ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2020, Vol. 41, No. 2, pp. 260–272.

To the Question of the Root-Class Residuality of Free Constructions of Groups

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Received September 13, 2019; revised October 5, 2019; accepted October 15, 2019

Abstract—Let $\mathcal{C}$ be a root class of groups and $\pi_1(\mathcal{G})$ be the fundamental group of a graph $\mathcal{G}$ of groups. We prove that if $\mathcal{G}$ has a finite number of edges and there exists a homomorphism of $\pi_1(\mathcal{G})$ onto a group of $\mathcal{C}$ acting injectively on all the edge subgroups, then $\pi_1(\mathcal{G})$ is residually a $\mathcal{C}$-group. The main result of the paper is that the inverse statement is not true for many root classes of groups. The proof of this result is based on the criterion for the fundamental group of a graph of isomorphic groups to be residually a $\mathcal{C}$-group, which is of independent interest.

DOI: 10.1134/S1995080220020158

Keywords and phrases: *Root-class residuality, Fundamental group of a graph of groups, Generalized free product, HNN-extension.*

1. INTRODUCTION. STATEMENT OF RESULTS

Let us recall that if $\mathcal{C}$ is a class of groups, then a group X is said to be *residually a $\mathcal{C}$-group* if, for any element $x \in X \setminus \{1\}$, there exists a homomorphism of X onto a group of $\mathcal{C}$ (*$\mathcal{C}$-group*) mapping x to a nonidentity element.

The first results on the residuality of groups appeared in the 30s of the XX century. Soon after, it became clear that when proving the residuality of one and the same group by different classes of groups, similar reasoning schemes were often used, and therefore it is natural to try to carry out the proofs once, using the general properties of the indicated classes. One of the first to implement this idea was K. Gruenberg [1], who proposed the concept of the root class of groups. Following his definition, we say that a class $\mathcal{C}$ of groups is *root* if it is nontrivial (i. e., contains at least one nonidentity group) and satisfies the following conditions:

1) $\mathcal{C}$ is closed under taking subgroups;

2) $\mathcal{C}$ is closed under taking direct products of a finite number of factors;

3) if $1 \leqslant Z \leqslant Y \leqslant X$ is a subnormal series of a group X and $X/Y,\ Y/Z \in \mathcal{C}$, then there exists a normal subgroup T of X such that $T \leqslant Z$ and $X/T \in \mathcal{C}$.

It is easy to see that in the above definition, the second condition follows from the third and therefore is redundant. The third condition, now commonly called *the Gruenberg condition*, is utilitarian in nature and allows one to prove a number of statements that turn out to be very useful in studying the residuality of free constructions of groups (see Proposition 5 below). At the same time, the Gruenberg condition makes it difficult to understand what the root classes of groups are as a whole. The situation is clarified by

Proposition 1 [2]. *Let $\mathcal{C}$ be a nontrivial class of groups closed under taking subgroups. Then the following statements are equivalent.*

1. *$\mathcal{C}$ satisfies the Gruenberg condition and therefore is root.*

2. *$\mathcal{C}$ is closed under taking Cartesian wreath products.*

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3. *$\mathcal{C}$ is closed under taking extensions and, together with any two groups X, Y, contains the Cartesian product $\prod_{y\in Y} X_y$, where X_y is an isomorphic copy of X for each $y \in Y$.*

It follows easily from Proposition 1 that the intersection of any number of root classes of groups again turns out to be a root class if it contains at least one nonidentity group. Concrete examples of root classes include classes of all finite groups, finite p-groups (where p is a prime number), periodic π-groups of finite period (where π is an arbitrary set of primes), solvable groups, and all torsion-free groups.

The first results on the residuality by an arbitrary root class of groups were obtained already in [1]. However, the path to a systematic study of the root-class residuality of free constructions of groups was opened only by the remark made by D.N. Azarov, that every free group is residually a $\mathcal{C}$-group for each root class $\mathcal{C}$ [3]. In recent years, quite a lot of results have been obtained on the root-class residuality of generalized free products and HNN-extensions [2–13]. This article is devoted to the study of the root-class residuality of the fundamental groups of arbitrary graphs of groups. Unless otherwise specified, all graphs considered below are assumed to be nonempty, non-oriented, and not necessarily connected. The number of vertices and edges in them does not have to be finite; multiple edges and loops are allowed.

Let $\Gamma = (V, E)$ be an arbitrary graph with the set of vertices V and the set of edges E. Denote the vertices of Γ that are the ends of an edge $e \in E$ by $e(1)$, $e(-1)$, and associate to each vertex $v \in V$ a group G_v and to each edge $e \in E$ a group H_e and injective homomorphisms $\varphi_{+e}\colon H_e \to G_{e(1)}$, $\varphi_{-e}\colon H_e \to G_{e(-1)}$. As a result, we obtain *the graph of groups*

$$\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}\ (e \in E)\big)$$

corresponding to the graph Γ. We call the groups G_v and the subgroups $H_{+e} = H_e\varphi_{+e}$, $H_{-e} = H_e\varphi_{-e}$ *the vertex groups* and *the edge subgroups* respectively.

We note that in $\mathcal{G}(\Gamma)$, two, in general, different homomorphisms φ_{+e}, φ_{-e} are associated with an edge e even if e is a loop, i. e., $e(1) = e(-1)$. We also note that, unlike the graph Γ, the graph of groups $\mathcal{G}(\Gamma)$ can be considered oriented if necessary, assuming that the homomorphism φ_{+e} corresponds to the beginning, while the homomorphism φ_{-e} does to the end of an edge e.

Let us fix some maximal forest $F = (V, E_F)$ of Γ. *The fundamental group* of the graph of groups $\mathcal{G}(\Gamma)$ is the group

$$\pi_1(\mathcal{G}(\Gamma)) = \big\langle G_v\ (v \in V),\ t_e\ (e \in E \setminus E_F);$$
$$h\varphi_{+e} = h\varphi_{-e}\ (e \in E_F,\ h \in H_e),\ t_e^{-1} h\varphi_{+e} t_e = h\varphi_{-e}\ (e \in E \setminus E_F,\ h \in H_e)\big\rangle,$$

whose generators are the generators of G_v $(v \in V)$ and symbols t_e $(e \in E \setminus E_F)$, and whose defining relations are the relations of G_v $(v \in V)$ and all possible relations of the forms $h\varphi_{+e} = h\varphi_{-e}$ $(e \in E_F$, $h \in H_e)$, $t_e^{-1} h\varphi_{+e} t_e = h\varphi_{-e}$ $(e \in E \setminus E_F,\ h \in H_e)$ where $h\varphi_{\varepsilon e}$ $(\varepsilon = \pm 1)$ is a word in the generators of $G_{e(\varepsilon)}$ defining the image of h under $\varphi_{\varepsilon e}$. Obviously, the presentation of $\pi_1(\mathcal{G}(\Gamma))$ depends on the choice of the maximal forest F. It is known, however, that all the groups with presentations described above corresponding to the different maximal forests of Γ are isomorphic [14, § 5.1]. This allows us to refer the fundamental group of a graph of groups without mentioning a particular maximal forest.

Note that if Γ contains two vertices v, w and an edge e connecting them, then $\pi_1(\mathcal{G}(\Gamma))$ is a free product of G_v and G_w with the amalgamated subgroups H_{+e} and H_{-e}; if Γ has one vertex v and a loop e at this vertex, then $\pi_1(\mathcal{G}(\Gamma))$ is an HNN-extension of G_v with one stable letter t_e and the associated subgroups H_{+e} and H_{-e} (the terminology regarding generalized free products and HNN-extensions, which is used here and further, is agreed with the monographs [15, 16]). In [3, 4], the quite useful sufficient conditions for the root-class residuality of the above constructions were proved; they can be formulated as follows.

Proposition 2 [3, Theorem 3; 4, Theorem 4.1]. *Let $\mathcal{C}$ be a root class of groups, G be a free product of residually $\mathcal{C}$-groups A and B with amalgamated subgroups $H \leqslant A$ and $K \leqslant B$ or an HNN-extension of a residually $\mathcal{C}$-group B with associated subgroups $H \leqslant B$ and $K \leqslant B$. If there exists a homomorphism of G onto a group of $\mathcal{C}$ acting injectively on H and K, then G is residually a $\mathcal{C}$-group.*

In this paper, Proposition 2 is generalized as follows.

Theorem 1. *Let $\mathcal{C}$ be a root class of groups, $\Gamma = (V, E)$ be a graph with a finite number of edges, $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}\ (e \in E)\big)$ be a corresponding graph of groups and all G_v $(v \in V)$ be residually $\mathcal{C}$-groups. If there exists a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto a group of $\mathcal{C}$ acting injectively on all the subgroups $H_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$, then $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group.*

The aim of this paper is to discuss the truth of the statement inverse to Theorem 1 (including the case when Γ is an arbitrary graph). Note that if there exists a homomorphism σ with the properties indicated in this theorem, then every subgroup $H_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$ embeds in the $\mathcal{C}$-group $\pi_1(\mathcal{G}(\Gamma))\sigma$ and itself belongs to $\mathcal{C}$ because this class is root and hence is closed under taking subgroups. Thus, it makes sense to formulate the question of interest to us as follows.

Question. *Let $\mathcal{C}$ be a root class of groups, $\Gamma = (V, E)$ be an arbitrary graph and $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}\ (e \in E)\big)$ be a corresponding graph of groups. Let also all G_v $(v \in V)$ be residually $\mathcal{C}$-groups and all $H_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$ belong to $\mathcal{C}$. Under what conditions does the $\mathcal{C}$-residuality of $\pi_1(\mathcal{G}(\Gamma))$ imply the existence of a homomorphism of this group onto a group of $\mathcal{C}$ acting injectively on all the subgroups $H_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$?*

The interest in the formulated question is explained as follows. The widely used approach to the study of the $\mathcal{C}$-residuality of free constructions of groups, going back to [17], includes two main steps. At step 1, the $\mathcal{C}$-residuality of some construction composed of groups belonging to the class $\mathcal{C}$ is explored. At step 2, we study the $\mathcal{C}$-residuality of the same construction, but formed already from arbitrary residually $\mathcal{C}$-groups, and the task is to find the conditions under which this construction is residually a construction explored in step 1. It turns out, however, that in step 2, the criterion for the $\mathcal{C}$-residuality of the studied construction, made up of $\mathcal{C}$-groups, may not be enough. It is required to know when such a construction has a homomorphism onto a $\mathcal{C}$-group that is injective on all its edge subgroups (see [7, 8]). Theorem 2 given below describes some cases in which the indicated homomorphism exists. Preceding its formulation, we recall that a group is said to be of *finite Hirsch–Zaitsev rank* if it has a finite subnormal series, all of whose factors are periodic or infinite cyclic groups [18].

Theorem 2. *Let $\mathcal{C}$ be a class of groups closed under taking subgroups and direct products of a finite number of factors, $\Gamma = (V, E)$ be a graph with a finite number of edges and $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}\ (e \in E)\big)$ be a corresponding graph of groups. Let also at least one of the following conditions take place:*

1) *all the subgroups $H_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$ are finite and $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group;*

2) *all the subgroups $H_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$ are of finite Hirsch–Zaitsev rank and $\pi_1(\mathcal{G}(\Gamma))$ is residually a torsion-free $\mathcal{C}$-group.*

Then there exists a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto a group of $\mathcal{C}$ acting injectively on all the subgroups $H_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$.

From Theorem 2 it follows, in particular, that if $\mathcal{C}$ is a root class consisting only of finite groups and all the edge subgroups of $\pi_1(\mathcal{G}(\Gamma))$ are contained in this class, then Theorem 1 turns into a criterion. This explains the fact that, despite many years of research on the property of residual finiteness, the question formulated above arose only now, with the beginning of a systematic study of the residuality of free constructions by arbitrary root classes of groups.

The main result of the present paper is Theorem 3 below, which states that for many root classes of groups, the residuality of the fundamental group of a graph of groups is a weaker statement than the existence of a homomorphism of this group injective on all its edge subgroups.

Theorem 3. *Let $\mathcal{C}$ be a root class of groups containing at least one infinite group and not containing some (absolutely) free group of finite or countable rank. Then, for any graph $\Gamma = (V, E)$, there exists a corresponding graph of groups $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}\ (e \in E)\big)$ such that:*

1) *all G_v $(v \in V)$ are residually $\mathcal{C}$-groups;*

2) *all $H_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$ belong to $\mathcal{C}$;*

3) *$\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group;*

4) *for any homomorphism σ of $\pi_1(\mathcal{G}(\Gamma))$ onto a group of $\mathcal{C}$ and for any $e \in E$, $\varepsilon = \pm 1$, the relation $\ker\sigma \cap H_{\varepsilon e} \neq 1$ holds.*

We note that the condition of Theorem 3 is satisfied, in particular, by any root class consisting only of periodic groups and containing at least one infinite group.

The proof of Theorem 3 is based on the criterion for the root-class residuality of the fundamental groups of graphs of isomorphic groups that generalizes a series of results from [3, 4] and is of independent interest. We give the necessary definitions.

Let $\Gamma = (V, E)$ be an arbitrary graph and $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}\ (e \in E)\big)$ be a corresponding graph of groups. Let also, for any $v, w \in V$, there exist an isomorphism $\alpha_{v,w}\colon G_v \to G_w$ and the set $\{\alpha_{v,w} | v, w \in V\}$ satisfy the following conditions:

1) $\forall v \in V\ \alpha_{v,v} = \mathrm{id}_{G_v}$ (where id_{G_v} is the identity map of G_v to itself);

2) $\forall u, v, w \in V\ \alpha_{u,v}\alpha_{v,w} = \alpha_{u,w}$ (in particular, $\forall v, w \in V\ \alpha_{w,v} = \alpha_{v,w}^{-1}$);

3) $\forall e \in E\ \forall \varepsilon = \pm 1\ \alpha_{e(\varepsilon),e(-\varepsilon)}|_{H_{\varepsilon e}} = \varphi_{\varepsilon e}^{-1}\varphi_{-\varepsilon e}$.

Then $\mathcal{G}(\Gamma)$ will be called *the graph of isomorphic groups*.

Recall also that a subgroup Y of a group X is said to be *separable* in this group *by a class of groups* $\mathcal{C}$ (or, briefly, $\mathcal{C}$*-separable*) if, for each $x \in X \setminus Y$, there exists a homomorphism σ of X onto a group of $\mathcal{C}$ such that $x\sigma \notin Y\sigma$ [19].

Theorem 4. *Let $\mathcal{C}$ be a root class of groups, $\Gamma = (V, E)$ be an arbitrary graph and $\mathcal{G}(\Gamma) = \big(\Gamma, G_v\,(v \in V), H_e, \varphi_{\pm e}\,(e \in E), \alpha_{v,w}\,(v, w \in V)\big)$ be a corresponding graph of isomorphic groups. Then $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group if and only if all G_v $(v \in V)$ are residually $\mathcal{C}$-groups and, for any $e \in E$, $\varepsilon = \pm 1$, $H_{\varepsilon e}$ is $\mathcal{C}$-separable in $G_{e(\varepsilon)}$.*

The second section of the paper contains a number of necessary auxiliary definitions and statements. The remaining sections are devoted to the proof of Theorems 1–4.

2. SOME KNOWN STATEMENTS

If $\mathcal{C}$ is a class of groups and X is an arbitrary group, then we denote by $\mathcal{C}^*(X)$ the family of all normal subgroups of X, the quotient groups by which belong to $\mathcal{C}$. The subgroups of $\mathcal{C}^*(X)$ will be called *co-$\mathcal{C}$-subgroups* of X.

Proposition 3 [10, Proposition 1]. *Let $\mathcal{C}$ be a class of groups closed under taking subgroups and direct products of a finite number of factors, X be a group. Then the intersection of a finite number of subgroups of $\mathcal{C}^*(X)$ again belongs to this family.*

Proposition 4. *Let $\mathcal{C}$ be a class of torsion-free groups closed under taking subgroups and direct products of a finite number of factors. Then the following statements take place.*

1. *If X is residually a $\mathcal{C}$-group and Y is a subgroup of X having finite Hirsch–Zaitsev rank, then there exists a subgroup $Z \in \mathcal{C}^*(X)$ such that $Z \cap Y = 1$* [10, Proposition 11].

2. *If X is residually a $\mathcal{C}$-group and Y is a polycyclic subgroup of X, then Y is $\mathcal{C}$-separable in X* [20, Proposition 1].

Proposition 5. *Let $\mathcal{C}$ be a root class of groups. Then the following statements take place.*

1. *Every free group is residually a $\mathcal{C}$-group* [3, Theorem 1].

2. *The direct and free products of an arbitrary number of residually $\mathcal{C}$-groups are also residually $\mathcal{C}$-groups* [1, Lemma 1.1, Theorem 4.1; 3, Theorem 2].

3. *Any extension of a residually $\mathcal{C}$-group by a group of $\mathcal{C}$ is residually a $\mathcal{C}$-group* [1, Lemma 1.5].

Suppose that until the end of this section, P denotes the free product of groups A and B with subgroups $H \leqslant A$ and $K \leqslant B$ amalgamated according to an isomorphism $\varphi\colon H \to K$, Q denotes the HNN-extension of a group C with a stable letter t and subgroups $L \leqslant C$ and $M \leqslant C$ associated by an isomorphism $\psi\colon L \to M$. Recall that the presentation of an element $x \in P$ in the form $x = x_1x_2\ldots x_m$, where $m \geqslant 1$, $x_1, x_2, \ldots, x_m \in A \cup B$, is called *reduced* if, for $m > 1$, no neighboring factors x_i, x_{i+1} lie simultaneously in A or B. Recall also that the presentation of an element $y \in Q$ in the form $y = y_0t^{\delta_1}y_1\ldots t^{\delta_n}y_n$, where $n \geqslant 0$, $y_0, y_1, \ldots, y_n \in C$, $\delta_1, \ldots, \delta_n \in \{1, -1\}$, is called *reduced* if, for any $i \in \{1, \ldots, n-1\}$, it follows from the relations $-\delta_i = 1 = \delta_{i+1}$ that $y_i \notin L$ and from the relations $\delta_i = 1 = -\delta_{i+1}$ that $y_i \notin M$. The numbers m and n are called *the lengths* of these forms. The following two statements can be deduced from the normal form theorem for generalized free products [15, Corollary 4.4.1] and Britton's lemma for HNN-extensions [16, Ch. IV, § 2].

Proposition 6. *An arbitrary element $x \in P$ having a reduced form of length greater than* 1 *is not equal to* 1.

Proposition 7. *An arbitrary element $y \in Q$ having a reduced form of length greater than* 0 *is not equal to* 1.

The following proposition is a special case of Theorem 4 and will be used in its proof.

Proposition 8 [3, Theorem 4]**.** *Let $\mathcal{C}$ be a root class of groups, A and B be isomorphic groups, $\alpha\colon A \to B$ be an isomorphism and $\varphi = \alpha|_H$. Then P is residually a $\mathcal{C}$-group if and only if A and B are residually $\mathcal{C}$-groups, H is $\mathcal{C}$-separable in A, and K is $\mathcal{C}$-separable in B.*

The next two statements follow from theorems on the structure of subgroups of generalized free products and HNN-extensions (see, for example, [21, Theorems 3, 4]).

Proposition 9. *Every normal subgroup N of P that intersects trivially with H and K decomposes into the free product of some free group and groups isomorphic to $N \cap A$ or $N \cap B$.*

Proposition 10. *Every normal subgroup N of Q that intersects trivially with L and M decomposes into the free product of some free group and groups isomorphic to $N \cap C$.*

3. PROOF OF THEOREMS 1, 2

Throughout what follows, if Γ is a graph, $\mathcal{G}(\Gamma)$ is a corresponding graph of groups, and Δ is a subgraph of Γ, then by $\mathcal{G}(\Delta)$ we denote the graph of groups obtained by associating to the vertices and edges of Δ the same groups and mappings, as in $\mathcal{G}(\Gamma)$.

Let $\Gamma = (V, E)$ be an arbitrary graph and $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}\ (e \in E)\big)$ be a corresponding graph of groups. It is well known (see, for example, [14, § 5.2]) that, for each $v \in V$, the identity map of the generators of G_v to $\pi_1(\mathcal{G}(\Gamma))$ defines an injective homomorphism. Therefore, for each edge $e \in E$, the maps φ_{+e} and φ_{-e} can be considered as embeddings of H_e in $\pi_1(\mathcal{G}(\Gamma))$, as well as in $\pi_1(\mathcal{G}(\Delta))$, where Δ is a subgraph of Γ containing $e(1)$ and (or) $e(-1)$. This remark allows us to formulate

Proposition 11. *Let Γ be a tree and Γ_1, Γ_{-1} be the connected components of the graph obtained from Γ by removing some edge $e \in E$. Then the groups $\pi_1(\mathcal{G}(\Gamma_1))$ and $\pi_1(\mathcal{G}(\Gamma_{-1}))$ are embedded in $\pi_1(\mathcal{G}(\Gamma))$ by the identity mapping of the generators, and the group $\pi_1(\mathcal{G}(\Gamma))$ decomposes into the free product of the groups $\pi_1(\mathcal{G}(\Gamma_1))$ and $\pi_1(\mathcal{G}(\Gamma_{-1}))$ with the subgroups H_{+e} and H_{-e} amalgamated according to the isomorphism $\varphi_{+e}^{-1}\varphi_{-e}\colon H_{+e} \to H_{-e}$.*

Proof. Let P denote the free product of the groups $\pi_1(\mathcal{G}(\Gamma_1))$ and $\pi_1(\mathcal{G}(\Gamma_{-1}))$ with the subgroups H_{+e} and H_{-e} amalgamated according to the isomorphism $\varphi_{+e}^{-1}\varphi_{-e}\colon H_{+e} \to H_{-e}$. Then P and $\pi_1(\mathcal{G}(\Gamma))$ have the same sets of generators and, as is easy to see, any defining relation of P is derived from the defining relations of $\pi_1(\mathcal{G}(\Gamma))$. Therefore, the identity mapping of the generators of P to $\pi_1(\mathcal{G}(\Gamma))$ defines an isomorphism of the first on the second.

As noted earlier, $\pi_1(\mathcal{G}(\Gamma_1))$ and $\pi_1(\mathcal{G}(\Gamma_{-1}))$ are embedded in P by the identity mapping of the generators. In view of the above, these embeddings can be continued to homomorphisms into $\pi_1(\mathcal{G}(\Gamma))$. □

Similarly, we can prove

Proposition 12. *Let Γ be an arbitrary graph, E' be a set of edges that do not belong to a fixed maximal forest F of Γ and Δ be a graph obtained from Γ by removing all the edges of E'. Then the group $\pi_1(\mathcal{G}(\Delta))$ is embedded in $\pi_1(\mathcal{G}(\Gamma))$ by the identity mapping of the generators, and $\pi_1(\mathcal{G}(\Gamma))$ is the HNN-extension of $\pi_1(\mathcal{G}(\Delta))$ with the set of stable letters $\{t_e | e \in E'\}$ and the pairs of subgroups (H_{+e}, H_{-e}) associated by the isomorphisms $\varphi_{+e}^{-1}\varphi_{-e}\colon H_{+e} \to H_{-e}$ $(e \in E')$. In particular, $\pi_1(\mathcal{G}(\Gamma))$ is an HNN-extension of the tree product $\pi_1(\mathcal{G}(F))$.*

Proposition 13. *Let $\Gamma = (V, E)$ be a finite graph, $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}\ (e \in E)\big)$ be a corresponding graph of groups and N be a normal subgroup of $\pi_1(\mathcal{G}(\Gamma))$ intersecting trivially with each subgroup $H_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$. Then N is the free product of some free group and groups, each of which is isomorphic to a subgroup of the form $N \cap G_v$ $(v \in V)$.*

Proof. We suppose first that Γ is a forest and use induction on the number n of its edges. Without loss of generality, we can assume that Γ is a tree (if this is not so, we add the missing edges and associate to them unit groups and obvious homomorphisms). Therefore, if $n = 0$, then Γ contains only one vertex, and the statement of the proposition is trivial.

Let $n > 0$, e be some edge of Γ and Γ_1, Γ_{-1} be the connected components of the graph obtained from Γ by removing the edge e. Then, by Proposition 11, $\pi_1(\mathcal{G}(\Gamma))$ is a free product of the groups $\pi_1(\mathcal{G}(\Gamma_1))$ and $\pi_1(\mathcal{G}(\Gamma_{-1}))$ with the amalgamated subgroups H_{+e} and H_{-e}. By Proposition 9, it follows from the equalities $N \cap H_{+e} = 1 = N \cap H_{-e}$ that N is the free product of some free group and groups isomorphic to $N \cap \pi_1(\mathcal{G}(\Gamma_1))$ or $N \cap \pi_1(\mathcal{G}(\Gamma_{-1}))$. Since $\pi_1(\mathcal{G}(\Gamma_1))$ and $\pi_1(\mathcal{G}(\Gamma_{-1}))$ are embedded in $\pi_1(\mathcal{G}(\Gamma))$ by the identity mappings of the generators, the subgroups $N \cap \pi_1(\mathcal{G}(\Gamma_1))$ and $N \cap \pi_1(\mathcal{G}(\Gamma_{-1}))$ intersect trivially with all the edge subgroups of $\pi_1(\mathcal{G}(\Gamma_1))$ and $\pi_1(\mathcal{G}(\Gamma_{-1}))$ respectively. Therefore, we can apply the inductive hypothesis to the groups $\pi_1(\mathcal{G}(\Gamma_1))$, $\pi_1(\mathcal{G}(\Gamma_{-1}))$ and their subgroups $N \cap \pi_1(\mathcal{G}(\Gamma_1))$, $N \cap \pi_1(\mathcal{G}(\Gamma_{-1}))$, and the required result follows.

We now consider the general case and use induction on the number m of the edges that do not belong to some fixed maximal forest F of Γ. If $m = 0$, the statement has already been proved; therefore, we assume that $m > 0$.

Let e be an edge of Γ that does not belong to F and Δ be the graph obtained from Γ by removing the edge e. Then, by Proposition 12, $\pi_1(\mathcal{G}(\Gamma))$ is an HNN-extension of $\pi_1(\mathcal{G}(\Delta))$ with the associated subgroups H_{+e} and H_{-e}. By Proposition 10, N is the free product of some free group and groups isomorphic to $N \cap \pi_1(\mathcal{G}(\Delta))$. The subgroup $N \cap \pi_1(\mathcal{G}(\Delta))$ intersects trivially with all the edge subgroups of $\pi_1(\mathcal{G}(\Delta))$ and, by the inductive hypothesis, in turn decomposes into the free product of a free group and groups isomorphic to some subgroups of the form $N \cap G_v$ $(v \in V)$. Therefore, N has the required form. □

Proof of Theorem 1. Let σ be a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto a group of $\mathcal{C}$ acting injectively on all the subgroups $H_{\varepsilon e}$ $(e \in E,\ \varepsilon = \pm 1)$. Let also $\Gamma_i = (V_i, E_i)$ $(i \in \mathcal{I})$ be all the connected components of Γ. It is easy to see that $\pi_1(\mathcal{G}(\Gamma))$ is the free product of the groups $\pi_1(\mathcal{G}(\Gamma_i))$ $(i \in \mathcal{I})$. We put $N_i = \ker \sigma \cap \pi_1(\mathcal{G}(\Gamma_i))$ for every $i \in \mathcal{I}$. Then, for any $i \in \mathcal{I}$, N_i intersects trivially with all the edge subgroups of $\pi_1(\mathcal{G}(\Gamma_i))$.

Since Γ has a finite number of edges, all Γ_i $(i \in \mathcal{I})$ are finite. Therefore, by Proposition 13, for every $i \in \mathcal{I}$, N_i is the free product of some free group and groups, each of which is isomorphic to a subgroup of the form $N_i \cap G_v$ $(v \in V_i)$. The class $\mathcal{C}$ is closed under taking subgroups; therefore, for any vertex $v \in V_i$, the subgroup $N_i \cap G_v$ of the residually $\mathcal{C}$-group G_v is also residually a $\mathcal{C}$-group. As it is follows from Proposition 5, any free group is residually a $\mathcal{C}$-group. Therefore, for each $i \in \mathcal{I}$, $\pi_1(\mathcal{G}(\Gamma_i))$ is an extension of the free product N_i of residually $\mathcal{C}$-groups by the $\mathcal{C}$-group $\pi_1(\mathcal{G}(\Gamma_i))\sigma$ and hence is residually a $\mathcal{C}$-group by Proposition 5. The same proposition claims that $\pi_1(\mathcal{G}(\Gamma))$, the free product of $\pi_1(\mathcal{G}(\Gamma_i))$ $(i \in \mathcal{I})$, is also residually a $\mathcal{C}$-group. □

Proof of Theorem 2. Suppose first that all the subgroups $H_{\varepsilon e}$ $(e \in E,\ \varepsilon = \pm 1)$ are finite. We use the fact that $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group and, for each element s of the set $S = \bigcup_{e \in E,\, \varepsilon = \pm 1} H_{\varepsilon e} \setminus \{1\}$, find a subgroup $N_s \in \mathcal{C}^*(\pi_1(\mathcal{G}(\Gamma)))$, which does not contain s. Since Γ has a finite number of edges, S is also finite and, by Proposition 3, the subgroup $N = \bigcap_{s \in S} N_s$ belongs to the family $\mathcal{C}^*(\pi_1(\mathcal{G}(\Gamma)))$. Therefore, the natural homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto the quotient group $\pi_1(\mathcal{G}(\Gamma))/N$ is the desired one.

Now let all the subgroups $H_{\varepsilon e}$ $(e \in E,\ \varepsilon = \pm 1)$ be of finite Hirsch–Zaitsev rank. The class of all torsion-free $\mathcal{C}$-groups, as well as the class $\mathcal{C}$, is closed under taking subgroups and direct products of finitely many factors. Therefore, by Proposition 4, for any $e \in E$, $\varepsilon = \pm 1$, there exists a subgroup $N_{\varepsilon e} \in \mathcal{C}^*(\pi_1(\mathcal{G}(\Gamma)))$ that intersects trivially with $H_{\varepsilon e}$. As above, we use the finiteness of the number of edges of Γ and conclude that the desired mapping is the natural homomorphism of the group $\pi_1(\mathcal{G}(\Gamma))$ onto its quotient group by the subgroup $\bigcap_{e \in E,\, \varepsilon = \pm 1} N_{\varepsilon e}$, which belongs to $\mathcal{C}^*(\pi_1(\mathcal{G}(\Gamma)))$ by Proposition 3. □

4. PROOF OF THEOREM 4

Proposition 14. *Let $\mathcal{C}$ be a root class of groups, $\Gamma = (V, E)$ be a finite graph and $\mathcal{G}(\Gamma) = \big(\Gamma,\, G_v\ (v \in V),\, H_e,\, \varphi_{\pm e}\ (e \in E),\, \alpha_{v,w}\ (v, w \in V)\big)$ be a corresponding graph of isomorphic groups. If all G_v $(v \in V)$ belong to $\mathcal{C}$, then $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group.*

Proof. Let us fix some maximal forest F of Γ and a vertex $v \in V$. We define a map σ_0 of the generators of $\pi_1(\mathcal{G}(\Gamma))$ to G_v as follows: if x is a generator of G_w, then $x\sigma_0 = x\alpha_{w,v}$; if $e \in E$ is an edge that does not belong to F, then $t_e\sigma_0 = 1$. Denote by σ the continuation of σ_0 to the mapping of words.

Let $e \in E$ and $h \in H_e$. Since $\varphi_{+e}\alpha_{e(1),e(-1)} = \varphi_{-e}$ and $\alpha_{e(1),e(-1)}\alpha_{e(-1),v} = \alpha_{e(1),v}$, then

$$h\varphi_{+e}\sigma = h\varphi_{+e}\alpha_{e(1),v} = h\varphi_{+e}\alpha_{e(1),e(-1)}\alpha_{e(-1),v} = h\varphi_{-e}\alpha_{e(-1),v} = h\varphi_{-e}\sigma.$$

Therefore, σ maps all the defining relations of $\pi_1(\mathcal{G}(\Gamma))$ into the equalities valid in G_v and hence determines a homomorphism of the first to the second. Being a continuation of the isomorphisms $\alpha_{w,v}$ $(w \in V)$, σ acts injectively on all the vertex groups. Thus, $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group by virtue of Theorem 1. □

Let $\Gamma = (V, E)$ be an arbitrary graph and $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}\ (e \in E)\big)$ be a corresponding graph of groups. Suppose also that in each group G_v $(v \in V)$, we fix a normal subgroup R_v and, for each edge $e \in E$, there exists a normal subgroup S_e of H_e such that $S_e\varphi_{\varepsilon e} = R_{e(\varepsilon)} \cap H_{\varepsilon e}$ $(\varepsilon = \pm 1)$. Then the set $\mathcal{R} = \{R_v | v \in V\}$ will be called *the system of compatible normal subgroups* of the groups G_v $(v \in V)$.

It is easy to verify that if $\mathcal{R} = \{R_v | v \in V\}$ is a system of compatible normal subgroups, then for any $e \in E$, $\varepsilon = \pm 1$ the map $\overline{\varphi}_{\varepsilon e}\colon H_e/S_e \to G_{e(\varepsilon)}/R_{e(\varepsilon)}$ taking the coset hS_e $(h \in H_e)$ to the coset $(h\varphi_{\varepsilon e})R_{e(\varepsilon)}$ is well defined and is an injective homomorphism. Therefore, we can consider the graph of groups $\mathcal{G}_{\mathcal{R}}(\Gamma)$, in which the vertices are associated with the groups G_v/R_v and the edges are associated with the groups H_e/S_e and the homomorphisms $\overline{\varphi}_{\varepsilon e}$ defined above.

It is easy to see that if some maximal forest of Γ is fixed, then the presentation of $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ can be obtained from the presentation of $\pi_1(\mathcal{G}(\Gamma))$ by adding to the last, for every $v \in V$, all possible relations of the form $r = 1$, where r is a word in the generators of G_v defining an element of R_v. Therefore, $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ is a quotient group of $\pi_1(\mathcal{G}(\Gamma))$ by the normal closure of the set of elements $\bigcup_{v \in V} R_v$. The natural homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ we denote by $\rho_{\mathcal{R}}$.

Now let $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}\ (e \in E),\ \alpha_{v,w}\ (v, w \in V)\big)$ be a graph of isomorphic groups and $\mathcal{R} = \{R_v | v \in V\}$ be a system of normal subgroups of G_v $(v \in V)$ such that, for any $v, w \in V$, the equality $R_w = R_v\alpha_{v,w}$ holds. Then it follows easily from the relations $\alpha_{e(\varepsilon),e(-\varepsilon)}|_{H_{\varepsilon e}} = \varphi_{\varepsilon e}^{-1}\varphi_{-\varepsilon e}$, valid for all $e \in E$, $\varepsilon = \pm 1$, that $\mathcal{R}$ is a system of compatible normal subgroups. Below we call such sets of subgroups *the systems of isomorphic compatible normal subgroups*.

Obviously, if $\mathcal{R}_1 = \{R_{v,1} | v \in V\}$ and $\mathcal{R}_2 = \{R_{v,2} | v \in V\}$ are two systems of isomorphic compatible normal subgroups, then the set $\mathcal{R} = \{R_{v,1} \cap R_{v,2} | v \in V\}$ is also a system of isomorphic compatible normal subgroups. We also note that if $\mathcal{R}_1$ and $\mathcal{R}_2$ are systems of isomorphic compatible co-$\mathcal{C}$-subgroups, i. e., for each $v \in V$, $R_{v,1}$, $R_{v,2} \in \mathcal{C}^*(G_v)$, then by virtue of Proposition 3, $R_{v,1} \cap R_{v,2} \in \mathcal{C}^*(G_v)$ for all $v \in V$, and therefore the intersection $\mathcal{R}$ of $\mathcal{R}_1$ and $\mathcal{R}_2$ also turns out to be a system of isomorphic compatible co-$\mathcal{C}$-subgroups.

Proposition 15. *Let $\mathcal{C}$ be a class of groups closed under taking subgroups and direct products of a finite number of factors, $\Gamma = (V, E)$ be a finite graph, $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}$ $(e \in E),\ \alpha_{v,w}\ (v, w \in V)\big)$ be a corresponding graph of isomorphic groups and, for any $e \in E$, $\varepsilon = \pm 1$, $H_{\varepsilon e}$ be $\mathcal{C}$-separable in $G_{e(\varepsilon)}$. Then the following statements take place.*

1. For any vertex $u \in V$, any $\mathcal{C}$-separable subgroup L_u of G_u and any element $g \in \pi_1(\mathcal{G}(\Gamma)) \setminus L_u$, there exists a system $\mathcal{R} = \{R_v | v \in V\}$ of isomorphic compatible co-$\mathcal{C}$-subgroups of G_v $(v \in V)$ such that $g\rho_{\mathcal{R}} \notin L_u\rho_{\mathcal{R}}$.

2. If all G_v $(v \in V)$ are residually $\mathcal{C}$-groups, then $\pi_1(\mathcal{G}(\Gamma))$ is also residually a $\mathcal{C}$-group.

Proof. 1. We suppose first that Γ is a forest and use induction on the number n of its edges. As in the proof of Proposition 13, we can assume that Γ is a tree. Therefore, if $n = 0$, then Γ contains only one vertex u and the existence of the subgroup $R_u \in \mathcal{C}^*(G_u)$ satisfying the condition $gR_u \notin L_uR_u$ follows from the $\mathcal{C}$-separability of L_u in G_u.

Now let $n > 0$; $e \in E$ be some edge of Γ; $\Gamma_1 = (V_1, E_1)$, $\Gamma_{-1} = (V_{-1}, E_{-1})$ be the connected components of the graph obtained from Γ by removing the edge e and $e(1) \in V_1$, $e(-1) \in V_{-1}$. Then, by Proposition 11, $\pi_1(\mathcal{G}(\Gamma))$ is a free product of the groups $\pi_1(\mathcal{G}(\Gamma_1))$ and $\pi_1(\mathcal{G}(\Gamma_{-1}))$ with the amalgamated subgroups H_{+e} and H_{-e}. For definiteness, we assume that $u \in V_1$, and consider two cases.

Case 1. $g \in \pi_1(\mathcal{G}(\Gamma_1))$.

By the inductive hypothesis, there exists a system $\mathcal{R}_1 = \{R_v | v \in V_1\}$ of isomorphic compatible co-$\mathcal{C}$-subgroups of G_v $(v \in V_1)$ such that $g\rho_{\mathcal{R}_1} \notin L_u\rho_{\mathcal{R}_1}$. We use the isomorphisms $\alpha_{v,w}$ $(v, w \in V)$ to extend $\mathcal{R}_1$ to a system $\mathcal{R}$ of isomorphic compatible co-$\mathcal{C}$-subgroups of G_v $(v \in V)$.

The group $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ is a generalized free product of $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma_1))$ and $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma_{-1}))$. The homomorphism $\rho_{\mathcal{R}}$ maps the subgroup $\pi_1(\mathcal{G}(\Gamma_1))$ onto $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma_1)) = \pi_1(\mathcal{G}_{\mathcal{R}_1}(\Gamma_1))$ and acts on its elements as $\rho_{\mathcal{R}_1}$. Therefore, it follows from the relation $g\rho_{\mathcal{R}_1} \notin L_u\rho_{\mathcal{R}_1}$ that $g\rho_{\mathcal{R}} \notin L_u\rho_{\mathcal{R}}$. Thus, $\mathcal{R}$ is the desired system.

Case 2. $g \notin \pi_1(\mathcal{G}(\Gamma_1))$.

Let $g = g_1g_2\ldots g_l$ be a reduced form of g considered as an element of the generalized free product of the groups $\pi_1(\mathcal{G}(\Gamma_1))$ and $\pi_1(\mathcal{G}(\Gamma_{-1}))$. Then, for any $i \in \{1, \ldots, l\}$, there exists $\varepsilon_i = \pm 1$ such that $g_i \in \pi_1(\mathcal{G}(\Gamma_{\varepsilon_i})) \setminus H_{\varepsilon_i e}$, and if $l = 1$, then $\varepsilon_1 = -1$.

For each $i \in \{1, \ldots, l\}$, we apply the inductive hypothesis to the group $\pi_1(\mathcal{G}(\Gamma_{\varepsilon_i}))$, the subgroup $H_{\varepsilon_i e}$ and the element g_i. As a result, we obtain a system $\mathcal{R}_i$ of isomorphic compatible co-$\mathcal{C}$-subgroups of G_v $(v \in V_{\varepsilon_i})$. As above, we extend each of $\mathcal{R}_i$ to a system $\overline{\mathcal{R}}_i$ of isomorphic compatible co-$\mathcal{C}$-subgroups of G_v $(v \in V)$ and denote by $\mathcal{R}$ the intersection of $\overline{\mathcal{R}}_i$ $(1 \leqslant i \leqslant l)$. Then $\mathcal{R} = \{R_v | v \in V\}$ is a system of isomorphic compatible co-$\mathcal{C}$-subgroups of G_v $(v \in V)$. Put $\mathcal{R}_1 = \{R_v \in \mathcal{R} | v \in V_1\}$ and $\mathcal{R}_{-1} = \{R_v \in \mathcal{R} | v \in V_{-1}\}$.

Let $i \in \{1, \ldots, l\}$. Since $\bigcup_{R \in \mathcal{R}_{\varepsilon_i}} R \subseteq \bigcup_{R \in \mathcal{R}_i} R$, then $\ker \rho_{\mathcal{R}_{\varepsilon_i}} \leqslant \ker \rho_{\mathcal{R}_i}$ and the relation $g_i\rho_{\mathcal{R}_i} \notin H_{\varepsilon_i e}\rho_{\mathcal{R}_i}$ implies $g_i\rho_{\mathcal{R}_{\varepsilon_i}} \notin H_{\varepsilon_i e}\rho_{\mathcal{R}_{\varepsilon_i}}$. As in case 1, we obtain from the latter that $g_i\rho_{\mathcal{R}} \notin H_{\varepsilon_i e}\rho_{\mathcal{R}}$. Therefore, $g\rho_{\mathcal{R}}$ has in $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ a reduced form of length l, and if $l = 1$, then $g\rho_{\mathcal{R}} \in \pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma_{-1})) \setminus H_{-e}\rho_{\mathcal{R}}$. It follows that $g\rho_{\mathcal{R}} \notin \pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma_1))$ and, in particular, $g\rho_{\mathcal{R}} \notin L_u\rho_{\mathcal{R}}$.

Thus, if Γ is a forest, the proposition is proved. We now consider the general situation and use induction on the number m of the edges that do not belong to some fixed maximal forest F of Γ, or, what is the same, on the number of the stable letters in the presentation of $\pi_1(\mathcal{G}(\Gamma))$ corresponding to F.

If $m = 0$, then Γ is a forest and the required statement has already been proved. Let $m > 0$, $e \in E$ be some edge that does not belong to F and $\Delta = (V, E_\Delta)$ be the graph obtained from Γ by removing the edge e. Then, by Proposition 11, $\pi_1(\mathcal{G}(\Gamma))$ is an HNN-extension of the group $\pi_1(\mathcal{G}(\Delta))$ with the stable letter t_e and the associated subgroups H_{+e} and H_{-e}. Again, consider two cases.

Case 1. $g \in \pi_1(\mathcal{G}(\Delta))$.

We apply the inductive hypothesis to the group $\pi_1(\mathcal{G}(\Delta))$ and find a system $\mathcal{S}$ of isomorphic compatible co-$\mathcal{C}$-subgroups of G_v $(v \in V)$ such that $g\rho_{\mathcal{S}} \notin L_u\rho_{\mathcal{S}}$ (where $\rho_{\mathcal{S}}$ is the natural homomorphism of $\pi_1(\mathcal{G}(\Delta))$ onto $\pi_1(\mathcal{G}_{\mathcal{S}}(\Delta))$). Obviously, the set $\mathcal{S}$ can be considered as a system $\mathcal{R}$ of isomorphic compatible co-$\mathcal{C}$-subgroups of the vertex groups of $\mathcal{G}(\Gamma)$, and $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ is an HNN-extension of $\pi_1(\mathcal{G}_{\mathcal{S}}(\Delta))$. The natural homomorphism $\rho_{\mathcal{R}}$ of $\pi_1(\mathcal{G}(\Gamma))$ onto $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ acts on the subgroup $\pi_1(\mathcal{G}(\Delta))$ in the same way as $\rho_{\mathcal{S}}$ and maps it onto the subgroup $\pi_1(\mathcal{G}_{\mathcal{R}}(\Delta))$. Therefore, it follows from the relation $g\rho_{\mathcal{S}} \notin L_u\rho_{\mathcal{S}}$ that $g\rho_{\mathcal{R}} \notin L_u\rho_{\mathcal{R}}$.

Case 2. $g \notin \pi_1(\mathcal{G}(\Delta))$.

Let $g = g_0t_e^{\delta_1}g_1\ldots t_e^{\delta_l}g_l$ be a reduced form of g in the HNN-extension $\pi_1(\mathcal{G}(\Gamma))$ of $\pi_1(\mathcal{G}(\Delta))$. Since $g \notin \pi_1(\mathcal{G}(\Delta))$, then $l \geqslant 1$.

If $i \in \{1, \ldots, l-1\}$ and there exists $\varepsilon_i = \pm 1$ such that $-\varepsilon_i\delta_i = 1 = \varepsilon_i\delta_{i+1}$, then $g_i \notin H_{\varepsilon_i e}$ and we can apply the inductive hypothesis to the group $\pi_1(\mathcal{G}(\Delta))$, the subgroup $H_{\varepsilon_i e}$ and the element g_i. As a result, we obtain a system $\mathcal{R}_i$ of isomorphic compatible co-$\mathcal{C}$-subgroups of G_v $(v \in V)$ such that $g_i\rho_{\mathcal{R}_i} \notin H_{\varepsilon_i e}\rho_{\mathcal{R}_i}$. Moreover, as in case 1, $\mathcal{R}_i$ and $\rho_{\mathcal{R}_i}$ can be considered as a system of subgroups of the vertex groups of $\mathcal{G}(\Gamma)$ and a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$. If $i \in \{0, l\}$ or the number ε_i with the above properties does not exist, we put $\mathcal{R}_i = \{G_v | v \in V\}$. Then the intersection $\mathcal{R} = \{R_v | v \in V\}$ of $\mathcal{R}_i$ $(0 \leqslant i \leqslant l)$ is also a system of isomorphic compatible co-$\mathcal{C}$-subgroups of G_v $(v \in V)$.

By Proposition 12, $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ is an HNN-extension of $\pi_1(\mathcal{G}_{\mathcal{R}}(\Delta))$. Since $\bigcup_{R\in\mathcal{R}} R \subseteq \bigcup_{R\in\mathcal{R}_i} R$ for any $i \in \{0, \ldots, l\}$, then $\ker \rho_{\mathcal{R}} \leqslant \ker \rho_{\mathcal{R}_i}$ and the relation $g_i\rho_{\mathcal{R}_i} \notin H_{\varepsilon_i e}\rho_{\mathcal{R}_i}$ implies $g_i\rho_{\mathcal{R}} \notin H_{\varepsilon_i e}\rho_{\mathcal{R}}$. Therefore, in the indicated HNN-extension, the product $(g_0\rho_{\mathcal{R}})t_e^{\delta_1}(g_1\rho_{\mathcal{R}}) \ldots t_e^{\delta_l}(g_l\rho_{\mathcal{R}})$ serves as a reduced form of the element $g\rho_{\mathcal{R}}$ of length $l \geqslant 1$, and hence $g\rho_{\mathcal{R}} \notin \pi_1(\mathcal{G}_{\mathcal{R}}(\Delta))$. This means, in particular, that $g\rho_{\mathcal{R}} \notin L_u\rho_{\mathcal{R}}$. Thus, $\mathcal{R}$ is the desired system.

2. Let $g \in \pi_1(\mathcal{G}(\Gamma)) \setminus \{1\}$ be an arbitrary element. Since all G_v $(v \in V)$ are residually $\mathcal{C}$-groups, then the identity subgroup of every group G_v $(v \in V)$ is $\mathcal{C}$-separable in this group. Choose some vertex $w \in V$ and apply statement 1 of the proposition to the group $\pi_1(\mathcal{G}(\Gamma))$, the identity subgroup of G_w and the element g. As a result, we find a system $\mathcal{R} = \{R_v | v \in V\}$ of isomorphic compatible co-$\mathcal{C}$-subgroups of G_v $(v \in V)$ such that $g\rho_{\mathcal{R}} \neq 1$.

It is easy to see that $\mathcal{G}_{\mathcal{R}}(\Gamma)$ is a graph of isomorphic $\mathcal{C}$-groups. Therefore, by Proposition 14, $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ is residually a $\mathcal{C}$-group and hence $\rho_{\mathcal{R}}$ can be continued to a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto a group of $\mathcal{C}$ mapping g to a nonidentity element. □

Proof of Theorem 4. Necessity. Since $\mathcal{C}$ is closed under taking subgroups, any subgroup of a residually $\mathcal{C}$-group is also residually a $\mathcal{C}$-group. Therefore, all G_v $(v \in V)$ are residually $\mathcal{C}$-groups. We show that, for any $e \in E$, $\varepsilon = \pm 1$, $H_{\varepsilon e}$ is $\mathcal{C}$-separable in $G_{e(\varepsilon)}$.

First, let the edge $e \in E$ be a loop, i. e., $e(1) = e(-1)$. Then $\varphi_{+e}^{-1}\varphi_{-e} = \alpha_{e(1),e(-1)}|_{H_{+e}} = \mathrm{id}_{G_{e(1)}}|_{H_{+e}}$ and hence $\varphi_{+e} = \varphi_{-e}$, $H_{+e} = H_{-e}$. In addition, e cannot belong to any maximal forest of Γ; therefore, every presentation of $\pi_1(\mathcal{G}(\Gamma))$ has a stable letter t_e corresponding to e.

Suppose that H_{+e} is not $\mathcal{C}$-separable in $G_{e(1)}$. Then there exists an element $x \in G_{e(1)} \setminus H_{+e}$ such that $x\theta \in H_{+e}\theta$ for every homomorphism θ of $G_{e(1)}$ onto a group of $\mathcal{C}$. Consider the element $g = t_e^{-1}xt_ex^{-1}$.

According to Proposition 12, $\pi_1(\mathcal{G}(\Gamma))$ is an HNN-extension of $\pi_1(\mathcal{G}(\Delta))$, where Δ is the graph obtained from Γ by removing the edge e. Since $x \in G_{e(1)} \setminus H_{+e}$, the element g in this HNN-extension has a reduced form of nonzero length and therefore differs from 1. However, if σ is an arbitrary homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto a group of $\mathcal{C}$, then its restriction to $G_{e(1)}$ is also a homomorphism of the latter onto a group of $\mathcal{C}$. Therefore, $x \equiv h\varphi_{+e} \pmod{\ker\sigma}$ for some $h \in H_e$, and

$$g \equiv t_e^{-1}(h\varphi_{+e})t_e(h\varphi_{+e})^{-1} = (h\varphi_{-e})(h\varphi_{+e})^{-1} = (h\varphi_{+e})(h\varphi_{+e})^{-1} = 1 \pmod{\ker\sigma}$$

what contradicts the fact that $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group.

Now suppose that the edge $e \in E$ is not a loop. Choose a maximal forest F of Γ so that it contains e. Then, according to Theorem 1 from [22], the free product P of the groups $G_{e(1)}$ and $G_{e(-1)}$ with the subgroups H_{+e} and H_{-e} amalgamated according to the isomorphism $\varphi_{+e}^{-1}\varphi_{-e}$ turns out to be a subgroup of $\pi_1(\mathcal{G}(F))$, which, in turn, embeds in $\pi_1(\mathcal{G}(\Gamma))$. It follows that P is residually a $\mathcal{C}$-group, and therefore the $\mathcal{C}$-separability of H_{+e} in $G_{e(1)}$ and H_{-e} in $G_{e(-1)}$ follows from Proposition 8.

Sufficiency. We fix some maximal forest F of Γ and the corresponding presentation of $\pi_1(\mathcal{G}(\Gamma))$. The same considerations as in the proof of Proposition 13 allow us to assume that the graph Γ is connected and therefore its maximal forest F is a tree.

Let $g \in \pi_1(\mathcal{G}(\Gamma))$ be an arbitrary nonidentity element, ω be some word in the generators of $\pi_1(\mathcal{G}(\Gamma))$ defining it and S be a finite subset of vertices of Γ defined as follows:

a) if ω includes a generator of some group G_v, then $v \in S$;

b) if ω includes the stable letter t_e corresponding to some edge $e \in E$, then $e(1), e(-1) \in S$;

c) there are no other vertices in S.

Let $\Gamma_1 = (W, E_1)$ be the smallest subtree of F containing all the vertices from S, $\Gamma_2 = (W, E_2)$ be the subgraph of Γ obtained from Γ_1 by adding each edge $e \in E$ such that the stable letter t_e is contained in ω. Let also $\Gamma_3 = (W, E_3)$ be the graph obtained from Γ_2 by adding to it, for any, not necessarily different vertices $u, v \in W$, two new edges e_1 and e_2 such that $e_1(1) = u = e_2(-1)$ and $e_1(-1) = v = e_2(1)$.

It is clear that the graph Γ_3 is finite. We turn it into a graph of groups $\mathfrak{G}(\Gamma_3)$ by associating to the vertices of W and the edges of E_2 the same groups and homomorphisms as in the graph Γ, and to each edge $e \in E_3 \setminus E_2$ the group $G_{e(1)}$ and the maps $\varphi_{+e} = \mathrm{id}_{G_{e(1)}}$, $\varphi_{-e} = \alpha_{e(1),e(-1)}$. It is easy to see that $\mathfrak{G}(\Gamma_3)$ is a graph of isomorphic groups satisfying the conditions of Proposition 15, and therefore $\pi_1(\mathfrak{G}(\Gamma_3))$ is residually a $\mathcal{C}$-group.

We choose Γ_1 as a maximal tree of Γ_2 and Γ_3 and fix the corresponding presentations of $\pi_1(\mathfrak{G}(\Gamma_2))$ and $\pi_1(\mathfrak{G}(\Gamma_3))$. Then $\pi_1(\mathfrak{G}(\Gamma_2))$ has the same presentation as $\pi_1(\mathcal{G}(\Gamma_2))$, and $\pi_1(\mathfrak{G}(\Gamma_3))$, according to Proposition 12, is an HNN-extension of $\pi_1(\mathfrak{G}(\Gamma_2))$ with the set of stable letters $\{t_e | e \in E_3 \setminus E_2\}$.

Since $\pi_1(\mathfrak{G}(\Gamma_2))$ includes all the generators from the word ω, the latter defines some element $\mathfrak{g}$ of this group. Suppose $\mathfrak{g} = 1$. Then there is a sequence of insertions and deletions of the defining relations of $\pi_1(\mathfrak{G}(\Gamma_2))$ and words trivially equal to one that transforms ω to an empty word. Since all the defining relations of $\pi_1(\mathfrak{G}(\Gamma_2))$ are also present in $\pi_1(\mathcal{G}(\Gamma))$, it turns out that the equality $g = 1$ holds what contradicts the choice of g. Thus, $\mathfrak{g} \neq 1$.

Since $\pi_1(\mathfrak{G}(\Gamma_3))$ is an HNN-extension of $\pi_1(\mathfrak{G}(\Gamma_2))$, then ω also defines a nonidentity element of it. Therefore, to complete the proof of the theorem, it remains to construct a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ to $\pi_1(\mathfrak{G}(\Gamma_3))$ acting identically on the generators of $\pi_1(\mathcal{G}(\Gamma_2))$, of which, in particular, ω is composed. Since, as proved above, $\pi_1(\mathfrak{G}(\Gamma_3))$ is residually a $\mathcal{C}$-group, this homomorphism can be continued to a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto a group of $\mathcal{C}$ mapping g to a nonidentity element.

If $v \in V$, we call the length of the shortest path in the tree F connecting v with some vertex of W *the distance from the vertex v to the graph* Γ_1. It is well known that any two vertices of a tree are connected by a single simple path. This easily implies that if the distance from v to Γ_1 is equal to d, then there exists exactly one vertex $w \in W$ such that v and w are connected by a path of length d; we call it *the vertex of W nearest to v*. It follows also from the uniqueness of a simple path connecting two vertices that if e is an edge of F that does not belong to Γ_1, then one and the same vertex of W is nearest to $e(1)$ and $e(-1)$.

Suppose that a map $\delta\colon V \to W$ associates a vertex $v \in V$ with the nearest to it vertex of W. Define a map σ_0 of the generators of $\pi_1(\mathcal{G}(\Gamma))$ to $\pi_1(\mathfrak{G}(\Gamma_3))$ as follows. If x is a generator of some group G_v ($v \in V$), then $x\sigma_0 = x\alpha_{v,\delta(v)}$. Let e be an edge of Γ that does not belong to F. If $e \in E_2$, then $t_e\sigma_0 = t_e$. If $e \notin E_2$, then $t_e\sigma_0 = t_{\mathfrak{e}}$, where $\mathfrak{e}$ is an edge from the set $E_3 \setminus E_2$ for which $\mathfrak{e}(1) = \delta(e(1))$ and $\mathfrak{e}(-1) = \delta(e(-1))$. We continue σ_0 to the mapping of words σ and show that it maps all the defining relations of $\pi_1(\mathcal{G}(\Gamma))$ into the equalities valid in $\pi_1(\mathfrak{G}(\Gamma_3))$.

By definition, σ acts identically on the generators of $\pi_1(\mathcal{G}(\Gamma_2))$ and continues the isomorphisms $\alpha_{v,\delta(v)}$ for all $v \in V$. Therefore, it maps into the valid equalities all the defining relations of the groups G_v ($v \in V$) and all the defining relations of $\pi_1(\mathcal{G}(\Gamma))$ corresponding to the edges of E_2.

Let $e \in E \setminus E_2$. If e is an edge of F, then, as noted above, $\delta(e(1)) = \delta(e(-1))$. Since $\alpha_{e(1),e(-1)}\alpha_{e(-1),\delta(e(1))} = \alpha_{e(1),\delta(e(1))}$ and $\varphi_{+e}\alpha_{e(1),e(-1)} = \varphi_{-e}$, then, for each element $h \in H_e$,

$$\begin{aligned}(h\varphi_{+e})\sigma = (h\varphi_{+e})\alpha_{e(1),\delta(e(1))} &= (h\varphi_{+e})\alpha_{e(1),e(-1)}\alpha_{e(-1),\delta(e(1))}\\ = (h\varphi_{-e})\alpha_{e(-1),\delta(e(1))} &= (h\varphi_{-e})\alpha_{e(-1),\delta(e(-1))} = (h\varphi_{-e})\sigma.\end{aligned}$$

Let e do not belong to F and $\mathfrak{e}$ be the edge of $E_3 \setminus E_2$ such that $t_e\sigma = t_{\mathfrak{e}}$ and hence $\mathfrak{e}(1) = \delta(e(1))$, $\mathfrak{e}(-1) = \delta(e(-1))$. Then the equalities $t_{\mathfrak{e}}^{-1}xt_{\mathfrak{e}} = x\alpha_{\mathfrak{e}(1),\mathfrak{e}(-1)} = x\alpha_{\delta(e(1)),\delta(e(-1))}$ hold in $\pi_1(\mathfrak{G}(\Gamma_3))$ for every $x \in G_{\delta(e(1))}$. Since $\varphi_{+e}\alpha_{e(1),e(-1)} = \varphi_{-e}$ and $\alpha_{e(1),\delta(e(1))}\alpha_{\delta(e(1)),\delta(e(-1))} = \alpha_{e(1),\delta(e(-1))} = \alpha_{e(1),e(-1)}\alpha_{e(-1),\delta(e(-1))}$, we have

$$\begin{aligned}(t_e^{-1}h\varphi_{+e}t_e)\sigma = t_{\mathfrak{e}}^{-1}(h\varphi_{+e})\alpha_{e(1),\delta(e(1))}t_{\mathfrak{e}} &= (h\varphi_{+e})\alpha_{e(1),\delta(e(1))}\alpha_{\delta(e(1)),\delta(e(-1))}\\ = (h\varphi_{+e})\alpha_{e(1),e(-1)}\alpha_{e(-1),\delta(e(-1))} &= (h\varphi_{-e})\alpha_{e(-1),\delta(e(-1))} = (h\varphi_{-e})\sigma.\end{aligned}$$

Thus, σ defines a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ to $\pi_1(\mathfrak{G}(\Gamma_3))$ acting identically on the generators of $\pi_1(\mathcal{G}(\Gamma_2))$. □

5. PROOF OF THEOREM 3

Proposition 16. *Let $\mathcal{C}$ be a root class of groups containing a group G whose cardinality is not less than the cardinality of some (absolutely) free group F that does not belong to $\mathcal{C}$. Then there exist a group X and its subgroup Y such that:*

1) *X is residually a $\mathcal{C}$-group;*

2) *Y belongs to $\mathcal{C}$;*

3) *Y is $\mathcal{C}$-separable in X;*

4) *the kernel of any homomorphism of X onto a group of $\mathcal{C}$ intersects Y nontrivially.*

Proof. Suppose first that $\mathcal{C}$ consists only of periodic groups. Then, due to its closeness under taking subgroups and extensions, it contains all cyclic groups whose orders are the powers of some prime p.

Let, for each $i \geqslant 1$, C_{p^i} be a cyclic group of order p^i. Denote by X the direct product of C_{p^i} ($i \geqslant 1$) and by Y the product of the subgroups $(C_{p^i})^{p^{i-1}}$. Then X is residually a $\mathcal{C}$-group according to Proposition 5. The subgroup Y is the direct product of a countable number of groups of order p and, since G is obviously infinite, embeds in the Cartesian product $\prod_{g \in G} C_p(g)$, where $C_p(g)$ is an isomorphic copy of C_p for each $g \in G$. The indicated Cartesian product belongs to $\mathcal{C}$ by virtue of Proposition 1. Therefore, Y is also contained in $\mathcal{C}$.

Note that Y is normal in X and $X/Y \cong X$. Therefore, if $x \in X \setminus Y$ is an arbitrary element, then xY is a nonidentity element of the residually $\mathcal{C}$-group X/Y. It follows that the natural homomorphism $X \to X/Y$ can be continued to a homomorphism σ of X onto a group of $\mathcal{C}$ such that $x\sigma \neq 1 = Y\sigma$. Thus, Y is $\mathcal{C}$-separable in X.

Finally, suppose that there exists a homomorphism of X onto some group $Z \in \mathcal{C}$ injective on Y. Then it must be injective on each subgroup C_{p^i} ($i \geqslant 1$), and therefore Z contains elements of arbitrarily large order.

On the other hand, according to Proposition 1, the Cartesian product $P = \prod_{z \in Z} Z_z$, where Z_z is an isomorphic copy of Z for each $z \in Z$, also belongs to $\mathcal{C}$ and hence is a periodic group. This means that the element of P, the function $f\colon Z \to Z$ defined by the rule $f(z) = z$, has some finite order q. Obviously, by the definition of f, the period of Z coincides with q and, in particular, is finite, contrary to what was established above. Therefore, a homomorphism with the indicated properties does not exist.

Thus, in the case when $\mathcal{C}$ consists of periodic groups, the proposition is proved.

Now suppose that $\mathcal{C}$ contains at least one non-periodic group and X is the direct wreath product of the free group F with the $\mathcal{C}$-group G. Recall how this construction is defined.

Let D be the direct product of isomorphic copies of F indexed by the elements of G, i. e., the set of all functions from G to F having nonidentity values only at a finite number of points with pointwise multiplication. Then X is the extension of D by G, in which conjugation by an element $g \in G$ maps a function $d \in D$ to the function d^g defined as follows: $d^g(x) = d(gx)$, $x \in G$.

Since the cardinality of F does not exceed the cardinality of G, there exists an injective map β of the first to the second. For each element $g \in G$, we define a function $d_g \in D$ as follows: if there exists an element $f \in F$ such that $\beta(f) = g$, then

$$d_g(x) = \begin{cases} f, & \text{if } x = g, \\ 1, & \text{if } x \neq g; \end{cases}$$

otherwise $d_g = 1$. Denote by Y the subgroup of D generated by all the elements d_g ($g \in G$).

By Proposition 5, F, D, and X are residually $\mathcal{C}$-groups. It is easy to see that Y is the direct product of cyclic subgroups generated by the elements d_g ($g \in G$). Since D is torsion-free, Y is a free Abelian group and therefore embeds in the Cartesian product $P = \prod_{g \in G} Z(g)$, where $Z(g)$ is an infinite cyclic group for each $g \in G$. Since $\mathcal{C}$ contains a non-periodic group and is closed under taking subgroups, it also includes an infinite cyclic group. Therefore, by Proposition 1, the group P together with its subgroup Y belong to $\mathcal{C}$.

Let us show that Y is $\mathcal{C}$-separable in X. To do this, we fix an arbitrary element $x \in X \setminus Y$ and indicate a homomorphism σ of X onto a group of $\mathcal{C}$ such that $x\sigma \notin Y\sigma$.

If $x \notin D$, the desired one is the natural homomorphism of X onto the quotient group X/D, isomorphic to the $\mathcal{C}$-group G. Let $x \in D$. Since $x \notin Y$, then there exists an element $h \in G$ such that $x(h)$ does not belong to the cyclic subgroup $\langle d_h(h)\rangle \leqslant F$ generated by $d_h(h)$.

The class of all torsion-free $\mathcal{C}$-groups is nontrivial (it contains, for example, an infinite cyclic group) and hence is root as the intersection of $\mathcal{C}$ and the root class of all torsion-free groups. Therefore, according to Proposition 5, F is residually a torsion-free $\mathcal{C}$-group, and, by Proposition 4, the subgroup $\langle d_h(h)\rangle$ is $\mathcal{C}$-separable in F.

Let $N \in \mathcal{C}^*(F)$ be such a subgroup that $x(h) \notin \langle d_h(h)\rangle N$ and $\overline{X}$ be the direct wreath product of $\overline{F} = F/N$ with G. It is easy to show that the function that maps the product gd $(g \in G,\ d \in D)$ to the product $g\overline{d}$, where $\overline{d}$ is the function from G to $\overline{F}$ given by the rule $\overline{d}(x) = d(x)N$, defines a homomorphism σ of X onto $\overline{X}$. The image of Y under this homomorphism is still the direct product of the subgroups generated by the elements $\overline{d}_g$ $(g \in G)$. Therefore, it follows from the relations $\overline{x}(h) = x(h)N \notin \langle d_h(h)\rangle N/N = \langle \overline{d}_h(h)\rangle$ that $x\sigma \notin Y\sigma$.

It remains to note that, by Proposition 1, the Cartesian wreath product of the $\mathcal{C}$-group $\overline{F}$ with the $\mathcal{C}$-group G is, in turn, a $\mathcal{C}$-group. Since the direct wreath product is a subgroup of the Cartesian one, it follows that $\overline{X} \in \mathcal{C}$. Thus, Y is $\mathcal{C}$-separable in X.

Suppose now that some homomorphism σ of X onto a group of $\mathcal{C}$ acts injectively on Y. Then $d_g \notin \ker\sigma$ for any $g \in G$ such that $d_g \neq 1$.

For each $f \in F$, we denote by $\dot{f}$ the element of D defined as follows:

$$\dot{f}(x) = \begin{cases} f, & \text{if } x = 1, \\ 1, & \text{if } x \neq 1. \end{cases}$$

It is easy to see that the set of functions $\dot{F} = \{\dot{f} \mid f \in F\}$ is a subgroup and the map $f \to \dot{f}$ defines an isomorphism of F onto $\dot{F}$.

Let $f \in F \setminus \{1\}$ and $g = \beta(f)$. Then $d_g = g\dot{f}g^{-1}$, and hence $\dot{f} \notin \ker\sigma$. Therefore, $\dot{F}$ embeds in the $\mathcal{C}$-group $X\sigma$ and hence is contained in $\mathcal{C}$. But this is impossible, since, by the condition of the proposition, $F \notin \mathcal{C}$.

Thus, the kernel of any homomorphism of X onto a group of $\mathcal{C}$ intersects Y nontrivially. □

The following statement is a strengthened form of Theorem 3.

Proposition 17. *Let $\mathcal{C}$ be a root class of groups containing a group G whose cardinality is not less than the cardinality of some (absolutely) free group F that does not belong to $\mathcal{C}$. Then, for any graph $\Gamma = (V, E)$, there exists a corresponding graph of groups $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in V),\ H_e,\ \varphi_{\pm e}\ (e \in E)\big)$ such that:*

1) *all G_v $(v \in V)$ are residually $\mathcal{C}$-groups;*

2) *all $H_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$ belong to $\mathcal{C}$;*

3) *$\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group;*

4) *for any homomorphism σ of $\pi_1(\mathcal{G}(\Gamma))$ onto a group of $\mathcal{C}$ and for any $e \in E$, $\varepsilon = \pm 1$, the relation $\ker\sigma \cap H_{\varepsilon e} \neq 1$ holds.*

Proof. This statement follows from Proposition 16 and Theorem 4. If X and Y are the group and the subgroup from Proposition 16, then we should take X, Y and the natural embeddings of Y in X as the groups G_v $(v \in V)$, H_e $(e \in E)$ and the maps $\varphi_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$ respectively. In this case, the statements 1, 2, and 4 follow from Proposition 16, and the statement 3 does from Theorem 4. □

Proof of Theorem 3. A free group of no more than countable rank is countable, and therefore its cardinality is not greater than the cardinality of any infinite group belonging to the class $\mathcal{C}$. Thus, the indicated class satisfies the condition of Proposition 17, from which the desired result follows. □

6. FINAL REMARKS

We continue to the discussion of Theorem 3 and note that the vertex groups of the graph of groups $\mathcal{G}(\Gamma)$ constructed during its proof are residually $\mathcal{C}$-groups, but do not belong to $\mathcal{C}$. The authors do not know whether this theorem will remain true if we change the statement 1 of it as follows: $G_v \in \mathcal{C}$ for all $v \in V$.

FUNDING

The second author was supported by the Russian Foundation for Basic Research (project no. 18-31-00187).

REFERENCES

1. K. W. Gruenberg, "Residual properties of infinite soluble groups," Proc. London Math. Soc. **7**, 29–62 (1957).
2. E. V. Sokolov, "A characterization of root classes of groups," Commun. Algebra **43**, 856–860 (2015).
3. D. N. Azarov and D. Tieudjo, "On the root-class residuality of a free product of groups with an amalgamated subgroup," Nauch. Tr. Ivanovsk. Univ., Mat. **5**, 6–10 (2002).
4. D. Tieudjo, "On root-class residuality of some free constructions," JP J. Algebra, Number Theory Appl. **18**, 125–143 (2010).
5. E. A. Tumanova, "Certain conditions of the root-class residuality of generalized free products with a normal amalgamated subgroup," Chebyshev. Sb. **14**, 134–141 (2013).
6. E. A. Tumanova, "On the root-class residuality of generalized free products," Model. Anal. Inform. Sist. **20**, 133–137 (2013).
7. E. A. Tumanova, "On the root-class residuality of HNN-extensions of groups," Model. Anal. Inform. Sist. **21** (4), 148–180 (2014).
8. E. A. Tumanova, "On the root-class residuality of generalized free products with a normal amalgamation," Russ. Math. (Iz. VUZ), No. 10, 27–44 (2015).
9. D. V. Gol'tsov, "Approximability of HNN-extensions with central associated subgroups by a root class of groups," Math. Notes **97**, 679–683 (2015).
10. E. V. Sokolov and E. A. Tumanova, "Sufficient conditions for the root-class residuality of certain generalized free products," Sib. Math. J. **57**, 135–144 (2016).
11. E. V. Sokolov and E. A. Tumanova, "Root class residuality of HNN-extensions with central cyclic associated subgroups," Math. Notes **102**, 556–568 (2017).
12. E. A. Tumanova, "The root class residuality of Baumslag–Solitar groups," Sib. Math. J. **58**, 546–552 (2017).
13. E. A. Tumanova, "The root class residuality of the tree product of groups with amalgamated retracts," Sib. Math. J. **60**, 699–708 (2019).
14. J.-P. Serre, *Trees* (Springer, Berlin, Heidelberg, New York, 1980).
15. W. Magnus, A. Karras, and D. Solitar, *Combinatorial Group Theory* (Dover, New York, 1976).
16. R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory* (Springer, Berlin, Heidelberg, New York, 1977).
17. G. Baumslag, "On the residual finiteness of generalized free products of nilpotent groups," Trans. Am. Math. Soc. **106**, 193–209 (1963).
18. M. R. Dixon, L. A. Kurdachenko, and I. Ya. Subbotin, "On various rank conditions in infinite groups," Algebra Discrete Math., No. 4, 23–43 (2007).
19. A. I. Mal'cev, "On homomorphisms onto finite groups," Uch. Zap. Ivanov. Ped. Inst. **18**, 49–60 (1958).
20. E. V. Sokolov, "A remark on subgroup separability in the class of finite π-groups," Math. Notes **73**, 855–858 (2003).
21. D. E. Cohen, "Subgroups of HNN groups," J. Aust. Math. Soc. **17**, 394–405 (1974).
22. A. Karras and D. Solitar, "The subgroups of a free product of two groups with an amalgamated subgroups," Trans. Am. Math. Soc. **150**, 227–255 (1970).