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RESIDUAL PROPERTIES OF GENERALIZED FREE PRODUCTS WITH CYCLIC AMALGAMATION

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Let $\mathcal{N}_0$ be the class of all finitely generated torsion-free nilpotent groups, and let A and B be residually $\mathcal{N}_0$-groups. It is proved that the generalized free product with cyclic amalgamation of groups A and B is a residually finite soluble group and is a virtually residually finite p-group for any prime p.

Key Words: Generalized free product of groups; Residually finite group; Residually nilpotent group; Virtually residually p-group.

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1. INTRODUCTION

If $\mathcal{K}$ is a class of groups then a group G will be said to be $\mathcal{K}$-residual if for any nonidentity element $a \in G$ there exists a homomorphism φ of group G onto some $\mathcal{K}$-group (i.e., onto group from class $\mathcal{K}$) such that the image $a\varphi$ of a is not equal to identity. A group G will be said to be virtually $\mathcal{K}$-residual if it contains a finite index subgroup which is $\mathcal{K}$-residual.

The following notation of classes of groups will be considered here:

$\mathcal{F}$— the class of all finite groups;
$\mathcal{F}_p$— the class of all finite p-groups;
$\mathcal{F}_s$— the class of all finite soluble groups;
$\mathcal{N}$— the class of all nilpotent groups;
$\mathcal{N}_0$— the class of all finitely generated torsion-free nilpotent groups.

Let $G = (A * B, H = K)$ be the free product of the groups A and B with amalgamated subgroups H and K. G. Baumslag [2] has proved that, if A and B are free groups and subgroups H and K are cyclic, then group G is $\mathcal{F}$-residual. Moreover, as it was shown in [1], in this case group G is $\mathcal{F}_s$-residual. Also Baumslag has proved in [2] that every free product of two $\mathcal{N}_0$-groups with cyclic amalgamation is an $\mathcal{F}$-residual group. In the present paper, we generalize these results as follows.

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Theorem 1. *Let $G = (A * B, H = K)$ be the free product of $\mathcal{N}_0$-residual groups A and B with cyclic amalgamated subgroups H and K. Then in group G there exists a normal subgroup M such that the quotient group G/M is a finite nilpotent group and the group M is $\mathcal{F}_p$-residual for any prime p. In particular, the following assertions are true:*

1. *The group G is $\mathcal{F}$-residual. Moreover, G is an $\mathcal{F}_s$-residual group;*
2. *The group G is virtually $\mathcal{F}_p$-residual for any prime p, and therefore, G is a virtually $\mathcal{N}$-residual group.*

By a well-known Magnus Theorem free groups are $\mathcal{N}_0$-residual. The arbitrary $\mathcal{N}_0$-groups also are $\mathcal{N}_0$-residual. Therefore, the following assertion is a special case of Theorem 1.

Corollary. *Let $G = (A * B, H = K)$ be the free product of the groups A and B with cyclic amalgamated subgroups H and K. If A and B are free groups or $\mathcal{N}_0$-groups, then the group G is $\mathcal{F}_s$-residual, and G is a virtually $\mathcal{F}_p$-residual group for any prime p.*

Note that the cited above Baumslag results in [2] are special cases of this corollary.

Under the assumptions in the corollary some residual properties of group $G = (A * B, H = K)$ were considered in [1, 3, 4, 7]. In particular, the following result was proved independently in works [1] and [7].

Theorem 2. *Let $G = (A * B, H = K)$ be the free product of the groups A and B with nonidentity cyclic amalgamated subgroups $H = (h)$ and $K = (k)$.*

1. *Let A and B be free groups. Also let m and n be the largest positive integers such that the equations $x^m = h$ and $y^n = k$ are solvable in the groups A and B, respectively. If one of the numbers m and n is 1, then the group G is $\mathcal{F}_p$-residual for any prime p. If $m > 1$ and $n > 1$, then the group G is $\mathcal{F}_p$-residual if and only if the numbers m and n are powers of the prime p.*
2. *Let A and B be $\mathcal{N}_0$-groups. And let m and n be the same as in assertion 1. If $H \neq A$ and $K \neq B$, then the group G is $\mathcal{F}_p$-residual if and only if the numbers m and n are powers of the prime p.*

In connection with the first assertion of Theorem 2 it should be noted that by [1] the free product $G = (A * B, H = K)$ of free groups A and B with amalgamated cyclic subgroups H and K is $\mathcal{N}$-residual if and only if G is an $\mathcal{F}_p$-residual group for some prime p. Therefore, in general this group need not be an $\mathcal{N}$-residual group. However, by the Theorem 1, G is a virtually $\mathcal{N}$-residual group.

To prove Theorem 1, we need a series of preliminary assertions.

2. PRELIMINARIES

Proposition 1. *Let F be an $\mathcal{N}_0$-residual group, and let $H = (h)$ be a cyclic subgroup of the group F. Also let X be a finite subset in F such that $X \cap H = \emptyset$. Then there exists a normal subgroup V of the group F such that $F/V \in \mathcal{N}_0$ and $X \cap HV = \emptyset$.*

Proof. Since when $H = 1$ the assertion is trivial, we can assume that the subgroup H is not an identity. We denote by Δ the set of all normal subgroups D of F such that $F/D \in \mathcal{N}_0$ and $h \notin D$. Note that for any subgroup $D \in \Delta$ the order of the element hD is infinite. Since F is an $\mathcal{N}_0$-residual group, it follows that $\bigcap_{D\in\Delta} D = 1$.

Let $x \in F\backslash H$. We claim that there exists a normal subgroup M_x of the group F such that $M_x \in \Delta$ and $x \notin HM_x$. Indeed, suppose the contrary. Then for any subgroup $D \in \Delta$ there exists an integer d such that $xD = h^d D$. Let $M \in \Delta$. Then $xM = h^m M$ for some integer m. Since $x \neq h^m$ and $\bigcap_{D\in\Delta} D = 1$, it follows that there exists a subgroup $L \in \Delta$ such that $xL \neq h^m L$. Since the subgroup $S = M \cap L$ belongs to the set Δ, it follows that there exists an integer s such that $xS = h^s S$. Therefore, $xM = h^s M$ and hence $h^m M = h^s M$. Since the order of the element hM is infinite, it follows that $m = s$. Thus $xS = h^s S = h^m S$. This contradicts the fact that $xL \neq h^m L$ and $S \subseteq L$.

Thus for any element $x \in X$ there exists a normal subgroup M_x in F such that $F/M_x \in \mathcal{N}_0$ and $x \notin HM_x$. Let $V = \bigcap_{x\in X} M_x$. Then $F/V \in \mathcal{N}_0$ and $X \cap HV = \emptyset$. This completes the proof of Proposition 1.

Let F be a free group or $\mathcal{N}_0$-group. Then F is $\mathcal{F}_p$-residual for any prime p [5]. If C is a maximal cyclic subgroup in F, then for any prime p and for any element $x \in F\backslash C$ there exists a homomorphism φ of the group F onto a finite p-group such that $x\varphi \notin C\varphi$ (see [7, Corollary 2.5, Lemma 3.1]).

Obviously, if F is an $\mathcal{F}_p$-residual group, then for any cyclic subgroup C of the group F and for any finite p-index subgroup H of the group C there exists a normal finite p-index subgroup S of the group F such that $S \cap C = H$. Hence, if F is an $\mathcal{F}_p$-residual group for any prime p, then for any cyclic subgroup C of the group F and for any nonidentity subgroup H of the group C there exists a normal finite index subgroup S of the group F such that $S \cap C = H$. In particular, this property holds for free groups and $\mathcal{N}_0$-groups.

Proposition 2. *Let $F \in \mathcal{N}_0$, and let H be a nonidentity cyclic subgroup in F. Then the following assertions are true*:

1. *There exists a normal finite index subgroup S of the group F such that $H \leq S$ and H is a maximal cyclic subgroup in S*;
2. *Let S be the same as in assertion* 1. *Then for any prime p and for any finite subset X of the set $F\backslash H$ there exists a normal subgroup P of the group F satisfying to the following conditions*: *$P \leq S$, $X \cap HP = \emptyset$, F/P is a finite group, S/P is a finite p-group.*

Proof. 1. Since every $\mathcal{N}_0$-group is a group with the maximal condition on subgroups, it follows that there exists a maximal cyclic subgroup C of the group F such that $H \leq C$. As noted above, there exists a normal finite index subgroup S of the group F such that $S \cap C = H$. We claim that H is a maximal cyclic subgroup in S. Indeed, let D be a cyclic subgroup in F such that $H \leq D$. It is obvious that $D \leq C$. Moreover, if $D \leq S$, then $D \leq S \cap C = H$ and in this case $D = H$. Thus H is a maximal cyclic subgroup in S.

2. Let S be a normal finite index subgroup in F such that $H \leq S$ and H is a maximal cyclic subgroup in S. Let p be a prime number, and let X be a finite subset in F such that $X \cap H = \emptyset$.

Let us show that for any element $x \in X$ there exists a normal subgroup P_x of the group F satisfying to the following conditions: $P_x \leq S$, $x \notin HP_x$, F/P_x is a finite group, S/P_x is a finite p-group. If $x \notin S$, then we choose $P_x = S$. We assume now that $x \in S$. Since H is a maximal cyclic subgroup in S and since $x \notin H$, it follows that there exists a homomorphism φ of the group S onto a finite p-group such that $x\varphi \notin H\varphi$ (see the result [7, Corollary 2.5] cited above). Then the subgroup

$$P_x = \bigcap_{a \in F} a^{-1}(\mathrm{Ker}\varphi)a$$

is a normal subgroup in F satisfying to the following conditions: $P_x \leq S$, $x \notin HP_x$, F/P_x is a finite group, S/P_x is a finite p-group.

Let $P = \bigcap_{x \in X} P_x$. It is obvious that P is a normal subgroup in F satisfying to the following conditions: $P \leq S$, $X \cap HP = \emptyset$, F/P is a finite group, S/P is a finite p-group. This completes the proof of Proposition 2.

Proposition 3. *Let F be an $\mathcal{N}_0$-residual group, and let $H = (h)$ be a nonidentity cyclic subgroup in F. Also let U be a normal subgroup in F such that $F/U \in \mathcal{N}_0$ and $HU/U \neq 1$. Then the following assertions are true*:

1. *There exists a normal finite index subgroup S of the group F such that $HU \leq S$ and HU/U is a maximal cyclic subgroup in S/U*;
2. *Let S be the same as in assertion* 1. *Then for any prime p and for any finite subset X of the set $F \backslash H$ there exists a normal subgroup P of the group F satisfying to the following conditions*: $P \leq S$, $X \cap HP = \emptyset$, *F/P is a finite nilpotent group, and S/P is a finite p-group.*

Proof. 1. By the Proposition 2, there exists a normal finite index subgroup S/U of the group F/U such that $HU/U \leq S/U$ and HU/U is a maximal cyclic subgroup in S/U. Thus the assertion 1 is true.

2. Let S be a normal finite index subgroup in F such that $HU \leq S$ and HU/U is a maximal cyclic subgroup in S/U. Let p be a prime number, and let X be a finite subset in F such that $X \cap H = \emptyset$.

By the Proposition 1, there exists a normal subgroup V of the group F such that $F/V \in \mathcal{N}_0$ and $X \cap HV = \emptyset$. Let $W = U \cap V$. Then $F/W \in \mathcal{N}_0$, $HW/W \neq 1$, $X \cap HW = \emptyset$, and hence $XW/W \cap HW/W = \emptyset$. Since $HW \leq HU \leq S$, it follows that $HW/W \leq S/W$.

We claim that HW/W is a maximal cyclic subgroup in S/W. Indeed, suppose the contrary. In this case, there exists $a \in S$ such that $a^m W = hW$, where $m > 1$. Since $W \leq U$, it follows that $a^m U = hU$. This contradicts the fact that HU/U is a maximal cyclic subgroup in S/U.

Thus the following assertions are true: $F/W \in \mathcal{N}_0$, HW/W is a nonidentity cyclic subgroup in F/W, S/W is a normal finite index subgroup in F/W, HW/W is a maximal cyclic subgroup in S/W, XW/W is a finite subset in F/W, and $XW/W \cap HW/W = \emptyset$.

Hence, by the Proposition 2, there exists a normal subgroup P/W of the group F/W satisfying to the following conditions: $P/W \leq S/W$, $XW/W \cap HW/W \cdot P/W = \emptyset$, $(F/W)/(P/W)$ is a finite group, and $(S/W)/(P/W)$ is a finite p-group. Then P is a normal subgroup in F satisfying to the following conditions: $P \leq S$, $X \cap HP = \emptyset$, F/P is a finite group, and S/P is a finite p-group. Note that $W \leq P$ and hence F/P is a nilpotent group. This completes the proof of Proposition 3.

In [6], Higman considered generalized free products of two finite p-groups. He showed that, if the amalgamated subgroup is cyclic, then these free products are $\mathcal{F}_p$-residual.

Let A and B be arbitrary groups. Also let h and k be elements of the groups A and B, respectively, and let the orders of these elements be equal. Further, let $G = (A * B,\ h = k)$ be the free product of the groups A and B with amalgamated cyclic subgroups $H = (h)$ and $K = (k)$. We need the following construction due to Baumslag [2]. Let P and Q be normal subgroups in the groups A and B, respectively. Let the orders of the elements $hP \in A/P$ and $kQ \in B/Q$ coincide. Then we can consider the amalgamated free product $G_{PQ} = (A/P * B/Q,\ hP = kQ)$ and a homomorphism $\rho_{PQ} : G \to G_{PQ}$ that extends the natural homomorphisms $A \to A/P$ and $B \to B/Q$. Denote by $[A, B]$ the subgroup in G generated by all elements $a^{-1}b^{-1}ab$, where $a \in A$ and $b \in B$.

Proposition 4. *Let $G = (A * B,\ h = k)$ be the free product of finite nilpotent groups A and B with amalgamated cyclic p-subgroups $H = (h)$ and $K = (k)$. Also let S and T be the largest p-subgroups of the groups A and B, respectively. Let M be a normal closure in G of the set $S \cup T \cup [A, B]$. Then the group M is $\mathcal{F}_p$-residual.*

Proof. Note that $H \leq S$ and $K \leq T$. Obviously, the group G/M is a direct product of the groups A/S and B/T. Hence $M \cap A = S$ and $M \cap B = T$.

As is well known, a finite nilpotent group is a direct product of the Sylow subgroups. Therefore, $A = S \times P$ and $B = T \times Q$ for some subgroups P and Q. Note that $|hP| = |h| = |k| = |kQ|$. We can now consider the free product

$$G_{PQ} = (A/P * B/Q,\ hP = kQ) \cong (S * T,\ h = k)$$

and the homomorphism $\rho_{PQ} : G \to G_{PQ}$ that extends the natural homomorphisms $A \to A/P$ and $B \to B/Q$. The group G_{PQ} is a free product of two finite p-groups with amalgamated cyclic subgroups, and hence it is $\mathcal{F}_p$-residual by the Higman Theorem [6] cited above. Therefore, there exists a homomorphism σ of the group G_{PQ} onto a finite p-group X that is injective on the factors A/P and B/Q. Let $L = \mathrm{Ker}\rho_{PQ}\sigma$. Since $\mathrm{Ker}\rho_{PQ} \cap A = P$ and $\mathrm{Ker}\rho_{PQ} \cap B = Q$ and since σ is injective on the factors A/P and B/Q, it follows that $L \cap A = P$ and $L \cap B = Q$.

Let $F = M \cap L$. Since

$$S \cap P = 1, \quad M \cap A = S, \quad L \cap A = P, \quad T \cap Q = 1, \quad M \cap B = T,$$
$$L \cap B = Q,$$

it follows that $F \cap A = 1$ and $F \cap B = 1$. By the H. Neumann theorem [8, Section 1, Proposition 11.22], the subgroup F is free. Since $M/F = M/(M \cap L) \cong ML/L \leq G/L$

and since $G/L \cong X$ is a finite p-group, it follows that M/F is a finite p-group. Thus the free group F is a normal finite p-index subgroup in M. Hence M is an $\mathscr{F}_p$-residual group. This completes the proof of Proposition 4.

Proposition 5. *Let $G = (A * B,\ h = k)$ be the free product of $\mathscr{N}_0$-residual groups A and B with amalgamated nonidentity cyclic subgroups $H = (h)$ and $K = (k)$. Let U and V be normal subgroups in A and B such that $A/U \in \mathscr{N}_0$, $B/V \in \mathscr{N}_0$ and $HU/U \neq 1 \neq KV/V$. Then the following assertions are true*:

1. *There exist normal finite index subgroups S and T of the groups A and B, respectively, such that $HU \leq S$, $KV \leq T$ and subgroups HU/U and KV/V are maximal cyclic subgroups in S/U and T/V.*
2. *Let S and T be the same as in assertion* 1. *Let M be a normal closure in G of the set $S \cup T \cup [A, B]$. Then the quotient group G/M is a finite nilpotent group and the group M is $\mathscr{F}_p$-residual for any prime p.*

Proof. 1. By the Proposition 3, there exist normal finite index subgroups S and T of the groups A and B, respectively, such that $HU \leq S$, $KV \leq T$ and subgroups HU/U and KV/V are maximal cyclic subgroups in S/U and T/V.

2. Let S and T be the same as in assertion 1. Let M be a normal closure in G of the set $S \cup T \cup [A, B]$. Since $G/M \cong A/S \times B/T$ and since A/S and B/T are finite nilpotent groups, it follows that G/M is a finite nilpotent group. Let us show that the group M is $\mathscr{F}_p$-residual for any prime p. Note that the factors A and B are $\mathscr{F}_p$-residual, because they are $\mathscr{N}_0$-residual. Assume that g is a nonidentity element of the group M. Let us show that there exists a homomorphism φ of the group M onto a finite p-group that sends the element g to a nonidentity element. Let g be presented by a reduced word

$$g = x_1 x_2 \dots x_r.$$

We assume first that $r = 1$. In this case, $g \in A$ or $g \in B$. To be definite, assume that $g \in A$. Since the group A is $\mathscr{F}_p$-residual, it follows that there exists a normal finite p-index subgroup P in the group A that does not contain the element g. Since the group B is $\mathscr{F}_p$-residual and since the order of the element hP of p-group A/P is a power of p, it follows that there exists a normal finite p-index subgroup Q of the group B such that $|hP| = |kQ|$. The group $G_{PQ} = (A/P * B/Q,\ hP = kQ)$ is a free product of two finite p-groups with cyclic amalgamation and hence, by the Higman Theorem [6] cited above, it is $\mathscr{F}_p$-residual. Let us consider the homomorphism $\rho_{PQ} : G \to G_{PQ}$ that extends the natural homomorphisms $A \to A/P$ and $B \to B/Q$. Since $g\rho_{PQ} \neq 1$ and since G_{PQ} is $\mathscr{F}_p$-residual, it follows that there exists a homomorphism τ of the group G_{PQ} onto a finite p-group such that $g\rho_{PQ}\tau \neq 1$. Let φ be a restriction of $\rho_{PQ}\tau$ to M. Then φ is a homomorphism of the group M onto a finite p-group that sends the element g to a nonidentity element.

We assume now that $r > 1$. To be definite, assume that $x_1, x_3, \dots \in A \backslash H$ and $x_2, x_4, \dots \in B \backslash K$. Let $X = \{x_1, x_3, \dots\}$ and $Y = \{x_2, x_4, \dots\}$. Since S and T are the same as in assertion 1 and since $X \cap H = \emptyset = Y \cap K$, by the Proposition 3, it follows that there exist normal subgroups P_0 and Q_0 of the groups A and B satisfying to the following conditions: $P_0 \leq S$, $Q_0 \leq T$, $X \cap HP_0 = \emptyset$, $Y \cap KQ_0 = \emptyset$, A/P_0 and B/Q_0 are finite nilpotent groups, and S/P_0 and T/Q_0 are finite p-groups. Since hP_0 and kQ_0 are elements of finite p-groups S/P_0 and T/Q_0, it follows that $|hP_0| = p^s$ and $|kQ_0| = p^t$. Let $m = \max(s, t)$. Since A and B are $\mathcal{F}_p$-residual, it follows that there exist normal finite p-index subgroups P_1 and Q_1 in the groups A and B such that $|hP_1| = p^m = |kQ_1|$. Let $P = P_0 \cap P_1$ and $Q = Q_0 \cap Q_1$. Then P and Q are normal subgroups in A and B satisfying to the following conditions: $P \leq S$, $Q \leq T$, $X \cap HP = \emptyset$, $Y \cap KQ = \emptyset$, A/P and B/Q are finite nilpotent groups, S/P and T/Q are finite p-groups, and $|hP| = p^m = |kQ|$. Hence we can consider the amalgamated free product $G_{PQ} = (A/P * B/Q,\ hP = kQ)$ and a homomorphism $\rho_{PQ} : G \to G_{PQ}$, that extends the natural homomorphisms $A \to A/P$ and $B \to B/Q$. Since $X \cap HP = \emptyset$ and $Y \cap KQ = \emptyset$, it follows that

$$x_1P \notin HP/P,\ x_2Q \notin KQ/Q,\ x_3P \notin HP/P,\ x_4Q \notin KQ/Q, \ldots$$

Therefore, the element $g\rho_{PQ}$ of the group G_{PQ} has a reduced expression

$$g\rho_{PQ} = x_1\rho_{PQ}x_2\rho_{PQ} \ldots x_r\rho_{PQ} = x_1P \cdot x_2Q \cdot x_3P \cdot x_4Q \cdot \ldots$$

of length $r > 1$. Hence $g\rho_{PQ} \neq 1$. Note that the subgroup $M\rho_{PQ}$ of the group G_{PQ} coincides with the normal closure in G_{PQ} of the set $S/P \cup T/Q \cup [A/P, B/Q]$. Since A/P and B/Q are finite nilpotent groups, and since S/P and T/Q are subgroups in the largest p-subgroups of the groups A/P and B/Q, by the Proposition 4, it follows that the group $M\rho_{PQ}$ is $\mathcal{F}_p$-residual. Since $g\rho_{PQ} \in M\rho_{PQ}\backslash 1$, it follows that there exists a homomorphism σ of the group $M\rho_{PQ}$ onto a finite p-group such that $(g\rho_{PQ})\sigma \neq 1$. Let ρ be a restriction of ρ_{PQ} to M. Let $\varphi = \rho\sigma$ Then φ is a homomorphism of the group M onto a finite p-group that sends the element g to a nonidentity element.

Thus, the group M is $\mathcal{F}_p$-residual for any prime p. This completes the proof of Proposition 5.

3. THE PROOF OF THE THEOREM 1

Gruenberg has proved in [5] that every free product of two $\mathcal{F}_p$-residual groups is an $\mathcal{F}_p$-residual group. In particular, every free product of two $\mathcal{N}_0$-residual groups is $\mathcal{F}_p$-residual for any prime p.

Let $G = (A * B, H = K)$ be the free product of $\mathcal{N}_0$-residual groups A and B with amalgamated cyclic subgroups H and K. Let us show that there exists a normal subgroup M of the group G such that G/M is a finite nilpotent group and the group M is $\mathcal{F}_p$-residual for any prime p.

If $H = 1$ and $K = 1$, then G is a free product of $\mathcal{N}_0$-residual groups A and B. In this case the group G is $\mathcal{F}_p$-residual for any prime p, and we choose $M = G$.

If $H \neq 1$ and $K \neq 1$, then by the Proposition 5 there exists a normal subgroup M of the group G such that G/M is a finite nilpotent group and the group M is $\mathcal{F}_p$-residual for any prime p. In particular, the following assertions are true:

1. The group G is $\mathcal{F}$-residual. Moreover, G is an $\mathcal{F}_s$-residual group;
2. The group G is virtually $\mathcal{F}_p$-residual for any prime p, and therefore, G is a virtually $\mathcal{N}$-residual group.

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