ON THE CONJUGACY SEPARABILITY IN THE CLASS OF FINITE P-GROUPS OF FINITELY GENERATED NILPOTENT GROUPS

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It is proved that for any prime p a finitely generated nilpotent group is conjugacy separable in the class of finite p-groups if and only if the torsion subgroup of it is a finite p-group and the quotient group by the torsion subgroup is abelian.

1. Let $\mathcal{K}$ be a class of groups. A group G is called residual $\mathcal{K}$ (or $\mathcal{K}$-residual) if for each non-unit element $a \in G$ there is a homomorphism φ of G onto some group X from the class $\mathcal{K}$ (or $\mathcal{K}$-group) such that $a\varphi$ is not the identity. The group G is called conjugacy separable in the class $\mathcal{K}$ (or conjugacy $\mathcal{K}$-separable), if whenever a and b are not conjugate in G, there is a homomorphism φ of G onto $\mathcal{K}$-group X such that $a\varphi$ and $b\varphi$ are not conjugate in X.

It is easy to see that a conjugacy $\mathcal{K}$-separable group is also $\mathcal{K}$-residual. Since in general the inverse statement is not true, it is interesting to find such classes of groups for which the property to be $\mathcal{K}$-residual implies conjugacy $\mathcal{K}$-separability. The most investigated (and chronologically first) is the case when class $\mathcal{K}$ is the class $\mathcal{F}$ of all finite groups; in this case one studies finite residuality and conjugacy separability respectively. K. Gruenberg [4] showed that finitely generated nilpotent groups are residually finite, and then N. Blackburn [5] proved that such groups are conjugacy separable. On the other hand the famous theorem of P. Hall states that every finitely generated metabelian group is residually finite, but there exists the example of finitely generated metabelian group that is not conjugacy separable constructed by M. I. Kargapolov and E. I. Timoshenko [1].

When a class of groups is proved to be $\mathcal{F}$-residual or conjugacy $\mathcal{F}$-separable, the question arises if these groups are $\mathcal{K}$-residual or conjugacy $\mathcal{K}$-separable for some subclass $\mathcal{K}$ of the class $\mathcal{F}$. From this point of view the class $\mathcal{F}_p$ of all finite p-groups is frequently considered. For example, A. I. Mal'cev [2] showed that free groups are residually finite. K. Gruenberg [4] proved that for any prime p every finitely generated torsion-free nilpotent group is $\mathcal{F}_p$-residual. From this assertion and from the theorem of W. Magnus about $\mathcal{N}$-residuality of free groups (where $\mathcal{N}$ is the class of all finitely generated torsion-free nilpotent groups) follows that every free group is $\mathcal{F}_p$-residual for all primes p. Since [3] these groups are conjugacy $\mathcal{F}_p$-separable .

As we have noted above any finitely generated torsion-free nilpotent group is $\mathcal{F}_p$-residual for every prime p. Nevertheless, here we'll show that for every prime p a finitely generated torsion-free nilpotent group is conjugacy $\mathcal{F}_p$-separable if and only of it is an abelian group. More exactly, we establish the following statement:

Theorem. *Suppose p is a prime. A finitely generated nilpotent group G is conjugacy $\mathcal{F}_p$-separable if and only if its torsion subgroup $\tau(G)$ is a p-group and the quotient group $G/\tau(G)$ is abelian.*

2. We'll begin the proof of this theorem from one general remark.

A subgroup H of a group G is called $\mathcal{K}$-separable if for every $a \in G$, $a \notin H$, there is a homomorphism φ of G onto $\mathcal{K}$-group such that $a\varphi \notin H\varphi$. It is well known that if a class $\mathcal{K}$ is closed under homomorphic images, then for every normal subgroup N of the group G the quotient group G/N is $\mathcal{K}$-residual if and only if the subgroup N is $\mathcal{K}$-separable. Consequently, if a class $\mathcal{K}$ is closed under homomorphic images, subgroups and direct products of finite number of factors, then a quotient group of the $\mathcal{K}$-residual group by any finite normal subgroup is $\mathcal{K}$-residual.

To receive the similar result for the property to be conjugacy $\mathcal{K}$-separable we'll introduce the following notion.

A subset M of a group G is called conjugacy $\mathcal{K}$-separable in the group G if whenever $a \in G$ is not conjugate to any element from M, there is a homomorphism of G onto $\mathcal{K}$-group X such that $a\varphi$ is not conjugate to any element from $M\varphi$. It is obvious that subset M is conjugacy $\mathcal{K}$-separable in G if and only if for every element $a \in G$ which is not conjugate to any element from M there exists a normal subgroup H in G that quotient group G/H is $\mathcal{K}$-group and a is not conjugate to any element from MH. Evidently also that a group G is conjugacy $\mathcal{K}$-separable iff every one-element subset of G is conjugacy $\mathcal{K}$-separable in G.

Proposition 1. *Suppose $\mathcal{K}$ is a class of groups which is closed under homomorphic images. For every group G and normal subgroup N of G the quotient group G/N is conjugacy $\mathcal{K}$-separable if and only if every coset of G modulo N is conjugacy $\mathcal{K}$-separable in G.*

Proof. Firstly we'll show that if some coset of G modulo N is not conjugacy $\mathcal{K}$-separable in G, then the quotient group G/N is not conjugacy $\mathcal{K}$-separable.

If $a \in G$ is not conjugate to any element from the coset bN, then aN and bN are not conjugate in the quotient group G/N. On the other hand suppose that for any homomorphism from G onto some $\mathcal{K}$-group the image of a is conjugate to image of some element from a coset bN. Let now φ be a homomorphism from G/N onto some $\mathcal{K}$-group X. Then the composition of the natural homomorphism $\varepsilon : G \to G/N$ and φ maps G onto X. Consequently, for some elements $g \in G$ and $c \in N$ we have

$$(g(\varepsilon\varphi))^{-1}a(\varepsilon\varphi)g(\varepsilon\varphi) = (bc)(\varepsilon\varphi),$$

i. e.

$$((gN)\varphi)^{-1}(aN)\varphi(gN)\varphi = (bN)\varphi.$$

Hence, the images of aN and bN under any homomorphism from G/N onto $\mathcal{K}$-group are conjugate.

Conversely, suppose any coset of G modulo N is conjugacy $\mathcal{K}$-separable in G. If elements aN and bN of the quotient group G/N are not conjugate in it, then a is not conjugate to any element from the coset bN in G. Consequently, a is not conjugate in G to any element from the coset bNH for some normal subgroup H of the group G such that the quotient group G/H is $\mathcal{K}$-group. It means that the elements aNH and bNH of the homomorphic image G/NH of G/N are not conjugate. Since the class $\mathcal{K}$ is closed under homomorphic images, the quotient group $G/NH \simeq (G/H)/(NH/H)$ is $\mathcal{K}$-group and the proposition 1 is proved.

It follows immediately from the proposition 1 that

Proposition 2. *Suppose $\mathcal{K}$ is a class of groups which is closed under homomorphic images, subgroups and direct products of the finite number of factors. If a group G is conjugacy $\mathcal{K}$-separable, then a quotient group G/N of G by any finite normal subgroup N of G is conjugacy $\mathcal{K}$-separable.*

Actually, it follows from the conditions of the proposition 2 that in any group G the family of all normal subgroups of G such that the quotient groups by them are $\mathcal{K}$-groups is closed under finite intersections. Hence, if group G is conjugacy $\mathcal{K}$-separable, then every finite subset of G is conjugacy $\mathcal{K}$-separable in G. So every coset of G modulo normal finite subgroup N of G is conjugacy $\mathcal{K}$-separable in G.

Now, let p be a prime and G be a finitely generated nilpotent group which is conjugacy $\mathcal{F}_p$-separable. Since in this case G is $\mathcal{F}_p$-residual, its torsion subgroup $\tau(G)$ must be a p-group. The quotient group $G/\tau(G)$ is torsion-free and by the proposition 2 must be conjugacy $\mathcal{F}_p$-separable. So to show that necessary condition of the theorem is true it is enough to prove

Proposition 3. *Suppose G is a finitely generated torsion-free nilpotent group. If for some prime p the group G is conjugacy $\mathcal{F}_p$-separable, then G is an abelian group.*

Proof. Let $1 = Z_0 \leqslant Z_1 \leqslant \cdots \leqslant Z_r = G$ be the upper central series of G. If, on the contrary, G is not abelian, then $r \geqslant 2$. Let a be an element from Z_2 but not from Z_1. Since for any element $g \in G$ the commutator $[a, g]$ is in Z_1, we have $g^{-1}ag = az$ for some $z \in Z_1$. Since the element a is not in the centre Z_1 of the group G, there is an element $b \in G$ such that $b^{-1}ab = ac$ for some non-unit element $c \in Z_1$.

Suppose q is a prime which is not equal to p. Since Z_1 is a free abelian group and $c \in Z_1$ is not equal to 1, for some integer $n \geqslant 1$ the equation $x^{q^n} = c$ has no solutions in Z_1. We assert that the elements a^{q^n} and $a^{q^n}c$ are not conjugate in G.

Indeed, suppose that for some $g \in G$ the equality $g^{-1}a^{q^n}g = a^{q^n}c$ is valid. As $g^{-1}ag = az$ for some $z \in Z_1$, we have

$$a^{q^n}c = (g^{-1}ag)^{q^n} = a^{q^n}z^{q^n}.$$

Thus, $c = z^{q^n}$, but this is impossible.

On the other hand let's show that the images of the elements a^{q^n} and $a^{q^n}c$ under every homomorphism from G onto finite p-group are conjugate.

Suppose N is a normal subgroup of G such that $c^{p^m} \equiv 1 \pmod{N}$ for some integer $m \geqslant 0$. Since numbers q^n and p^m are coprime, there exists integer k such that $q^n k \equiv 1 \pmod{p^m}$. As c is a central element, from the equality $b^{-1}ab = ac$ follows that $b^{-k}ab^k = ac^k$, and therefore

$$a^{q^n} c \equiv a^{q^n} c^{q^n k} = (ac^k)^{q^n} = b^{-k} a^{q^n} b^k \pmod{N}.$$

Thus, the group G is not conjugacy $\mathcal{F}_p$-separable and the proposition 3 is proved.

Note that if torsion subgroup $\tau(G)$ of a finitely generated nilpotent group G is a finite p-group, then from [4] the group G is $\mathcal{F}_p$-residual. So the inverse statement of the theorem is contained in the following more general result:

Proposition 4. *Suppose a group G has a finite normal subgroup F such that the quotient group G/F is a finitely generated abelian group. If for some prime p the group G is $\mathcal{F}_p$-residual, then G is conjugacy $\mathcal{F}_p$-separable.*

Proof. Let a and b be non-conjugate elements of G. We'll show that there exists a homomorphism φ of G onto some finite p-group such that the images $a\varphi$ and $b\varphi$ are not conjugate in $G\varphi$.

By the remark above the quotient group G/F is $\mathcal{F}_p$-residual. Consequently, if $aF \neq bF$, then the existence of required homomorphism is evident.

Suppose then that $aF = bF$, i. e. $b = af$ for some $f \in F$. Since G is $\mathcal{F}_p$-residual, there is a normal subgroup N of G that G/N is finite p-group and $N \cap F = 1$. We claim that the elements aN and bN of the quotient group G/N are not conjugate in it.

Indeed, suppose on the contrary that for some element $g \in G$ the equality $g^{-1}ag = b$ modulo N is valid. As the quotient group G/F is abelian, we have $g^{-1}ag = ax$ for some $x \in F$. Then $x = f$ modulo N, and as $N \cap F = 1$, we get $x = f$. But it means that the elements a and b are conjugate in G.

The proposition 4 is proved and so the theorem is.

References

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