{A} \cap \bar{B} = \bar{1}$ in $\bar{G}$. Since $\bar{x}, \bar{y} \in \bar{A}$ and $\bar{A}, \bar{B} \subseteq Z(\bar{K})$, it follows that $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done. Similarly, we treat the case where $x, y \in B$.

Now suppose x and y are in different associated subgroups, say, $x \in A$ and $y \in B$. Let $z = t^{-1}xt$. Since $A, B \subseteq Z(K)$ and $A \cap B = 1$, we see that $x \nsim_G y$ if and only if $z \neq y$. Since K is residually finite, there exists $M \lhd_f K$ such that $zy^{-1} \notin M$. By Lemma 3.1, we can find $N \lhd_f K$ such that $N \subseteq M$, $(N \cap A)\varphi = N \cap B$, and $AN \cap BN = N$. Again, we form $\bar{G}$. Then $\bar{z} \neq \bar{y}$ and $\bar{A} \cap \bar{B} = \bar{1}$ in $\bar{G}$. Since $\bar{x} \in \bar{A}$, $\bar{y} \in \bar{B}$, and $\bar{A}, \bar{B} \subseteq Z(\bar{K})$, it follows that $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done. Similarly, we treat the case where $x \in B$ and $y \in A$.

Case 2. $\|x\| = \|y\| = n \geq 1$.

Let $x = t^{e_1}x_1 \cdots t^{e_n}x_n$ and $y = t^{\epsilon_1}y_1 \cdots t^{\epsilon_n}y_n$, where $x_i, y_i \in K$ and $e_i, \epsilon_i = \pm 1$ for $1 \leq i \leq n$. Since K is subgroup-separable, by Lemma 3.1, we can find $M_1 \lhd_f K$ such that $\tilde{x}$ and $\tilde{y}$ are cyclically reduced and $\|\tilde{x}\| = \|x\| = \|y\| = \|\tilde{y}\|$ in $\tilde{G} = \langle t, \tilde{K} \,|\, t^{-1}\tilde{A}t = \tilde{B}, \tilde{\varphi} \rangle$, where $\tilde{K} = K/M_1$, $\tilde{A} = AM_1/M_1$, $\tilde{B} = BM_1/M_1$, and $\tilde{\varphi}$ is the isomorphism from $\tilde{A}$ onto $\tilde{B}$ induced by φ.

First, suppose $e_i \neq \epsilon_i$ for some i. Since $\tilde{x}$ and $\tilde{y}$ are cyclically reduced and $\|\tilde{x}\| = \|\tilde{y}\|$ in $\tilde{G}$, clearly, $\tilde{x} \nsim_{\tilde{G}} \tilde{y}$ and the result follows.

Therefore, we may assume $e_i = \epsilon_i$ for all i. Since $x \nsim_G y$, either some equation in (1), say, $x_i = u_i^{-1}y_iv_i$, has no admissible solution in K or there exists a set of admissible solutions to (1), which is not complete.

Suppose the equation $x_i = u_i^{-1}y_iv_i$ has no admissible solution in K. This implies that $x_i \notin H_1y_iH_2$, where $H_i = A$ or B for $i = 1, 2$. So $x_iy_i^{-1} \notin H_1H_2$ since $A, B \subseteq Z(K)$. Because K is subgroup-separable, there exists $M_2 \lhd_f K$

such that $x_iy_i^{-1} \notin (H_1H_2)M_2$. Let $M = M_1 \cap M_2$. Then $M \lhd_f K$. By Lemma 3.1, we can find $N \lhd_f K$ such that $N \subseteq M$, $(N \cap A)\varphi = N \cap B$, and $AN \cap BN = N$. As in Case 1, we form $\bar{G}$. Then $\bar{x}_i\bar{y}_i^{-1} \notin \bar{H}_1\bar{H}_2$ and $\bar{A} \cap \bar{B} = \bar{1}$ in $\bar{G}$. Since $\bar{A}, \bar{B} \subseteq Z(\bar{K})$, we have $\bar{x}_i \notin (\bar{H}_1\bar{H}_2)\bar{y}_i = \bar{H}_1\bar{y}_i\bar{H}_2$. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and our result follows.

So we may assume there exists a set of admissible solutions $\rho_1, \sigma_1, \dots, \rho_n, \sigma_n \in A \cup B$ to (1), which is not complete.

(i) Suppose $x_i = \rho_i^{-1}y_i\sigma_i$, where $\rho_i \in A$ and $\sigma_i \in B$ for some i. Let $x_i = \alpha_i^{-1}y_i\beta_i$ be another expression of x_i, where $\alpha_i \in A$ and $\beta_i \in B$. Then $\alpha_i\rho_i^{-1} = \beta_i\sigma_i^{-1} \in A \cap B$. Since $A \cap B = 1$, we have $\rho_i = \alpha_i$ and $\sigma_i = \beta_i$. Therefore, the equation $x_i = u_i^{-1}y_iv_i$ has a unique solution in K.

Since the set of solutions $\rho_1, \sigma_1, \dots, \rho_n, \sigma_n$ is not complete, we have $t^{-e_i}\sigma_{i-1}t^{e_i} \neq \rho_i$ for some i. Since K is residually finite, there exists $M_2 \lhd_f K$ such that $t^{-e_i}\sigma_{i-1}t^{e_i}\rho_i^{-1} \notin M_2$. Let $M = M_1 \cap M_2$. Then $M \lhd_f K$ and, by Lemma 3.1, we can find $N \lhd_f K$ such that $N \subseteq M$, $(N \cap A)\varphi = N \cap B$, and $AN \cap BN = N$. As before, we form $\bar{G}$. Then $t^{-e_i}\bar{\sigma}_{i-1}t^{e_i} \neq \bar{\rho}_i$ and $\bar{A} \cap \bar{B} = \bar{1}$ in $\bar{G}$. Since $\bar{A} \cap \bar{B} = \bar{1}$, the expression $\bar{x}_i = \bar{\rho}_i^{-1}\bar{y}_i\bar{\sigma}_i$ is unique in $\bar{K}$. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows.

(ii) Now suppose $x_i = \rho_i^{-1}y_i\sigma_i$, where $\sigma_i, \rho_i \in A$ or $\sigma_i, \rho_i \in B$ for each i. Note that this case occurs only if $e_i = -e_{i+1}$ and $\epsilon_i = -\epsilon_{i+1}$ for all i. Without loss of generality, let $\sigma_1, \rho_1 \in A$. Then $\sigma_2, \rho_2 \in B$, $\sigma_3, \rho_3 \in A$, and so on, as well as $e_1, \epsilon_1 = -1$, $e_2, \epsilon_2 = 1$, etc. This implies that $x = t^{-1}x_1tx_2 \cdots t^{-1}x_n$ and $y = t^{-1}y_1ty_2 \cdots t^{-1}y_n$ if n is odd, or $x = t^{-1}x_1tx_2 \cdots tx_n$ and $y = t^{-1}y_1ty_2 \cdots ty_n$ if n is even.

If $\|x\| = \|y\| = n$ is odd, then $t\sigma_nt^{-1} \neq \rho_1$ since $t\sigma_nt^{-1} \notin A$ and $\rho_1 \in A$. We can proceed as above to prove our result.

If $\|x\| = \|y\| = n$ is even, then from (1), we have the following:

$$\begin{cases} x_1 = \rho_1^{-1}y_1\sigma_1, \\ x_2 = \rho_2^{-1}y_2\sigma_2, \\ \qquad \cdots \\ x_n = \rho_n^{-1}y_n\sigma_n. \end{cases} \tag{2}$$

Since $A, B \subseteq Z(K)$, we can assume $t^{-1}\sigma_1t = \rho_2$, $t\sigma_2t^{-1} = \rho_3, \dots, t^{-1}\sigma_{n-1}t = \rho_n$, but $t\sigma_nt^{-1} \neq \rho_1$ since $x \nsim_G y$.

Now we make adjustments on (2) by replacing ρ_1 by $\tilde{\rho}_1 = \rho_1c$ for some $c \in A$. Then σ_1 is replaced by $\tilde{\sigma}_1 = \sigma_1c$. This implies that ρ_2 is replaced by $\tilde{\rho}_2 = t^{-1}\tilde{\sigma}_1t = \rho_2t^{-1}ct$ and so on. Continuing in this way, we obtain the following:

$$\begin{cases} x_1 = \tilde{\rho}_1^{-1}y_1\tilde{\sigma}_1 = \rho_1^{-1}c^{-1}y_1\sigma_1c, \\ x_2 = \tilde{\rho}_2^{-1}y_2\tilde{\sigma}_2 = \rho_2^{-1}(t^{-1}c^{-1}t)y_2\sigma_2(t^{-1}ct), \\ \qquad \cdots \\ x_n = \tilde{\rho}_n^{-1}y_n\tilde{\sigma}_n = \rho_n^{-1}(t^{-1}c^{-1}t)y_n\sigma_n(t^{-1}ct). \end{cases}$$

This implies that $t^{-1}\tilde{\sigma}_1 t = \tilde{\rho}_2$, $t\tilde{\sigma}_2 t^{-1} = \tilde{\rho}_3$, $\ldots$, $t^{-1}\tilde{\sigma}_{n-1} t = \tilde{\rho}_n$, but $t\tilde{\sigma}_n t^{-1}\tilde{\rho}_1^{-1} = t\sigma_n(t^{-1}ct)t^{-1}\rho_1^{-1}c^{-1} = t\sigma_n t^{-1}\rho_1^{-1} \neq 1$. Let $z = t\sigma_n t^{-1}\rho_1^{-1}$. Since K is residually finite, there exists $M_2 \lhd_f K$ such that $z \notin M_2$. Let $M = M_1 \cap M_2$. Then $M \lhd_f K$ and, as above, there exists $N \lhd_f K$ such that $N \subseteq M$, $(N \cap A)\varphi = N \cap B$, and $AN \cap BN = N$. Again, we form $\bar{G}$. Then $\bar{z} \neq \bar{1}$ and $\bar{A} \cap \bar{B} = \bar{1}$ in $\bar{G}$. This implies that $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows. □

Next, we prove that G is conjugacy-separable when $A \cap B \lhd_f A$, $A \cap B \lhd_f B$, and $(A \cap B)\varphi = A \cap B$. We need the following two lemmas.

Lemma 3.3. *Let K be a subgroup-separable group and A a finitely generated subgroup of $Z(K)$. Then for each $R \lhd_f A$, there exists $N \lhd_f K$ such that $N \cap A = R$.*

Proof. Let $B = 1$ in Lemma 3.1 and the result follows. □

Lemma 3.4. *Let $G = \langle t, K \mid t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a subgroup-separable group. Suppose A and B are finitely generated subgroups of $Z(K)$ such that $A \cap B \lhd_f A$, $A \cap B \lhd_f B$, and $(A \cap B)\varphi = A \cap B$. Then G is C-separable for every finitely generated subgroup C of K.*

Proof. Let C be a finitely generated subgroup of K and x a reduced element in $G \backslash C$.

Case 1. $\|x\| = 0$, i.e., $x \in K$. Since K is subgroup-separable, there exists $M \lhd_f K$ such that $x \notin CM$. Let $H = A \cap B$ and $R = M \cap H$. Then $R \lhd_f H$. Suppose R is of index k in H. Since H is finitely generated, there exists only a finite number of subgroups of index k in H. Let R_1 be the intersection of all these subgroups of index k in H. Then $R_1 \subseteq R$ is characteristic and has finite index in H. It follows that $R_1\varphi = R_1$. By Lemma 3.3, there exist $P \lhd_f K$ and $Q \lhd_f K$ such that $P \cap A = R_1$ and $Q \cap B = R_1$. Let $N = M \cap P \cap Q$. Then $N \lhd_f K$, $N \cap A = R_1$, and $N \cap B = R_1$. Since $R_1\varphi = R_1$, $(N \cap A)\varphi = N \cap B$. Now we form $\bar{G} = \langle t, \bar{K} \mid t^{-1}\bar{A}t = \bar{B}, \bar{\varphi}\rangle$, where $\bar{K} = K/N$, $\bar{A} = AN/N$, $\bar{B} = BN/N$, and $\bar{\varphi}$ is the isomorphism from $\bar{A}$ onto $\bar{B}$ induced by φ. Then $\bar{x} \in \bar{K} \backslash \bar{C}$ in $\bar{G}$. Since $\bar{G}$ is residually finite and $\bar{C}$ is finite, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{x} \notin \bar{C}\bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ and $x \notin CP$. The result now follows.

Case 2. $\|x\| = n \geq 1$. Suppose $x = t^{e_1}x_1 \cdots t^{e_n}x_n$, where $x_i \in K$ and $e_i = \pm 1$. Let a_i denote those x_i in $K \backslash A$, b_i those x_i in $K \backslash B$, and c_i those x_i in $(A \cap B)\backslash\{1\}$. Since K is subgroup-separable, there exists $M \lhd_f K$ such that $a_i \notin AM$, $b_i \notin BM$, and $c_i \notin M$ for all i. As in Case 1, we can find $M \lhd_f K$ such that $N \subseteq M$ and $(N \cap A)\varphi = A \cap B$. Now we form $\bar{G} = \langle t, \bar{K} \mid t^{-1}\bar{A}t = \bar{B}, \bar{\varphi}\rangle$. Then $\bar{x}$ is reduced and $\|\bar{x}\| = \|x\|$ in $\bar{G}$. Since $\|\bar{x}\| \geq 1$ and $\bar{C}$ is a subgroup of $\bar{K}$, clearly, $\bar{x} \notin \bar{C}$ and we proceed as before. □

Theorem 3.5. *Let $G = \langle t, K \mid t^{-1}At = B, \varphi\rangle$ be an HNN extension, where*

K is a conjugacy-separable and subgroup-separable group. Suppose A and B are finitely generated subgroups of $Z(K)$ such that $A \cap B \lhd_f A$, $A \cap B \lhd_f B$, and $(A \cap B)\varphi = A \cap B$. Then G is conjugacy-separable.

Proof. As in Theorem 3.2, we can assume A and B are infinite. Let $H = A \cap B$ and $x, y \in G$ be such that $x \nsim_G y$. Without loss of generality, we may assume x and y are cyclically reduced in G.

First, suppose $\|x\| \neq \|y\|$. Let $x = t^{e_1}x_1 \cdots t^{e_n}x_n$ and $y = t^{\epsilon_1}y_1 \cdots t^{\epsilon_m}y_m$, where $x_i, y_i \in K$, $e_i, \epsilon_i = \pm 1$, and $n \neq m$. Let a_i denote those x_i, y_i in $K \backslash A$, b_i those x_i, y_i in $K \backslash B$, and c_i those x_i, y_i in $(A \cap B)\backslash\{1\}$. Since K is subgroup-separable, there exists $M \lhd_f K$ such that $a_i \notin AM$, $b_i \notin BM$, and $c_i \notin M$ for all i. As in the proof of Lemma 3.4, we can find $N \lhd_f K$ such that $N \subseteq M$ and $(N \cap A)\varphi = N \cap B$. Now we form $\bar{G} = \langle t, \bar{K} \mid t^{-1}\bar{A}t = \bar{B}, \bar{\varphi}\rangle$, where $\bar{K} = K/N$, $\bar{A} = AN/N$, $\bar{B} = BN/N$, and $\bar{\varphi}$ is the isomorphism from $\bar{A}$ onto $\bar{B}$ induced by φ. Then $\bar{x}$ and $\bar{y}$ are cyclically reduced and $\|\bar{x}\| = \|x\| \neq \|y\| = \|\bar{y}\|$ in $\bar{G}$. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ by Lemma 2.1 and the result follows as in Theorem 3.2. Hence, we can assume $\|x\| = \|y\|$.

Case 1. $\|x\| = \|y\| = 0$, i.e., $x, y \in K$.

Subcase 1. $x \notin A \cup B$ or $y \notin A \cup B$. Without loss of generality, let $x \notin A \cup B$. Since $A, B \subseteq Z(K)$, we see that $x \nsim_G y$ if and only if $x \nsim_K y$. Since K is subgroup-separable and conjugacy-separable, there exists $M \lhd_f K$ such that $\tilde{x} \nsim_{\tilde{K}} \tilde{y}$ and $\tilde{x} \notin \tilde{A} \cup \tilde{B}$ in $\tilde{K} = K/M$. As above, we can find $N \lhd_f K$ such that $N \subseteq M$ and $(N \cap A)\varphi = N \cap B$. Now we form $\bar{G} = \langle t, \bar{K} \mid t^{-1}\bar{A}t = \bar{B}, \bar{\varphi}\rangle$. Then $\bar{x} \notin \bar{A} \cup \bar{B}$ and $\bar{x} \nsim_{\bar{K}} \bar{y}$ in $\bar{G}$. Since $\bar{A}, \bar{B} \subseteq Z(\bar{K})$, it follows that $\bar{x} \nsim_{\bar{G}} \bar{y}$ and the result follows.

Subcase 2. $x, y \in A \cup B$. First, suppose x and y are in the same associated subgroup. Without loss of generality, let $x, y \in A$. We consider the following cases:

(i) $x \notin H$ or $y \notin H$. Without loss of generality, let $x \notin H$. Since $A, B \subseteq Z(K)$ and $(A \cap B)\varphi = A \cap B$, we see that $x \nsim_G y$ if and only if $x \neq y$. Since K is subgroup-separable, there exists $M \lhd_f K$ such that $x \notin HM$ and $xy^{-1} \notin M$. As above, we can find $N \lhd_f K$ such that $N \subseteq M$ and $(N \cap A)\varphi = N \cap B$. Again, we form $\bar{G} = \langle t, \bar{K} \mid t^{-1}\bar{A}t = \bar{B}, \bar{\varphi}\rangle$.

Next, we show that $\bar{H} = \bar{A} \cap \bar{B}$ in $\bar{G}$, where $\bar{H} = HN/N$. In fact, $\bar{H} = (A \cap B)N/N \subseteq (AN \cap BN)/N = (AN/N) \cap (BN/N) = \bar{A} \cap \bar{B}$. Since $N \cap A = N \cap B$, we have that $\bar{A} \cap \bar{B} = (AN/N) \cap (BN/N) \simeq (A/(N \cap A)) \cap (B/(N \cap B)) = (A \cap B)/(N \cap A \cap B) \simeq (A \cap B)N/N = HN/N = \bar{H}$. From $\bar{H} \subseteq \bar{A} \cap \bar{B}$ and $\bar{H} \simeq \bar{A} \cap \bar{B}$, we have $\bar{H} = \bar{A} \cap \bar{B}$.

Then $\bar{x} \notin \bar{H}$, $\bar{x} \neq \bar{y}$, and $\bar{H} = \bar{A} \cap \bar{B}$ in $\bar{G}$. Since $\bar{A}, \bar{B} \subseteq Z(\bar{K})$ and $(\bar{A} \cap \bar{B})\bar{\varphi} = A \cap B$, it follows that $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

(ii) $x, y \in H$. Let E denote the splitting extension of H by $\langle t\rangle$. We will show that $x^E = \cap\{x^E N \mid N \lhd_f K\}$. Since E is polycyclic, E is x^E-separable by [3]. Hence, $x^E = \cap\{x^E M \mid M \lhd_f E\} = \cap\{x^E H^n \mid n \in \mathbb{N}\}$. Since K is subgroup-separable and H is finitely generated, by Lemma 3.3, there exists

$N_n \lhd_f K$ such that $N_n \cap H = H^n$. This implies that $\cap\{x^E H^n \mid n \in \mathbb{N}\} \supseteq \cap\{x^E(N \cap H) \mid N \lhd_f K\} = \cap\{x^E N \cap H \mid N \lhd_f K\} = \cap\{x^E N \mid N \lhd_f K\} \cap H$. Since K is H-separable, it follows that $\cap\{HN \mid N \lhd_f K\} = H$, and hence, $\cap\{x^E N \mid N \lhd_f K\} \subseteq H$. So $\cap\{x^E N \mid N \lhd_f K\} \subseteq \cap\{x^E H^n \mid n \in \mathbb{N}\} = x^E$. Clearly, $x^E \subseteq \cap\{x^E N \mid N \lhd_f K\}$, and hence, $x^E = \cap\{x^E N \mid N \lhd_f K\}$. Now $x \nsim_G y$ if and only if $y \notin x^G$. Since $x \in H$ and $H \subseteq Z(K)$, we have $x^G = x^E$. Hence, there exists $M \lhd_f K$ such that $y \notin x^E M$. As above, we can find $N \lhd_f K$ such that $N \subseteq M$ and $(N \cap A)\varphi = N \cap B$. Again, we form $\bar{G}$. Then $\bar{y} \notin \bar{x}^{\bar{E}} = \bar{x}^{\bar{G}}$. This implies that $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

Now suppose x and y are in different associated subgroups. Without loss of generality, let $x \in A$ and $y \in B$. Clearly, $x \notin H$ and $y \notin H$. Since $A, B \subseteq Z(K)$, $x \nsim_G y$ if and only if $t^{-1}xt \neq y$. Since K is subgroup-separable, there exists $M \lhd_f K$ such that $x, y \notin HM$ and $t^{-1}xty^{-1} \notin M$. As above, we can find $N \lhd_f K$ such that $N \subseteq M$ and $(N \cap A)\varphi = N \cap B$. Now we form $\bar{G}$. Then $\bar{x} \in \bar{A} \backslash \bar{H}$, $\bar{y} \in \bar{B} \backslash \bar{H}$, and $t^{-1}\bar{x}t \neq \bar{y}$ in $\bar{G}$. Since $\bar{A}, \bar{B} \subseteq Z(\bar{K})$, it follows that $\bar{x} \nsim_{\bar{G}} \bar{y}$ and our result follows.

Case 2. $\|x\| = \|y\| = r \geq 1$. Let $x = t^{e_1}x_1 \cdots t^{e_r}x_r$ and $y = t^{\epsilon_1}y_1 \cdots t^{\epsilon_r}y_r$, where $x_i, y_i \in K$ and $e_i, \epsilon_i = \pm 1$ for $1 \leq i \leq r$. Since K is subgroup-separable, by Lemma 3.3, we can find $M_1 \lhd_f K$ such that $\tilde{x}$ and $\tilde{y}$ are cyclically reduced and $\|\tilde{x}\| = \|x\| = \|y\| = \|\tilde{y}\|$ in $\tilde{G} = \langle t, \tilde{K} \mid t^{-1}\tilde{A}t = \tilde{B}, \tilde{\varphi}\rangle$, where $\tilde{K} = K/M_1$, $\tilde{A} = AM_1/M_1$, $\tilde{B} = BM_1/M_1$, and $\tilde{\varphi}$ is the isomorphism from $\tilde{A}$ onto $\tilde{B}$ induced by φ.

As in Case 2 of Theorem 3.2, we may assume $e_i = \epsilon_i$ for all i. Since $x \nsim_G y$, either one equation in (1) has no admissible solution in K or there exists a set of admissible solutions to (1), which is not complete.

If one of the equations in (1) has no admissible solution in K, we may proceed as in the proof of Theorem 3.2 and the result follows.

So we may assume there exists a set of admissible solutions $\sigma_0, \rho_1, \ldots, \rho_r, \sigma_r \in A \cup B$ which is not complete to the set of equations (1).

Let $u_i = t^{e_1}x_1 \cdots x_{i-1}t^{e_i}$ and $v_i = t^{e_1}y_1 \cdots y_{i-1}t^{e_i}$ for $1 \leq i \leq r$. If $x \sim_G y$, then there exists an element $z \in A \cup B$ such that $z^{-1}xz = y$, i.e., $x^{-1}zy = z$. This implies that $u_1^{-1}zv_1$, $u_2^{-1}zv_2$, $\ldots$, $u_r^{-1}zv_r \in A \cup B$, and $x_r^{-1}u_r^{-1}zv_r y_r = z$. Since $x \nsim_G y$, for each element $w \in A \cup B$, either there exists an integer s with $1 \leq s < r$ such that $u_1^{-1}wv_1, \ldots, u_s^{-1}wv_s \in A \cup B$ but $u_{s+1}^{-1}wv_{s+1} \notin K$, or $u_1^{-1}wv_1, u_2^{-1}wv_2, \ldots, u_r^{-1}wv_r \in A \cup B$ but $x_r^{-1}u_r^{-1}wv_r y_r \neq w$. We consider the following two subcases:

Subcase 1. Suppose for each element $w \in A \cup B$, there exists an integer s with $1 \leq s < r$ such that $u_1^{-1}wv_1, \ldots, u_s^{-1}wv_s \in A \cup B$ but $u_{s+1}^{-1}wv_{s+1} \notin K$. Let i denote the largest value of all these integers s. Then we have $u_{i+1}^{-1}hv_{i+1} \notin K$ for all $h \in A \cup B$. We continue as follows:

(i) $e_i = 1 = e_{i+1}$. Recall that $x_i = \rho_i^{-1}y_i\sigma_i$, where $\rho_i, \sigma_i \in A \cup B$. Since $A, B \subseteq Z(K)$ and $A \cap B \neq 1$, we can assume $\sigma_i \in A$ and $\rho_i \in B$. First, we show that $u_i\rho_i^{-1}v_i^{-1} \notin A \cup B$. Suppose $u_i\rho_i^{-1}v_i^{-1} = w'$ for some $w' \in A \cup B$. Then $u_{i+1}^{-1}w'v_{i+1} = t^{-1}x_i^{-1}u_i^{-1}w'v_i y_i t = t^{-1}x_i^{-1}\rho_i^{-1}y_i t = t^{-1}\sigma_i^{-1}t \in K$, a contradiction. So $u_i\rho_i^{-1}v_i^{-1} \notin A \cup B$. By Lemma 3.4, there exists $P \lhd_f G$ such that $u_i\rho_i^{-1}v_i^{-1} \notin AP \cup BP$. Let $M = M_1 \cap (P \cap K)$. Then $M \lhd_f K$ and, as above, there exists $N \lhd_f K$ such that $N \subseteq M$ and $(N \cap A)\varphi = N \cap B$. Again, we form $\bar{G}$. Then $\|\bar{x}\| = \|x\| = \|y\| = \|\bar{y}\|$ and $\bar{u}_i\bar{\rho}_i^{-1}\bar{v}_i^{-1} \notin \bar{A} \cup \bar{B}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. Then there exists a set of complete admissible solutions $\bar{h}_0, \bar{k}_1, \ldots, \bar{k}_r, \bar{h}_r \in \bar{A} \cup \bar{B}$ to (1) such that $\bar{x}_j = \bar{k}_j^{-1}\bar{y}_j\bar{h}_j$ and $t^{-e_j}\bar{h}_{j-1}t^{e_j} = \bar{k}_j$ for $1 \leq j \leq r$, where $\bar{h}_0 = \bar{h}_r$. Let $\bar{u}_j = t^{e_1}\bar{x}_1 \cdots \bar{x}_{j-1}t^{e_j}$ and $\bar{v}_j = t^{e_1}\bar{y}_1 \cdots \bar{y}_{j-1}t^{e_j}$. Then $\bar{u}_j^{-1}\bar{h}_0^{-1}\bar{v}_j = \bar{k}_j$. Furthermore, $\bar{v}_j^{-1}\bar{h}\bar{v}_j \in \bar{A} \cap \bar{B}$ for all $\bar{h} \in \bar{A} \cap \bar{B}$ since $(\bar{A} \cap \bar{B})\bar{\varphi} = \bar{A} \cap \bar{B}$ and $\bar{A}, \bar{B} \subseteq Z(\bar{K})$. Since $t^{-e_i}\bar{h}_{i-1}t^{e_i} = \bar{k}_i$ and $t^{-e_{i+1}}\bar{h}_i t^{e_{i+1}} = \bar{k}_{i+1}$, we must have $\bar{h}_i \in \bar{A}$ and $\bar{k}_i \in \bar{B}$. Since $\bar{x}_i = \bar{\rho}_i^{-1}\bar{y}_i\bar{\sigma}_i = \bar{k}_i^{-1}\bar{y}_i\bar{h}_i$, we see that $\bar{h}_i\bar{\sigma}_i^{-1} = \bar{k}_i\bar{\rho}_i^{-1} \in \bar{A} \cap \bar{B}$. This implies that $\bar{\rho}_i^{-1} = \bar{k}_i^{-1}\bar{h}$ for some $\bar{h} \in \bar{A} \cap \bar{B}$. It follows that $\bar{u}_i\bar{\rho}_i^{-1}\bar{v}_i^{-1} = \bar{u}_i\bar{k}_i^{-1}\bar{h}\bar{v}_i^{-1} = (\bar{u}_i\bar{k}_i^{-1}\bar{v}_i^{-1})(\bar{v}_i\bar{h}\bar{v}_i^{-1}) = \bar{h}_0^{-1}\bar{h}' \in \bar{A} \cup \bar{B}$ for some $\bar{h}' \in \bar{A} \cap \bar{B}$, a contradiction. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and our result now follows. Similarly, we treat the case where $e_i = -1 = e_{i+1}$.

(ii) $e_i = 1 = -e_{i+1}$. Now let w be an element, where $u_1^{-1}wv_1, \ldots, u_i^{-1}wv_i \in A \cup B$ but $u_{i+1}^{-1}wv_{i+1} \notin K$. Since $u_i^{-1}wv_i \in A \cup B$ and $u_i^{-1}wv_i$ has the form $u_i^{-1}wv_i = t^{-1}x_{i-1}^{-1}u_{i-1}^{-1}wv_{i-1}y_{i-1}t$, we have $u_i^{-1}wv_i \in B$. Since $u_{i+1}^{-1}wv_{i+1} = tx_i^{-1}u_i^{-1}wv_iy_it^{-1} \notin K$, we must have $x_i^{-1}u_i^{-1}wv_iy_i \notin B$. These facts imply that $x_i^{-1}y_i \notin B$ since $B \subseteq Z(K)$. By the B-separability of K, there exists $M_2 \lhd_f K$ such that $x_i^{-1}y_i \notin BM_2$. As before, there exists $N \lhd_f K$ such that $N \subseteq M_1 \cap M_2$ and $(N \cap A)\varphi = N \cap B$. Now we form $\bar{G}$, which is conjugacy-separable as above. Then $\|\bar{x}\| = \|x\| = \|y\| = \|\bar{y}\|$ and $\bar{x}_i^{-1}\bar{y}_i \notin \bar{B}$ in $\bar{G}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. Then there exists a set of complete admissible solutions $\bar{h}_0, \bar{k}_1, \ldots, \bar{k}_r, \bar{h}_r \in \bar{A} \cup \bar{B}$ to (1) such that $\bar{x}_j = \bar{k}_j^{-1}\bar{y}_j\bar{h}_j$ and $t^{-e_j}\bar{h}_{j-1}t^{e_j} = \bar{k}_j$ for j with $1 \leq j \leq r$, where $\bar{h}_0 = \bar{h}_r$. Since $t^{-e_i}\bar{h}_{i-1}t^{e_i} = \bar{k}_i$ and $t^{-e_{i+1}}\bar{h}_i t^{e_{i+1}} = \bar{k}_{i+1}$, we must have $\bar{h}_i \in \bar{B}$ and $\bar{k}_i \in \bar{B}$. Since $\bar{x}_i = \bar{k}_i^{-1}\bar{y}_i\bar{h}_i$, $\bar{x}_i^{-1}\bar{y}_i = \bar{k}_i\bar{h}_i^{-1} \in \bar{B}$, a contradiction. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done. Similarly, we treat the case where $-e_i = 1 = e_{i+1}$.

Subcase 2. Suppose there exists an element $w \in A \cup B$ such that $u_1^{-1}wv_1, u_2^{-1}wv_2, \ldots, u_r^{-1}wv_r \in A \cup B$ but $x_r^{-1}u_r^{-1}wv_ry_r \neq w$.

(i) $e_1 = 1 = e_r$. Recall that $x_r = \rho_r^{-1}y_r\sigma_r$, where $\rho_r, \sigma_r \in A \cup B$. Since $A, B \subseteq Z(K)$ and $A \cap B \neq 1$, we can assume $\sigma_r \in A$ and $\rho_r \in B$. First, suppose $u_r\rho_r^{-1}v_r^{-1} \notin A$. Hence, by Lemma 3.4, there exists $P \lhd_f G$ such that $u_r\rho_r^{-1}v_r^{-1} \notin AP$. Let $M = M_1 \cap (P \cap K)$. Then $M \lhd_f K$. Again, there exists $N \lhd_f K$ such that $N \subseteq M$ and $(N \cap A)\varphi = N \cap B$. As before, we form $\bar{G}$. Then $\|\bar{x}\| = \|x\| = \|y\| = \|\bar{y}\|$ and $\bar{u}_r\bar{\rho}_r^{-1}\bar{v}_r^{-1} \notin \bar{A}$ in $\bar{G}$. Moreover, $\bar{H} = \bar{A} \cap \bar{B}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. Then there exists a set of complete admissible solutions $\bar{h}_0, \bar{k}_1, \ldots, \bar{k}_r, \bar{h}_r \in \bar{A} \cup \bar{B}$ to (1) such that $\bar{x}_j = \bar{k}_j^{-1}\bar{y}_j\bar{h}_j$ and $t^{-e_j}\bar{h}_{j-1}t^{e_j} = \bar{k}_j$ for j with $1 \leq j \leq r$, where $\bar{h}_0 = \bar{h}_r$. Since $t^{-e_1}\bar{h}_0t^{e_1} = \bar{k}_1$ and $t^{-e_r}\bar{h}_{r-1}t^{e_r} = \bar{k}_r$, we must have $\bar{h}_0 \in \bar{A}$ and $\bar{k}_r \in \bar{B}$. Since $\bar{x}_r = \bar{\rho}_r^{-1}\bar{y}_r\bar{\sigma}_r = \bar{k}_r^{-1}\bar{y}_r\bar{h}_r$, $\bar{h}_r\bar{\sigma}_r^{-1} = \bar{k}_r\bar{\rho}_r^{-1} \in \bar{A} \cap \bar{B}$, which implies that $\bar{\rho}_r^{-1} = \bar{k}_r^{-1}\bar{h}$ for some

$\bar{h} \in \bar{A} \cap \bar{B}$. It follows that $\bar{u}_r \bar{\rho}_r^{-1} \bar{v}_r^{-1} = \bar{u}_r \bar{k}_r^{-1} \bar{h} \bar{v}_r^{-1} = (\bar{u}_r \bar{k}_r^{-1} \bar{v}_r^{-1})(\bar{v}_r \bar{h} \bar{v}_r^{-1}) = \bar{h}_r^{-1} \bar{h}' \in \bar{A}$ for some $\bar{h}' \in \bar{A} \cap \bar{B}$, a contradiction. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and our result follows.

Finally, suppose $u_r \rho_r^{-1} v_r^{-1} \in A$, say, $u_r \rho_r^{-1} v_r^{-1} = a_1^{-1}$ for some $a_1 \in A$. Hence, $x^{-1} a_1^{-1} y = x_r^{-1} u_r^{-1} a_1^{-1} v_r y_r = \sigma_r^{-1}$, and therefore, $x = a_1^{-1} y \sigma_r$. Since $x \nsim_G y$, we must have $\sigma_r \neq a_1$.

Suppose $x = a_2^{-1} y a_3$ is another expression of x, where $a_2, a_3 \in A$. Then $y^{-1} a_2 a_1^{-1} y = a_3 \sigma_r^{-1}$, i.e., $x_r^{-1} t^{-e_r} \cdots x_1^{-1} t^{-e_1} a_2 a_1^{-1} t^{e_1} x_1 \cdots t^{e_r} x_r = a_3 \sigma_r^{-1}$. This implies that $t^{-e_1} a_2 a_1^{-1} t^{e_1} \in A \cup B$. Since $A, B \subseteq Z(K)$, we see that $x_r^{-1} t^{-e_r} \cdots x_1^{-1} t^{-e_1} a_2 a_1^{-1} t^{e_1} x_1 \cdots t^{e_r} x_r = a_3 \sigma_r^{-1}$. Again, we must have $t^{-(e_1+e_2)} a_2 a_1^{-1} t^{e_1+e_2} \in A \cup B$. Continuing in this way, we have $t^{-e} a_2 a_1^{-1} t^{e} = a_3 \sigma_r^{-1}$, where $e = e_1 + \cdots + e_r$.

If $e = 0$, then $a_2 a_1^{-1} = a_3 \sigma_r^{-1}$, and hence, $a_1^{-1} \sigma_r = a_2^{-1} a_3$. Let $z = a_1^{-1} \sigma_r$. Since $z \neq 1$ and K is residually finite, there exists $M_2 \lhd_f K$ such that $z \notin M_2$. As before, we can find $N \lhd_f K$ such that $N \subseteq M_1 \cap M_2$ and $(N \cap A)\varphi = N \cap B$. Again, we form $\bar{G}$. Then $\|\bar{x}\| = \|x\| = \|y\| = \|\bar{y}\|$ and $\bar{z} \neq \bar{1}$ in $\bar{G}$. It follows that $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

If $e \neq 0$, from $t^{-e} a_2 a_1^{-1} t^{e} = a_3 \sigma_r^{-1}$, we have $t^{-e} a_1^{-1} t^{e} \sigma_r = t^{-e} a_2^{-1} t^{e} a_3$. Furthermore, $a_2 a_1^{-1} \in A \cap B$ and $a_3 \sigma_r^{-1} \in A \cap B$ because $a_2 a_1^{-1}, a_3 \sigma_r^{-1} \in A$ and $(A \cap B)\varphi = A \cap B$. Since $x \nsim_G y$, $x \neq a^{-1} y a$ for all $a \in A$. This implies that $t^{-e} a_1^{-1} t^{e} \sigma_r \neq t^{-e} a^{-1} t^{e} a$ for all $a \in A$, where $a a_1^{-1} \in A \cap B$. Let $z = t^{-e} a_1^{-1} t^{e} \sigma_r$ and $L = \{t^{-e} a^{-1} t^{e} a \mid a a_1^{-1} \in A \cap B\}$. Then $z \notin L$.

Let $L' = \{t^{-e} a^{-1} t^{e} a \mid a \in A \cap B\}$ and $w = t^{-e} a_1^{-1} t^{e} a_1$. Then $L = wL'$. To see this, let $x \in L$. Then $x = t^{-e} a^{-1} t^{e} a$, where $a a_1^{-1} = h \in A \cap B$. So $x = t^{-e} a_1^{-1} h^{-1} t^{e} a_1 h = t^{-e} a_1^{-1} t^{e} t^{-e} h^{-1} t^{e} a_1 h = t^{-e} a_1^{-1} t^{e} a_1 t^{-e} h^{-1} t^{e} h \in wL'$. Hence, $L \subseteq wL'$, and similarly, $wL' \subseteq L$.

Note that L' is a finitely generated subgroup of K since $L' \subseteq A \cap B$. Now $z \notin L$ implies that $w^{-1} z \notin L'$. Since K is subgroup-separable, there exists $M_2 \lhd_f K$ such that $w^{-1} z \notin L' M_2$. As before, we can find $N \lhd_f K$ such that $N \subseteq M_1 \cap M_2$ and $(N \cap A)\varphi = N \cap B$. Again, we form $\bar{G}$. Then $\|\bar{x}\| = \|x\| = \|y\| = \|\bar{y}\|$ and $\bar{w}^{-1} \bar{z} \notin \bar{L}'$ in $\bar{G}$. This implies that $\bar{z} \notin \bar{L}$. It follows that $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

Similarly, we treat the case where $e_1 = -1 = e_r$.

(ii) $e_1 = 1 = -e_r$. First, suppose $x^{-1} w y \notin A$. Since $u_r^{-1} w v_r \in A \cup B$ and $u_r^{-1} w v_r$ has the form $u_r^{-1} w v_r = t x_{r-1}^{-1} u_{r-1}^{-1} w v_{r-1} y_{r-1} t^{-1}$, we have $u_r^{-1} w v_r \in A$. Since $x_r^{-1} u_r^{-1} w v_r y_r \notin A$ and $A \subseteq Z(K)$, we see that $x_r^{-1} y_r \notin A$. By the A-separability of K, there exists $M_2 \lhd_f K$ such that $x_r^{-1} y_r \notin A M_2$. As before, there exists $N \lhd_f K$ such that $N \subseteq M_1 \cap M_2$ and $(N \cap A)\varphi = N \cap B$. Now we form $\bar{G}$ which is conjugacy-separable as above. Then $\|\bar{x}\| = \|x\| = \|y\| = \|\bar{y}\|$ and $\bar{x}_r^{-1} \bar{y}_r \notin \bar{A}$ in $\bar{G}$.

Suppose $\bar{x} \sim_{\bar{G}} \bar{y}$. Then there exists a set of complete admissible solutions $\bar{h}_0, \bar{k}_1, \ldots, \bar{k}_r, \bar{h}_r \in \bar{A} \cup \bar{B}$ to (1) such that $\bar{x}_j = \bar{k}_j^{-1} \bar{y}_j \bar{h}_j$ and $t^{-e_j} \bar{h}_{j-1} t^{e_j} = \bar{k}_j$ for j with $1 \leq j \leq r$, where $\bar{h}_0 = \bar{h}_r$. Since $t^{-e_r} \bar{h}_{r-1} t^{e_r} = \bar{k}_r$ and $t^{-e_1} \bar{h}_r t^{e_1} = \bar{k}_0$, we must have $\bar{h}_r \in \bar{A}$ and $\bar{k}_r \in \bar{A}$. Since $\bar{x}_r = \bar{k}_r^{-1} \bar{y}_r \bar{h}_r$, $\bar{x}_r^{-1} \bar{y}_r = \bar{k}_r \bar{h}_r^{-1} \in \bar{A}$, a contradiction. Therefore, $\bar{x} \nsim_{\bar{G}} \bar{y}$ and we are done.

Now suppose $x^{-1}wy \in A$. Then $x^{-1}wy = a$ for some $a \in A$, i.e., $x = wya^{-1}$. Since $x \nsim_G y$, we have $w \neq a$. Now we can proceed as in Subcase 2(i) above to obtain our result. □

Two easy consequences of Theorem 3.5 are as follows:

Corollary 3.6. *Let $G = \langle t, K \,|\, t^{-1}At = A, \varphi \rangle$ be an HNN extension, where K is a conjugacy-separable and subgroup-separable group. Suppose A is a finitely generated subgroup of $Z(K)$. Then G is conjugacy-separable.*

Proof. Let $A = B$ in Theorem 3.5 and we are done. □

Corollary 3.7. *Let $G = \langle t, K \,|\, t^{-1}at = b \rangle$ be an HNN extension, where K is a conjugacy-separable and subgroup-separable group. Suppose $\langle a \rangle$ and $\langle b \rangle$ are infinite cyclic subgroups of $Z(K)$. If $a^n = b^{\pm n}$ for some $n > 0$, then G is conjugacy-separable.*

Proof. Let $A = \langle a \rangle$ and $B = \langle b \rangle$ in Theorem 3.5 and we are done. □

Next, we will give a characterization for HNN extensions of conjugacy-separable and subgroup-separable groups with cyclic associated subgroups to be conjugacy-separable again. The following two lemmas will be used in our proof.

Lemma 3.8. [6] *The group $\langle t, a \,|\, t^{-1}a^m t = a^n \rangle$ is residually finite if and only if $|m| = 1$, or $|n| = 1$, or $m = \pm n$.*

Lemma 3.9. [4, Lemma 2.14] *Let $G = \langle t, K \,|\, t^{-1}a^n t = a \rangle$ be an HNN extension, where K is not a cyclic group and $Z(K) \neq 1$. If $|n| > 1$, then G is not residually finite.*

Theorem 3.10. *Let $G = \langle t, K \,|\, t^{-1}at = b \rangle$ be an HNN extension, where K is a conjugacy-separable and subgroup-separable group. Suppose K is non-cyclic and $\langle a \rangle$, $\langle b \rangle$ are infinite cyclic subgroups of $Z(K)$. Then G is conjugacy-separable if and only if $\langle a \rangle \cap \langle b \rangle = 1$ or $a^n = b^{\pm n}$ for some $n > 0$.*

Proof. If $\langle a \rangle \cap \langle b \rangle = 1$ or $a^n = b^{\pm n}$ for some $n > 0$, then G is conjugacy-separable by Theorem 3.2 or Corollary 3.7, respectively. Conversely, suppose G is conjugacy-separable. If $\langle a \rangle \cap \langle b \rangle \neq 1$, then $a^n = b^m$, where $m > 0$. Since G is residually finite, the subgroup $\langle t, a \rangle$ with $t^{-1}a^m t = a^n$ is residually finite. By Lemma 3.8, we have $|m| = 1$, or $|n| = 1$, or $m = \pm n$. Suppose $n = \pm 1$. Then $a = b^{\pm m}$, and hence, $G = \langle t, K \,|\, t^{-1}b^{\pm m}t = b \rangle$. Since G is residually finite, we have $|m| = 1$ by Lemma 3.9. Similarly, if $|m| = 1$, then $|n| = 1$. Therefore, if G is conjugacy-separable and $\langle a \rangle \cap \langle b \rangle \neq 1$, then $a^n = b^{\pm n}$ for some $n > 0$. □

4 Applications to Finitely Generated Polycyclic-by-Finite Groups

Since finitely generated polycyclic-by-finite groups are conjugacy-separable [3] and subgroup-separable [5], from Theorems 3.2 and 3.5, and Corollary 3.6, we have the following:

Corollary 4.1. *Let* $G = \langle t, K \,|\, t^{-1}At = B, \varphi\rangle$ *be an HNN extension, where* K *is a finitely generated polycyclic-by-finite group. Suppose* A *and* B *are finitely generated subgroups of* $Z(K)$*, and one of the following conditions is satisfied*:

(i) $A \cap B = 1$,
(ii) $A = B$,
(iii) $A \cap B \lhd_f A$, $A \cap B \lhd_f B$, *and* $(A \cap B)\varphi = A \cap B$.

Then G *is conjugacy-separable.*

Corollary 4.2. *Let* $G = \langle t, K \,|\, t^{-1}At = B, \varphi\rangle$ *be an HNN extension, where* K *is a finitely generated abelian group. Suppose one of the following conditions is satisfied*:

(i) $A \cap B = 1$,
(ii) $A = B$,
(iii) $A \cap B \lhd_f A$, $A \cap B \lhd_f B$, *and* $(A \cap B)\varphi = A \cap B$.

Then G *is conjugacy-separable.*

Note that Corollary 4.2 is a special case of Theorem 1 in [7].

From Theorem 3.10, we have the following:

Corollary 4.3. *Let* $G = \langle t, K \,|\, t^{-1}at = b\rangle$ *be an HNN extension, where* K *is a finitely generated polycyclic-by-finite group. Suppose* $\langle a\rangle$ *and* $\langle b\rangle$ *are infinite cyclic subgroups of* $Z(K)$*. Then* G *is conjugacy-separable if and only if* $\langle a\rangle \cap \langle b\rangle = 1$ *or* $a^n = b^{\pm n}$ *for some* $n > 0$.

Corollary 4.4. *Let* $G = \langle t, K \,|\, t^{-1}at = b\rangle$ *be an HNN extension, where* K *is a finitely generated abelian group and* K *is non-cyclic. Then* G *is conjugacy-separable if and only if* $\langle a\rangle \cap \langle b\rangle = 1$ *or* $a^n = b^{\pm n}$ *for some* $n > 0$.

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