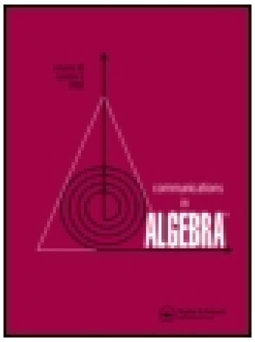

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On the residual properties of Baumslag–Solitar groups

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ABSTRACT
A survey of results on the residual properties of Baumslag–Solitar groups that have been obtained to date. Some unpublished results are included and for certain results new proofs are given

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1. Introduction

The Baumslag–Solitar group (BS-group) is an one-relator group with presentation

$$G(m,n) = \langle a,\ b;\ a^{-1}b^m a = b^n\rangle,$$

where m and n are non-zero integers. Note at once that since groups $G(m,n)$, $G(n,m)$ and $G(-m,-n)$ are pairwise isomorphic we can assume without loss of generality (and when it is convenient) that integers m and n in the presentation of group $G(m,n)$ satisfy the condition $|n| \geqslant m > 0$.

Recall that the family of groups $G(m,n)$ was singled out in 1962 in the paper of Baumslag and Solitar [2]. Just in this family authors discovered the first examples of finitely generated one-relator groups that are non-Hopfian (i. e. are isomorphic to some own proper quotient group) and therefore are not residually finite; specifically, it was shown that the group $G(2,3)$ is non-Hopfian. Thus, the supposition that every finitely generated one-relator group is Hopfian turned out to be disproved. At that time some mathematicians believed that this assumption, as well as the assumption of the residual finiteness of all one-relator groups, is correct (perhaps, because of the purely formal nearness of one-relator groups and free groups). It should be noted also that the properties of group $G(2,3)$ have given an answer to the question of Neumann [18, p. 545] whether a 2-generator non-Hopfian group can be defined by finite set of relations.

The study of properties of BS-groups became the permanent object of many investigations. This family of groups is of interest to researchers, in particular, because some natural questions about the properties of one-relator groups in the case of BS-groups can be answered in a more completed form than in the general case. For example, the isomorphism problem for groups of this family is trivial in view of following result (see [9]): groups $G(m,n)$ and $G(m',n')$, where $|n| \geqslant m > 0$ and $|n'| \geqslant m' > 0$, are isomorphic if and only if $m = m'$ and $n = n'$. To some extent the same is valid for questions about residual properties of one-relator groups. This article is an extended version of [14] and [13] and contains a survey of the results in this area that have been received to date.

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Some results are presented here with proofs. This generally happens in cases where the relevant publication is inaccessible now or (the new and more simple) proof has not been published. Very small number of previously unpublished results are also provided with proofs.

Let us agree on the following terminology. If $\mathcal{K}$ is a class of groups then a group G will be said to be *$\mathcal{K}$-residual* if for any non-identity element $a \in G$ there exists a homomorphism φ of group G onto some group from class $\mathcal{K}$ such that the image $a\varphi$ of a is not equal to identity. A group G will be said to be *conjugacy $\mathcal{K}$-separable* if for any elements $a, b \in G$, that are not conjugate in G, there exists a homomorphism φ of group G onto some group X from class $\mathcal{K}$ such that the images $a\varphi$ and $b\varphi$ of a and b are not conjugate in X. Subgroup H of group G is said to be *$\mathcal{K}$-separable* if for any element $g \in G \setminus H$ there exists a homomorphism φ of group G onto some group from class $\mathcal{K}$ such that the image $g\varphi$ of element g does not belong to image $H\varphi$ of subgroup H. It is obvious that if a group is conjugacy $\mathcal{K}$-separable then it is $\mathcal{K}$-residual (but the converse in general is not true) and that group is $\mathcal{K}$-residual if and only if its identity subgroup is $\mathcal{K}$-separable.

Let $\mathcal{F}$ denote the class of all finite groups and if p is a prime number and π is a set of prime numbers then let $\mathcal{F}_p$ and $\mathcal{F}_\pi$ denote the class of all finite p-groups and the class of all finite π-groups respectively. It is clear that the property of $\mathcal{F}$-residuality coincides with classical property of residually finite and the property of conjugacy $\mathcal{F}$-separability coincides with classical property of conjugacy separability. Group G is said to be *subgroup separable* if all of its finitely generated subgroups are $\mathcal{F}$-separable.

Recall also that for any set π of primes the integer n is called a π-number if $\pi(n) \subseteq \pi$ where $\pi(n)$ is the set of all prime divisors of n; n is a p-number when $\pi(n) = \{p\}$.

2. Residuality of BS-groups

The attempt to characterize $\mathcal{F}$-residual groups $G(m, n)$ made in [2] was refined by Meskin [8] as follows:

Theorem 1. *The group $G(m, n)$ is $\mathcal{F}$-residual if and only if (under the condition $|n| \geqslant m > 0$) either $m = 1$ or $|n| = m$.*

Theorem 1 implies, of course, that if the group $G(m, n)$ (where again it is supposed that $|n| \geqslant m > 0$) is $\mathcal{F}_p$-residual for some prime p then either $m = 1$ or $|n| = m$. The criterion of $\mathcal{F}_p$-residuality of groups $G(m, n)$ gives

Theorem 2 (see [10, Theorem 3])**.** *Let p be a prime number. Then*
(1) *the group $G(1, n)$ is $\mathcal{F}_p$-residual if and only if $n \equiv 1 \pmod p$;*
(2) *the group $G(m, m)$ is $\mathcal{F}_p$-residual if and only if $m = p^r$ for some $r \geqslant 0$;*
(3) *the group $G(m, -m)$ is $\mathcal{F}_p$-residual if and only if $p = 2$ and $m = 2^r$ for some $r \geqslant 0$.*

It makes sense to give a direct and quite elementary proofs of these theorems. To do this we first note that any group $G(m, n)$ is an *HNN*-extension of infinite cyclic base group B, generated by b, with stable letter a and with associated subgroups B^m and B^n that are generated by elements b^m and b^n respectively. Secondly, we introduce a family of finite homomorphic images of group $G(1, n)$; namely, for arbitrary positive integers k and l, such that $n^k \equiv 1 \pmod l$, we set

$$H_n(k, l) = \langle a, b;\ a^{-1}ba = b^n,\ a^k = b^l = 1\rangle.$$

Since the order of automorphism of cyclic group $\langle b;\ b^l = 1\rangle$, that is defined by the mapping $b \mapsto b^n$, divides the integer k, the group $H_n(k, l)$ is a split extension of cyclic group $\langle b;\ b^l = 1\rangle$ by cyclic group $\langle a;\ a^k = 1\rangle$. Hence, the order of group $H_n(k, l)$ is kl, orders of its elements a and b are k and l respectively and any element $g \in H_n(k, l)$ can be uniquely written in the form $g = a^i b^j$, where $0 \leqslant i < k$ and $0 \leqslant j < l$.

Now, let g be non-identity element of group $G(1,n)$. It is easy to see (using relations $ba = ab^n$ and $a^{-1}b = b^n a^{-1}$) that element g can be written as $g = a^q b^s a^{-r}$, where $q, r \geqslant 0$, and therefore g is conjugate to element $a^t b^s$, where $t = q - r$. If $t \neq 0$ then the image of element g under obvious homomorphism of $G(1,n)$ onto infinite cyclic group with generator a is not equal to identity. If $t = 0$ and hence $s \neq 0$, then the image of element g in group $H_n(k,l)$, where the number l is chosen coprime to n and not dividing s and where $k = \varphi(l)$ is the value of the Euler function, is not equal to identity.

Thus, the $\mathcal{F}$-residuality of any group $G(1,n)$ is proved. Moreover, if for some prime number p the congruence $n \equiv 1 \pmod{p}$ is fulfilled then for any number $s > 0$ we have $n^{p^{s+1}} \equiv 1 \pmod{p^s}$ and therefore the image of any non-identity element $g \in G(1,n)$ in the suitable finite p-group $H_n(p^{s+1}, p^s)$ is not equal to identity.

If $|n| = m$, i. e. $n = m\varepsilon$ for some $\varepsilon = \pm 1$, then in group $G(m, m\varepsilon)$ subgroup B^m is normal and the quotient group $G(m, m\varepsilon)/B^m$ is the free product of two cyclic groups, infinite and finite of order m. Therefore, if non-identity element g of group $G(m, m\varepsilon)$ does not belong to subgroup B^m then its image in $\mathcal{F}$-residual quotient group $G(m, m\varepsilon)/B^m$ is not equal to identity. To consider the remaining case when $g = b^{ms}$ for some $s \neq 0$ let φ be homomorphism of group $G(m, m\varepsilon)$ onto group $G(1,\varepsilon)$ defined by identity mapping of generators. Since the group $G(1,\varepsilon)$ is by above $\mathcal{F}$-residual and homomorphism φ on subgroup B acts injectively the proof of $\mathcal{F}$-residuality of group $G(m, m\varepsilon)$ is completed.

If $m = p^r$ for some prime number p then the quotient group $G(m, m\varepsilon)/B^m$ is $\mathcal{F}_p$-residual [4]. Moreover, the group $G(1,1)$ is free Abelian and therefore is $\mathcal{F}_p$-residual for any prime p. The group $G(1,-1)$ is $\mathcal{F}_2$-residual since its elements a^2 and b generate free Abelian normal subgroup of index 2.

So, the sufficiency of conditions in Theorems 1 and 2 is proved. Let us show that these conditions are necessary.

If $|n| > m > 1$ then element b does not belong to subgroup B^m. Also, if $d = (m,n)$ is the greatest common divisor of integers m and n then element b^d does not belong to subgroup B^n. Therefore the commutator $\left[ab^d a^{-1},\, b\right]$ is not equal to 1 since its expression $ab^{-d}a^{-1}b^{-1}ab^d a^{-1}b$ is reduced in HNN-extension $G(m,n)$. On the other hand, turns out to be that this commutator goes into the identity under any homomorphism of group $G(m,n)$ onto finite group. This assertion can be obtained from the following observation:

Proposition 1. *Let elements x and y of a group have the same finite order and let $x^n = y^m$ for some integers n and m. Then $\left[x^d, y\right] = 1$ where $d = (m,n)$ is the greatest common divisor of m and n.*

Really, let $r = |x| = |y|$. Since $x^n = y^m$ we must have $(r,n) = (r,m)$ and hence (r,n) divides d. Consequently, there exists an integer s such that $ns \equiv d \pmod{r}$. Then $x^d = x^{ns} = y^{ms}$ and therefore $\left[x^d, y\right] = 1$ as required.

Returning to the element $\left[ab^d a^{-1},\, b\right]$ of group $G(m,n)$ it is sufficient to remark that if φ is a homomorphism of group $G(m,n)$ onto finite group then elements $x = (aba^{-1})\varphi$ and $y = b\varphi$ satisfy the assumptions of the Proposition 1.

So, the proof of Theorem 1 is complete. Now, let us suppose that group $G(1,n)$ is $\mathcal{F}_p$-residual for some prime p. Then there exists a homomorphism φ of group $G(1,n)$ onto finite p-group X such that $y = b\varphi \neq 1$. Let also $x = a\varphi$. Since in group $G(1,n)$ for any number $k > 0$ the equality $a^{-k}ba^k = b^{n^k}$ holds, we have $n^{p^r} \equiv 1 \pmod{p^s}$ where p^r is the order of element x and p^s is the order of element y. Since $s > 0$ this implies the congruence $n^{p^r} \equiv 1 \pmod{p}$. But as by Fermat Theorem $n^{p-1} \equiv 1 \pmod{p}$ and as the numbers p^r and $p-1$ are coprime we obtain the required congruence $n \equiv 1 \pmod{p}$.

Next, let us show that if group $G(m, m\varepsilon)$ is $\mathcal{F}_p$-residual then m is a p-number. Indeed, otherwise there exists a prime $q \neq p$ dividing m, $m = m_1 q$. Then $m > m_1$ and therefore the commutator $\left[a^{-1}b^{m_1}a, b\right]$ is a non-identity element of group $G(m, m\varepsilon)$. On the other hand, let φ be a homomorphism of group $G(m, m\varepsilon)$ onto finite p-group X, $x = a\varphi$ and $y = b\varphi$. Let also p^s be the order of element y. Since numbers q and p^s are coprime there exists an integer k such that $qk \equiv 1 \pmod{p^s}$. Then $x^{-1}y^{m_1}x = (x^{-1}y^m x)^k = y^{m\varepsilon k}$ and hence $\left[a^{-1}b^{m_1}a, b\right]\varphi = 1$.

Finally, we note that for any integer $k \geqslant 0$ in group $G(m, m\varepsilon)$ the equality $a^{-k}b^m a^k = b^{m\varepsilon^k}$ holds. Hence if $\varepsilon = -1$ and if modulo some finite index normal subgroup N of group $G(m, m\varepsilon)$ the order k of element a is an odd number then $b^{2m} \in N$. Therefore if a group $G(m, -m)$ is $\mathcal{F}_p$-residual then $p = 2$ and Theorem 2 is proved.

Theorems 1 and 2 can be generalized in different directions. One of them is as follows.

Let $\mathcal{K}$ be again a class of groups and let for any group G the symbol $\sigma_{\mathcal{K}}(G)$ denote the intersection of all normal subgroups N of group G such that quotient group G/N belongs to $\mathcal{K}$. It is clear that a group G is $\mathcal{K}$-residual if and only if $\sigma_{\mathcal{K}}(G)$ coincides with identity subgroup. Moreover, $\sigma_{\mathcal{K}}(G)$ is the smallest normal subgroup of G the quotient group by which is $\mathcal{K}$-residual. If $\mathcal{K} = \mathcal{F}$, or $\mathcal{K} = \mathcal{F}_p$, or $\mathcal{K} = \mathcal{F}_\pi$ then in place of $\sigma_{\mathcal{K}}(G)$ we shall write $\sigma(G)$, or $\sigma_p(G)$, or $\sigma_\pi(G)$ respectively.

Theorem 3 (see [11, Theorem 1]). *Let $d = (m, n)$ be the greatest common divisor of integers m and n. Subgroup $\sigma(G(m, n))$ coincides with the normal closure in group $G(m, n)$ of the set of all commutators of form $\left[a^k b^d a^{-k}, b\right]$ where $k \in \mathbb{Z}$.*

Theorem 4 (see [12]). *Let p be a prime number and let $m = p^r m_1$ and $n = p^s n_1$ where $r, s \geqslant 0$ and integers m_1 and n_1 are not divided by p. Let also d be the greatest common divisor of integers m_1 and n_1 and let $m_1 = du$ and $n_1 = dv$ for suitable integers u and v. Then*

(1) *if $r \neq s$ or if integers m_1 and n_1 are not congruent modulo p then subgroup $\sigma_p(G(m, n))$ coincides with the normal closure in group $G(m, n)$ of element b^{p^t} where $t = \min\{r, s\}$;*

(2) *if $r = s$ and $m_1 \equiv n_1 \pmod p$ then subgroup $\sigma_p(G(m, n))$ coincides with the normal closure in group $G(m, n)$ of set consisting of element $a^{-1}b^{p^r u}ab^{-p^r v}$ and of all commutators of form $\left[a^k b^{p^r} a^{-k}, b\right]$ $(k \in \mathbb{Z})$.*

It should be emphasize that in proofs of Theorems 3 and 4 criterions of $\mathcal{F}$-residuality and $\mathcal{F}_p$-residuality of group $G(m, n)$ stated in Theorems 1 and 2 are not used. Vice versa, Theorems 1 and 2 can be deduced from Theorems 3 and 4 respectively.

To demonstrate this let us show, at first, how the sufficiency of conditions in Theorem 1 for group $G(m, n)$ (where $|n| \geqslant m > 0$) to be $\mathcal{F}$-residual can be derived from Theorem 3. It is well known (and easily to see) that if $m = 1$ then the normal closure in group $G(m, n)$ of element b is the locally cyclic and therefore Abelian group. Hence, all commutators of form $\left[a^k b^d a^{-k}, b\right]$ are equal to 1. If $|n| = m$ then d, the greatest common divisor of integers m and n, is equal to m and the defining relation of group $G(m, n)$ is of form $a^{-1}b^d a = b^{d\varepsilon}$ for some $\varepsilon = \pm 1$. Consequently, for any integer k in group $G(m, n)$ we have the equality $a^k b^d a^{-k} = b^{d\varepsilon^k}$ which implies that again $\left[a^k b^d a^{-k}, b\right] = 1$. Thus, we see that if either $m = 1$ or $|n| = m$ then by Theorem 3 subgroup $\sigma(G(m, n))$ of group $G(m, n)$ is equal to identity and therefore the group $G(m, n)$ is $\mathcal{F}$-residual.

Conversely, if $|n| > m > 1$ then, as was shown above, the commutator $\left[ab^d a^{-1}, b\right]$ is not equal to 1. Consequently, Theorem 3 implies that subgroup $\sigma(G(m, n))$ is not equal to identity and therefore the group $G(m, n)$ is not $\mathcal{F}$-residual.

Now, let us deduce Theorem 2 from Theorem 4.

Suppose that group $G(m, n)$ is $\mathcal{F}_p$-residual, i. e. $\sigma_p(G(m, n))$ coincides with identity subgroup. Since for any $t \geqslant 0$ element b^{p^t} differs from identity and therefore does not belong to subgroup $\sigma_p(G(m, n))$, the structure of this subgroup should be described in item (2) of Theorem 4. Consequently, we see that (in notations from the statement of Theorem 4) $r = s$ and $m_1 \equiv n_1 \pmod p$. So, if $m = 1$ and therefore $r = s = 0$, $m_1 = 1$ and $n = n_1$, then we obtain $n \equiv 1 \pmod p$.

Further, we claim that if $m > 1$ then $m_1 = 1 = |n_1|$. Indeed, since $\sigma_p(G(m, n)) = 1$ then by item (2) in group $G(m, n)$ all commutators of form $\left[a^k b^{p^r} a^{-k}, b\right]$ must be equal to identity. But if $m_1 > 1$ then element b^{p^r} does not belong to subgroup B^m. Also, since $|n| > 1$ element b does not belong to subgroup B^n. Hence the expression

$$\left[a^{-1}b^{p^r}a,b\right]=a^{-1}b^{-p^r}ab^{-1}a^{-1}b^{p^r}ab$$

of commutator $\left[a^{-1}b^{p^r}a,b\right]$ is reduced in *HNN*-extension $G(m,n)$ and therefore this commutator cannot be equal to identity. Similarly, assumption that $|n_1| > 1$ implies impossibility of equation $\left[ab^{p^r}a^{-1},b\right]=1$.

Thus, we have $m = p^r$ and $n = p^r\varepsilon$ for some $\varepsilon = \pm 1$. Finally, if $\varepsilon = -1$ then the congruence $m_1 \equiv n_1 \pmod p$ implies that $p = 2$.

Conversely, if $m = 1$ and $n \equiv 1 \pmod p$ then $r = 0$, $s = 0$, $m_1 = 1$ and $n_1 = n$. Hence the congruence $m_1 \equiv n_1 \pmod p$ is fulfilled. Therefore, in this case subgroup $\sigma_p(G(m,n))$ is the normal closure in group $G(m,n)$ of set of elements stated in item (2) of Theorem 4. As under $m = 1$ the normal closure in group $G(m,n)$ of element b is Abelian group, all commutators in this set are equal to identity. Since in this case we also have $p^r u = m$ and $p^r v = n$, element $a^{-1}b^{p^r u}ab^{-p^r v}$ is equal to identity too. Consequently, subgroup $\sigma_p(G(m,n))$ coincides with identity, i. e. group $G(m,n)$ is $\mathcal{F}_p$-residual.

If either $m = n = p^r$ or $m = 2^r$ and $n = -2^r$ then subgroup $\sigma_p(G(m,n))$ is again the normal closure in group $G(m,n)$ of set of elements stated in item (2) of Theorem 4 and it is clear that all these elements are equal to identity. Thus, in these cases group $G(m,n)$ is $\mathcal{F}_p$-residual and $\mathcal{F}_2$-residual respectively.

Another way to generalize Theorems 1 and 2 consists of study of conditions for group $G(m,n)$ to be $\mathcal{K}$-residual for some class $\mathcal{K}$. When $\mathcal{K}$ coincides with class $\mathcal{F}_\pi$ the following assertion is valid:

Theorem 5 (see [6, Theorem 1]). *Let π be a set of prime numbers. The group $G(1,n)$ is $\mathcal{F}_\pi$-residual if and only if there exists a π-number $s > 1$ coprime to n and such that the order modulo s of integer n is a π-number too.*

The criterion in Theorem 2 for group $G(1,n)$ to be $\mathcal{F}_p$-residual is a special case of Theorem 5. Indeed, if group $G(1,n)$ is $\mathcal{F}_p$-residual then by Theorem 5 we have $n^{p^t} \equiv 1 \pmod{p^r}$ for some numbers t and $r > 0$. Then $n^{p^t} \equiv 1 \pmod p$ and since by Fermat Theorem $n^{p-1} \equiv 1 \pmod p$ it follows that $n \equiv 1 \pmod p$. Conversely, if $n \equiv 1 \pmod p$ then the order modulo p of integer n is equal to 1 and therefore is a p-number. Consequently, group $G(1,n)$ is $\mathcal{F}_p$-residual by Theorem 5.

Theorem 2 implies certainly that group $G(1,n)$ is $\mathcal{F}_\pi$-residual if the set π contains at least one prime divisor of integer $n-1$. On the other hand, the Theorem 5 can be applied also to prove the existence of 2-element set π that contains no numbers from $\pi(n-1)$ and such that group $G(1,n)$ is $\mathcal{F}_\pi$-residual.

Corollary 1 (see [6, Theorems 2 and 3]). *Let $\pi = \{p,q\}$ be a set consisting of two prime numbers p and q such that $p < q$ and both p and q do not divide the integer $n-1$. Then group $G(1,n)$ is $\mathcal{F}_\pi$-residual if and only if $(n,q) = 1$, p divides $q-1$ and the order modulo q of integer n is a p-number. Moreover, if $|n| > 1$ then for any prime number p that does not belong to set $\pi(n-1)$ there exists a prime number $q > p$ such that $q \notin \pi(n-1)$ and group $G(1,n)$ is $\mathcal{F}_\pi$-residual where $\pi = \{p,q\}$.*

Thus, Corollary 1 characterizes 2-element sets π of primes such that group $G(1,n)$ is $\mathcal{F}_\pi$-residual and π is minimal with this property. Furthermore, if $|n| > 1$ then such sets exist. (It is easy to see that any set π, that is minimal with the property that the group $G(1,\pm 1)$ is $\mathcal{F}_\pi$-residual, consists of only one prime number.) The next proposition gives characterization of 3-element such sets.

Corollary 2. *Let $\pi = \{p,q,r\}$ be a set consisting of three prime numbers p, q and r where $p < q < r$. If the integer n is not divided by r and the order modulo r of n is a $\{p,q\}$-number then group $G(1,n)$ is $\mathcal{F}_\pi$-residual.*

Conversely, if the group $G(1,n)$ is $\mathcal{F}_\pi$-residual and also the set π is minimal with this property then

(1) *numbers p, q, r do not divide the integer $n-1$;*

(2) *if integer n is not divided by q, then the order modulo q of n is not a p-number;*

(3) *numbers p and q divide the number $r-1$, the integer n is not divided by r and the order modulo r of n is a $\{p,q\}$-number and is neither p-number nor q-number.*

We begin the proof of this Corollary by remarking that the first assertion is an immediate consequence of the theorem 5, where we set $s = r$.

Further, suppose that the group $G(1,n)$ is $\mathcal{F}_\pi$-residual and π is a minimal set with this property. It is obvious that the statement of item (1) is a direct consequence of Theorem 2 and minimality of the set π.

If the integer n is not divided by q and, nevertheless, the order modulo q of n is a p-number then $n^{p^t} \equiv 1 \pmod{q}$ for some integer $t \geqslant 0$. Since in addition $n^{q-1} \equiv 1 \pmod{q}$ and by (1) q do not divide the integer $n-1$ the numbers p^t and $q-1$ are not coprime and therefore $q-1$ is divided by p. But then the Corollary 1 implies that group $G(1,n)$ is $\mathcal{F}_{\pi_1}$-residual, where $\pi_1 = \{p,q\}$. Since this contradicts the assumption of minimality π the statement of item (2) is proved too.

Let us turn to the proof of the statement of item (3). By Theorem 5 there exists a π-number $s > 1$ that is coprime to n and satisfies the congruence $n^m \equiv 1 \pmod{s}$ for some π-number m. We first show that the numbers p and q are not divisors of the number s.

Indeed, if $p \mid s$ then $n^m \equiv 1 \pmod{p}$. Since numbers m and $p-1$ are coprime this congruence and Fermat Theorem imply the congruence $n \equiv 1 \pmod{p}$ that contradicts to item (1). Similarly, if $q \mid s$, then by item (1) numbers m and $q-1$ are not coprime and therefore the greatest common divisor of them is p^t for some $t > 0$. So, we have the congruence $n^{p^t} \equiv 1 \pmod{q}$ which is impossible by item (2).

Thus, the integer s is an r-number. Consequently, $n^m \equiv 1 \pmod{r}$ and therefore numbers n and r are coprime. If $d = (m, r-1)$ is the greatest common divisor of numbers m and $r-1$ then $n^d \equiv 1 \pmod{r}$ and assertion of item (1) implies that $d > 1$. It is also obvious that d is an $\{p,q\}$-number and so the order modulo r of integer n is a $\{p,q\}$-number too.

Let us show now that both numbers p and q divide d and hence divide $r-1$. It is clear that from this, taking in account of Corollary 1 and the minimality of the set π, it follows that the order modulo r of integer n is neither p-number nor q-number.

Really, if p is not divisor of d then d is a q-number and since $d > 1$ the integer $r-1$ is divided by q. In addition, the congruence $n^d \equiv 1 \pmod{r}$ implies that the order modulo r of integer n is a p-number. By Corollary 1 this contradicts the assumption of minimality π. Supposition that $q \nmid d$ leads to contradiction in the same way.

So, the Corollary 2 is proved. The existence of sets satisfying the conditions of this Corollary is confirmed by the following two examples:

Since the order modulo 7 of integer 2 equals to 3 and 2 is a primitive root modulo 29, the set $\pi = \left\{2,7,29\right\}$ is minimal with the property that group $G(1,2)$ is $\mathcal{F}_\pi$-residual.

Since the order modulo 5 of integer 38 equals to 4 and the order modulo 31 of integer 38 equals to 15, the set $\pi = \{3,5,31\}$ is minimal with the property that group $G(1,38)$ is $\mathcal{F}_\pi$-residual.

When $|n| = m$, the criterion of $\mathcal{F}_\pi$-residuality of group $G(m,n)$ can be expressed in more complete form:

Theorem 6 (see [23, Theorem 2]). *Let π be a set of prime numbers. Then*

(1) *the group $G(m,m)$ is $\mathcal{F}_\pi$-residual if and only if m is a π-number;*

(2) *the group $G(m,-m)$ is $\mathcal{F}_\pi$-residual if and only if m is a π-number and π contains the integer 2.*

In contrast to the corollaries of Theorem 5, for any group $G(m,n)$, where $|n| = m$, there exists exactly one set π of primes which is the minimal such that group $G(m,n)$ is $\mathcal{F}_\pi$-residual. Indeed, we have the obvious

Corollary 3. *A set π of primes is the minimal such that group $G(m,m)$ is $\mathcal{F}_\pi$-residual if and only if $\pi = \pi(m)$. A set π of primes is the minimal such that group $G(m,-m)$ is $\mathcal{F}_\pi$-residual if and only if $\pi = \pi(m) \cup \{2\}$.*

Theorem 6 also enables us to describe the subgroup $\sigma_\pi(G(m,n))$ in the case when $|n| = m$. Recall that for any set π of primes by π' is denoted the complement of π in the set of all primes.

Corollary 4. *Let $G = G(m, m\varepsilon)$ where $m > 0$ and $\varepsilon = \pm 1$ and let π be a set of primes. Suppose also that $m = m_1 m_2$ where m_1 is a π-number and m_2 is a π'-number. If either $\varepsilon = 1$ or $\varepsilon = -1$ and $2 \in \pi$ then subgroup $\sigma_\pi(G)$ coincides with the normal closure in group G of element $a^{-1}b^{m_1}ab^{-m_1\varepsilon}$. If $\varepsilon = -1$ and $2 \notin \pi$ then subgroup $\sigma_\pi(G)$ coincides with the normal closure in group G of element b^{m_1}.*

Indeed, let N be a normal subgroup of finite π-index of group G and let r and s be the orders modulo N of elements a and b respectively. Since the numbers m_2 and s are coprime, there exists an integer x such that $m_2 x \equiv 1 \pmod{s}$. Then, as

$$a^{-1}b^{m_1}a \equiv a^{-1}b^{m_1 m_2 x}a \equiv b^{m\varepsilon x} \equiv b^{m_1 m_2 \varepsilon x} \equiv b^{m_1\varepsilon} \pmod{N},$$

element $a^{-1}b^{m_1}ab^{-m_1\varepsilon}$ belongs to any normal subgroup of finite π-index of group G. Therefore the the normal closure C of this element is contained in subgroup $\sigma_\pi(G)$. Since the quotient group G/C is isomorphic to group $G(m_1, m_1\varepsilon)$, in the case when either $\varepsilon = 1$ or $\varepsilon = -1$ and $2 \in \pi$ according to Theorem 6 the group G/C is $\mathcal{F}_\pi$-residual. Hence $C \subseteq \sigma_\pi(G)$ and consequently $\sigma_\pi(G) = C$.

Further. if N is as above, a simple induction shows that for any integer $k > 0$ in group G the congruence $a^{-1}b^{m_1}a \equiv b^{m_1\varepsilon^k} \pmod{N}$ holds. If $\varepsilon = -1$ and r is an odd number this congruence implies that $b^{2m_1} \in N$. Moreover, if s is an odd number too, then we have $b^{m_1} \in N$ because $b^{m_1} \equiv (b^{2m_1})^y \pmod{N}$, where y is a solution of congruence $2y \equiv 1 \pmod{s}$. So, if $\varepsilon = -1$ and $2 \notin \pi$ then the normal closure D of element b^{m_1} is contained in $\sigma_\pi(G)$. Since the quotient group G/D is a free product of infinite cyclic group and cyclic group of order m_1 and therefore is $\mathcal{F}_\pi$-residual (by [4]), we have inclusion $D \subseteq \sigma_\pi(G)$. Thus, in this case $\sigma_\pi(G) = D$, and the proof of Corollary is complete. (Note that here, in contrast to the proofs of Theorems 3 and 4, we used the criterion of the $\mathcal{F}_\pi$-residuallity of these groups.)

It is clear that the assertion of Theorem 6 generalizes the second assertion of Theorem 2. In turn, Theorem 6 is a special case of a more general result of Tumanova. To formulate it, recall that by Gruenberg [4] the class of groups $\mathcal{K}$ is said to be a *root class* if every subgroup of any group from $\mathcal{K}$ belongs to class $\mathcal{K}$ and for any group X and for any subnormal series $Z \leqslant Y \leqslant X$ with factors from $\mathcal{K}$ there exists a normal subgroup N of X such that $N \leqslant Z$ and quotient group X/N belongs to $\mathcal{K}$. It is evident that every class $\mathcal{F}_\pi$ is a root class.

Theorem 7 (see [22, Corollary 8]). *Let $\mathcal{K}$ be a root class of groups such that every quotient group of any group from $\mathcal{K}$ belongs to $\mathcal{K}$. Then*

(1) *If class $\mathcal{K}$ contains at least one non-periodic group then the group $G(m, \pm m)$ is $\mathcal{K}$-residual.*
(2) *If all groups from $\mathcal{K}$ are periodic then*
  a) *the group $G(m,m)$ is $\mathcal{K}$-residual if and only if $\mathcal{K}$ contains group $\mathbb{Z}_m$;*
  b) *the group $G(m,-m)\mathcal{K}$-residual if and only if $\mathcal{K}$ contains groups $\mathbb{Z}_m$ and $\mathbb{Z}_2$.*

We conclude this section with two results of Azarov [1] about virtually residuality of BS-groups. Recall that for any class of groups $\mathcal{K}$ a group G is said to be virtually $\mathcal{K}$-residual if it contains a finite index subgroup which is $\mathcal{K}$-residual. It is obvious that if the class $\mathcal{K}$ consists only of finite groups, then any virtually $\mathcal{K}$-residual group is $\mathcal{F}$-residual.

Theorem 8 (see [1, theorem 1]). *The group $G(1,n)$ is virtually $\mathcal{F}_p$-residual if and only if the prime p does not divide n. The group $G(m, \pm m)$ is virtually $\mathcal{F}_p$-residual for any prime p.*

Theorem 9 (see [1, theorem 2]). *For any set π of prime numbers the group $G(m,n)$ is virtually $\mathcal{F}_\pi$-residual if and only if it is virtually $\mathcal{F}_p$-residual for some $p \in \pi$.*

3. Conjugacy separability of BS-groups

As it was noted above, any conjugacy $\mathcal{F}$-separable group is $\mathcal{F}$-residual. For BS-groups the converse is also true:

Theorem 10. *If the group $G(m,n)$ is $\mathcal{F}$-residual then it is conjugacy $\mathcal{F}$-separable.*

Conjugacy $\mathcal{F}$-separability of groups $G(1,n)$ was proved in [15]. This assertion is contained also in more general result that was obtained in [21] and affirms that any descending *HNN*-extension of finitely generated Abelian group is a conjugacy $\mathcal{F}$-separable group.

Conjugacy $\mathcal{F}$-separability of groups $G(m,n)$ under $|n| = m$ can be deduced from the result of work [24] or from generalization of it that was obtained in [20]. It should be also noted that since under $n = m$ the center of group $G(m,n)$ is non-trivial, the statement on conjugacy $\mathcal{F}$-separability of group $G(m,n)$ in this case follows as well from Armstrong's theorem which states that any one-relator group with non-trivial center is conjugacy $\mathcal{F}$-separable (see e.g. [3]).

However, we shall show here that in the case $|n| = m$ the statement on the conjugacy $\mathcal{F}$-separability of group $G(m,n)$ can be easily proved having applied ideas of Kargapolov [7] and result of Dyer [3]. The original proof of conjugacy $\mathcal{F}$-separability of group $G(1,n)$ given in [15] will also be reproduced.

The proof of Theorem 10 in the case $m = 1$.

Suppose that the coprime integers $n \neq \pm 1$ and $k > 0$ are fixed. Then integers r and s will be said to be (n,k)-equivalent if there exists a number $x \geqslant 0$ such that the congruence $rn^x \equiv s \pmod{k}$ holds; it is obvious that this relation is indeed an equivalence. It will allow us to give the necessary and sufficient conditions for certain elements of groups $G(1,n)$ and $H_n(r,s)$ (introduced above) to be conjugate. For any number $t > 0$ we set $u_t = |n^t - 1|$.

Proposition 2. *For any integer $n \neq \pm 1$ the following assertions are true:*

(1) *every element of group $G(1,n)$ is conjugate to element of form $a^t b^r$ for suitable integers t and r where integer r is not divisible by n if it is not 0;*

(2) *if $t > 0$ then elements $a^t b^r$ and $a^t b^s$ are conjugate in group $G(1,n)$ if and only if the integers r and s are (n, u_t)-equivalent;*

(3) *elements b^r and b^s are conjugate in group $G(1,n)$ if and only if either $r = sn^x$ or $s = rn^x$ for some $x \geqslant 0$; in particular, if integers r and s are different and not divisible by n then elements b^r and b^s are not conjugate.*

The verity of the first assertion of item (1) was noted above (in the proof of Theorem 1). If $r \neq 0$ and $r = nr_1$ then element $a^t b^r$ is conjugate to element $a(a^t b^r)a^{-1} = a^t b^{r_1}$ of the same form with $|r_1| < |r|$. So, the truth of the second assertion of (1) is also established.

To prove item (2) we first assume that the elements $a^t b^r$ and $a^t b^s$ are conjugate in group $G(1,n)$, i. e. $g^{-1}(a^t b^r)g = a^t b^s$ for some $g \in G(1,n)$. Let, as above, $g = a^p b^v a^{-q}$, where $p,q \geqslant 0$. Then $b^{-v}a^{-p}(a^t b^r)a^p b^v = a^{-q}(a^t b^s)a^q$, and therefore

$$a^t \cdot (a^{-t}ba^t)^{-v} \cdot (a^{-p}ba^p)^r \cdot b^v = a^t \cdot (a^{-q}ba^q)^s.$$

Hence $b^{rn^p - v(n^t-1)} = b^{sn^q}$ and since the order of element b is infinite we have the equality $rn^p - v(n^t - 1) = sn^q$. Since then $rn^p \equiv sn^q \pmod{|n^t - 1|}$, the integers r and s are (n, u_t)-equivalent.

Conversely, if for some number x the congruence $rn^x \equiv s \pmod{u_t}$ is valid then for suitable integer y we have $rn^x = s + y(n^t - 1)$. Hence

$$(a^x b^y)^{-1}(a^t b^r)(a^x b^y) = a^t \cdot (a^{-t}ba^t)^{-y} \cdot (a^{-x}ba^x)^r \cdot b^y = a^t b^{rn^x - y(n^t-1)} = a^t b^s,$$

and item (2) is proved.

Finally, if the element $g = a^p b^v a^{-q}$ is as above, then the equalities $g^{-1} b^r g = b^s$ and $rn^p = sn^q$ are equivalent.

Proposition 3. *The elements b^r and b^s of group $H_n(p, q)$ are conjugate if and only if the integers r and s are (n, q)-equivalent.*

Indeed, for any element $g = a^i b^j$ of group $H_n(p, q)$ the equality $g^{-1} b^r g = b^s$ is equivalent to equality $a^{-i} b^r a^i = b^s$ which, in turn, can be rewritten in the form $b^{n^i r} = b^s$. Thus, the elements b^r and b^s are conjugate if and only if for some integer $i \geqslant 0$ the congruence $n^i r \equiv s \pmod q$ holds.

A crucial role in the proof of the assertion of Theorem 10 in the case $m = 1$ plays the following statement from elementary number theory.

Proposition 4. *Let n be an integer $\neq \pm 1$. Then for any integers r and s, where $r \neq s$ and both r and s are not divisible by n, there exists a number $t > 0$ such that the exponential congruence $n^x r \equiv s \pmod{u_t}$ has no solution.*

The proof of Proposition 4 will be given below, after we use it to complete the proof of conjugacy $\mathcal{F}$-separability of groups $G(1, n)$.

It is obvious that the (free Abelian) group $G(1, 1)$ is conjugacy $\mathcal{F}$-separable. Since the center of group $G(1, -1)$ is non-trivial, the conjugacy $\mathcal{F}$-separability of this group follows from the result of Armstrong mentioned above. So, we can assume that $n \neq \pm 1$.

Let f and g be the non-conjugate elements of group $G(1, n)$. By the item (1) of Proposition 2 we may suppose that $f = a^{t_1} b^r$ and $g = a^{t_2} b^s$ for some integers t_1, t_2, r and s, where if any of numbers r and s is not equal to 0, then it is not divisible by n. If $t_1 \neq t_2$ then the images of elements f and g under the evident homomorphism of group $G(1, n)$ onto some finite cyclic group are distinct and therefore are non-conjugate. Thus, it remains to consider the case when $f = a^t b^r$ and $g = a^t b^s$. Here we can assume also (replacing, if it is necessary, elements f and g by f^{-1} and g^{-1}) that $t \geqslant 0$.

If $t > 0$ then by item (2) of Proposition 2 the integers r and s are not (n, u_t)-equivalent. Therefore, by Proposition 3 the images b^r and b^s of elements f and g under natural homomorphism of group $G(1, n)$ onto finite group $H_n(t, u_t)$ are not conjugate in this group.

Finally, let $f = b^r$ and $g = b^s$. Since the group $G(1, n)$ is $\mathcal{F}$-residual we can assume that both integers r and s are not equal to 0 and therefore are not divisible by n. Then by Proposition 4 there exists a number $t > 0$ such that numbers r and s are not (n, u_t)-equivalent. Consequently, the images of elements f and g under homomorphism of group $G(1, n)$ onto finite group $H_n(t, u_t)$ are not conjugate in this group. So, the conjugacy $\mathcal{F}$-separability of groups $G(1, n)$ is proved.

Now, proceed to the proof of Proposition 4. It states that for any integer $n \neq \pm 1$ and for any integers r and s, $r \neq s$, that are not divisible by n, there exists a number $t > 0$ such that the exponential congruence

$$n^x r \equiv s \pmod{u_t} \tag{1}$$

has no solutions. To prove this, let us consider two cases depending on the sign of n.

Case 1, $n > 0$. We shall show that in this case there exists an integer $t_0 > 0$ such that for any $t \geqslant t_0$ the congruence (1) does not have solution.

Assuming (without loss of generality) that the integer r is positive, we can write it in the number system with base n:

$$r = c_0 n^k + c_1 n^{k-1} + \cdots + c_{k-1} n + c_k,$$

where $k \geqslant 0$, $0 \leqslant c_i < n$ for any $i = 0, 1, \ldots, k$ and $c_0 \neq 0$. Remark that, since r is not divisible by n, we have also $c_k \neq 0$.

Next, let l be a positive integer and $R = n^l r$. Then

$$R = d_0 n^{k+l} + d_1 n^{k+l-1} + \cdots + d_{k+l-1} n + d_{k+l},$$

where, of course,

$$d_i = \begin{cases} c_i, & \text{if } 0 \leqslant i \leqslant k, \\ 0, & \text{if } k+1 \leqslant i \leqslant k+l. \end{cases}$$

Further, for every $i = 0, 1, \ldots, k$ let the symbol r_i denote the number that is obtained from number r by cyclic permutation of digits beginning with c_i; thus, $r_0 = r$ and for $i > 0$

$$r_i = c_i n^k + c_{i+1} n^{k-1} + \cdots + c_k n^i + c_0 n^{i-1} + \cdots + c_{i-1}.$$

Similarly, for every $i = 0, 1, \ldots, k+l$ let the number R_i be obtained by cyclic permutation of digits of number R beginning with d_i. Thus, $R_0 = R$ and if $i > 0$

$$R_i = \sum_{j=0}^{k+l-i} d_{i+j} n^{k+l-j} + \sum_{j=0}^{i-1} d_j n^{i-1-j}.$$

One can easily show that under $t = k+l+1$ for any $i = 0, 1, \ldots, k+l$ we have the congruence

$$n^i R \equiv R_i \pmod{u_t}. \tag{2}$$

Moreover, it is not difficult to see that

$$R_i = \begin{cases} n^l r, & \text{if } i = 0, \\ n^l r_i + p_i(1 - n^l), & \text{if } 1 \leqslant i \leqslant k, \\ n^{i-k-1} r, & \text{if } k+1 \leqslant i \leqslant k+l, \end{cases} \tag{3}$$

where for $1 \leqslant i \leqslant k p_i = c_0 n^{i-1} + c_1 n^{i-2} + \cdots + c_{i-1}$.

Congruences (2) obviously imply that any integer of form $n^i R$, $i \geqslant 0$, is congruent modulo u_t (where, recall, $t = k+l+1$) to one of numbers $R_0, R_1, \ldots, R_{k+l}$. From this and from (3) it follows that the same holds also for any number of form $n^i r$. Indeed, if $i \geqslant l$ this is evident as $n^i r = n^{i-l} R$. In the case $0 \leqslant i \leqslant l-1$ we set $j = i+k+1$. Then $k+1 \leqslant j \leqslant k+l$ and therefore by (3) we have $n^i r = n^{j-k-1} r = R_j$. Remark also that $0 < R_i < n^{k+l+1}$ for any $i = 0, 1, \ldots, k+l$.

Now, if in the case when $s > 0$ we choose the number l such that $n^l > s$ then all numbers s and $R_0, R_1, \ldots, R_{k+l}$ will belong to complete system of (the smallest non-negative) residues modulo u_t. In addition, number s is not equal to any number R_i ($0 \leqslant i \leqslant k+l$). Really, if $i = 0$ or $k+1 \leqslant i \leqslant k+l$ this follows directly from (3) since s is different from r and is not divisible by n. If $1 \leqslant i \leqslant k$ then again by (3) we have

$$R_i = n^l(r_i - p_i) + p_i > n^l(c_i n^k + c_{i+1} n^{k-1} + \cdots + c_k n^i) \geqslant n^{l+i} c_k \geqslant n^{l+i} > s.$$

Thus, if $s > 0$ and if we set $t_0 = k + l_0 + 1$, where $n^{l_0} > s$, then for any $t > t_0$ the congruence (1) does not have solution.

In the case when $s < 0$ it is sufficient to show that there exists a number $l_0 > 0$ such that

$$R_i < (n^{k+l+1} - 1) + s \qquad (0 \leqslant i \leqslant k+l)$$

for any $l \geqslant l_0$. Indeed, then all numbers s and $R_0, R_1, \ldots, R_{k+l}$ will belong to complete system $\{y \mid s \leqslant y < u_t + s\}$ of residues modulo u_t with $s < 0 < R_i$.

It follows from (3) that

$$n^{k+l+1} - R_i = \begin{cases} n^l(n^{k+1} - r), & \text{if } i = 0, \\ n^l(n^{k+1} - r_i + p_i) - p_i, & \text{if } 1 \leqslant i \leqslant k, \end{cases}$$

and if $k+1 \leqslant i \leqslant k+l$, then $n^{k+l+1} - R_i \geqslant n^{l-1}(n^{k+2} - r)$. Since all numbers $n^{k+1} - r$, $n^{k+1} - r_i + p_i$, $n^{k+2} - r$ are positive the existence of the required number l_0 is evident.

Case 2, $n < 0$. If the integers r^2 and s^2 are distinct then, since they are not divisible by n^2, it follows by the Case 1 that there exists a number $l > 0$ such that the congruence $(n^2)^x r^2 \equiv s^2 \pmod{((n^2)^l - 1)}$ has no solution. Then clearly that under $t = 2l$ the congruence $n^x r \equiv s \pmod{u_t}$ has no solution too. So, since $r \neq s$ it remains to consider the case $s = -r$.

Let us suppose, arguing by contradiction, that for every number $t > 0$ the congruence $n^x r \equiv -r \pmod{u_t}$ is solvable. By the Case 1 there exists a number t_0 such that for any number $t > t_0$ the congruence $(n^2)^x r \equiv -r \pmod{((n^2)^t - 1)}$ has no solution. Therefore, if the number p satisfies the inequality $2^{p-1} > t_0$, then the solution x_0 of congruence $n^x r \equiv -r \pmod{((n^{2^p} - 1))}$ must be an odd number.

Since the numbers x_0 and 2^p are coprime the greatest common divisor of numbers $n^{x_0}+1$ and $n^{2^p}-1$ is $-n-1$. Consequently, the number r must be divided by any number of form

$$(-n)^{2^p-1} + (-n)^{2^p-2} + \cdots + (-n) + 1,$$

where $p > \log_2 t_0 + 1$. But this is impossible since $r \neq 0$. The proof of Proposition 4 is complete.

The proof of Theorem 10 in the case $|n| = m$

The following statement was actually proved by Kargapolov [7] but was not stated explicitly:

Proposition 5. *Let C be an infinite cyclic normal subgroup of group G. If for every integer $r > 0$ the quotient group G/C^r is conjugacy $\mathcal{F}$-separable then group G is conjugacy $\mathcal{F}$-separable too.*

In order to derive from this proposition the conjugacy $\mathcal{F}$-separability of groups $G(m,n)$ under $|n| = m$ it is enough to note that in this case the cyclic subgroup $C = B^m$ of group $G(m,n)$ is infinite and normal in $G(m,n)$. It is clear also that for any integer $r > 0$ the quotient group

$$G(m,n)/C^r = \langle a,\ b;\ a^{-1}b^m a = b^{\pm m},\ b^{mr} = 1\rangle$$

is an HNN-extension of finite cyclic group. It remains to recall that by [3] any HNN-extension with finite base group is a conjugacy $\mathcal{F}$-separable group.

For the completeness of account let me give an outline of proof of Proposition 5.

So, let G be a group with infinite cyclic normal subgroup C (generated by element c) such that for every integer $r > 0$ the quotient group G/C^r is conjugacy $\mathcal{F}$-separable. To prove that group G is conjugacy $\mathcal{F}$-separable it is enough to show that for any elements f and g of group G which are not conjugate in G there exists an integer $r > 0$ such that elements f and g are not conjugate modulo subgroup C^r.

Since in the case when elements f and g are not conjugate modulo subgroup C we can put $r = 1$, it remains to consider the case when for some integer k element f is conjugate with element gc^k. Obviously, it is sufficient to prove that for some integer $r > 0$ elements g and gc^k are not conjugate modulo subgroup C^r. In order to make this let us introduce the set of integers

$$U = \left\{ n \in \mathbb{Z} \;\middle|\; (\exists x \in G)(x^{-1}gx = gc^n) \right\}$$

and its subset

$$V = \left\{ n \in \mathbb{Z} \;\middle|\; (\exists x \in G)(x^{-1}gx = gc^n \wedge xc = cx) \right\}.$$

It is easy to see that V is a subgroup of additive group $\mathbb{Z}$ of integers and if $U \neq V$ then U is the union of V and some another coset $V + n_0$. Note that since elements g and gc^k are not conjugate in G the integer k does not belong to U.

Now, for some integer $m \geqslant 0$ we must have $V = m\mathbb{Z}$. It is asserting that if $m > 0$ then we can put $r = m$, i. e. elements g and gc^k are not conjugate modulo subgroup C^m. Indeed, if, on the contrary, for some element $x \in G$ and for some integer s we have $x^{-1}gx = gc^{k+ms}$, then the integer $k + ms$ belong to

U and therefore $k \in U$ but this is impossible. If $m = 0$ then $U = \{0\}$ or $U = \{0, n_0\}$. If $U = \{0\}$ then let r be any positive integer that does not divide k and if $U = \{0, n_0\}$ then let r be any positive integer that does not divide both integers k and $k - n_0$. It is clear that then for any integer s the integer $k + rs$ does not belong to U, i. e. elements g and gc^k are not conjugate modulo subgroup C^r.

The proof of Theorem 10 is complete.

In connection with Theorem 10, the following question naturally arises: if π is a set of primes, will the group $G(m, n)$, which is $\mathcal{F}_\pi$-residual, be conjugacy $\mathcal{F}_\pi$-separable? Above results (Theorem 2 and Corollary 1) exhibit the existence of 1- and 2-elements sets π of prime numbers such that the group $G(1, n)$ is $\mathcal{F}_\pi$-residual. Nevertheless, for the property to be conjugacy $\mathcal{F}_\pi$-separable is valid the

Theorem 11 (see [5]). *If $n \neq \pm 1$ then for any set π consisting of two prime numbers the group $G(1, n)$ is not conjugacy $\mathcal{F}_\pi$-separable.*

Thus, for any integer $n \neq \pm 1$ there exists a set π of prime numbers such that the group $G(1, n)$ is $\mathcal{F}_\pi$-residual but is not conjugacy $\mathcal{F}_\pi$-separable. By contrast, when $|n| = m$, we have:

Theorem 12 (see [23]). *For any set π of prime numbers and for any group $G(m, n)$, where $|n| = m$, if group $G(m, n)$ is $\mathcal{F}_\pi$-residual then it is conjugacy $\mathcal{F}_\pi$-separable.*

4. Subgroup separability of BS-groups

It is well known and easily to see that if $|n| > 1$ then in group $G(1, n)$ the cyclic subgroup B generated by element b is not $\mathcal{F}$-separable. Indeed, element $g = aba^{-1}$ does not belong to B since in *HNN*-extension $G(1, n)$ it is reduced of length 2. Let N be a finite index normal subgroup of group $G(1, n)$ and let r be the order of element b modulo N. Since elements b and b^n are conjugate and therefore have the same order modulo N, the integers r and n are coprime. Hence there exists an integer k such that $nk \equiv 1 \pmod r$ and therefore, $g = aba^{-1} \equiv ab^{nk}a^{-1} = b^k \pmod N$. Thus, element g belongs to subgroup BN for every normal subgroup N of finite index of group $G(1, n)$ and hence subgroup B is not $\mathcal{F}$-separable. Remark that, on the other hand, an arbitrary non-cyclic finitely generated subgroup of group $G(1, n)$ is of finite index and therefore is $\mathcal{F}$-separable.

In the case $|n| = m$ the situation again appears to be more definite:

Theorem 13. *If $|n| = m$ then the group $G(m, n)$ is subgroup separable.*

It should be noted that in the case when $n = m$ this assertion was long known by the result of [16], which states that any one-relator group with non-trivial center is subgroup separable. In general this Theorem was proved in [17].

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