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# The Cyclic Subgroup Separability of Certain Generalized Free Products of Two Groups

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**Abstract.** Free products of two residually finite groups with amalgamated retracts are considered. It is proved that a cyclic subgroup of such a group is not finitely separable if, and only if, it is conjugated with a subgroup of a free factor which is not finitely separable in this factor. A similar result is obtained for the case of separability in the class of finite $p$-groups.

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## 1 Statement of Results

The object of this paper is to study the cyclic subgroup separability of free products of two groups with amalgamated subgroups, which are retracts of free factors. We recall that a subgroup $H$ of a group $G$ is a *retract* of this group if there exists a subgroup $F$, normal in $G$, such that $G = HF$ and $H \cap F = 1$. In other words, a subgroup $H$ is a retract of a group $G$ if this group is a splitting extension of a group $F$ by $H$.

Now let us recall that a subgroup $H$ is said to be *finitely separable* in a group $G$ if for any element $g \in G \backslash H$ there exists a homomorphism $\varphi$ of $G$ onto a finite group such that $g\varphi \notin H\varphi$. A group $G$ is called *residually finite* if its trivial subgroup is finitely separable. Fixing a prime number $p$ and considering in the definitions above only homomorphisms onto finite $p$-groups instead of homomorphisms onto arbitrary finite groups, we obtain the notions of the $p$-separability and the residual $p$-finiteness.

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At last, we shall call a subgroup $H$ *$p'$-isolated* in a group $G$ if for each element $g \in G$ and for each prime number $q \neq p$, $g \in H$ whenever $g^q \in H$. It is easy to see that the property "to be $p'$-isolated" is necessary for the $p$-separability. Therefore, we may consider only $p'$-isolated subgroups while studying $p$-separable cyclic subgroups.

If $F$ is a group, then we shall denote by $\Delta(F)$ the family of all cyclic subgroups of $F$ which are not finitely separable in $F$, and by $\Delta_p(F)$ the family of all $p'$-isolated cyclic subgroups of $F$ which are not $p$-separable in this group. The main result of this paper is the following.

**Theorem 1.1.** *Let $G = \langle A * B;\, H = K,\, \varphi\rangle$ be the free product of groups $A$ and $B$ with subgroups $H$ and $K$ amalgamated according to an isomorphism $\varphi$. Also let $H$ be a retract of $A$ and $K$ be a retract of $B$.*

(1) *If $A$ and $B$ are residually finite, then a cyclic subgroup of $G$ is finitely separable in this group if, and only if, it is not conjugated with any subgroup of the family $\Delta(A) \cup \Delta(B)$.*

(2) *If $A$ and $B$ are residually $p$-finite, then a $p'$-isolated cyclic subgroup of $G$ is $p$-separable in this group if, and only if, it is not conjugated with any subgroup of the family $\Delta_p(A) \cup \Delta_p(B)$.*

The first part of this theorem generalizes the result of Allenby and Gregorac [1] who showed that a free product with amalgamated retracts of two $\pi_c$-groups (i.e., groups with all cyclic subgroups being finitely separable) is a $\pi_c$-group. A similar statement for an arbitrary number of free factors was proved by Kim [4].

We note that for any generalized free product $G = \langle A * B;\, H = K,\, \varphi\rangle$, if a cyclic subgroup is conjugated with a subgroup of the family $\Delta(A) \cup \Delta(B)$ (resp., the family $\Delta_p(A) \cup \Delta_p(B)$), it is certainly not finitely separable (resp., $p$-separable) in $G$. Thus, the theorem formulated states in fact the maximality of the families of finitely separable and $p$-separable cyclic subgroups of $G$.

**Corollary 1.2.** *The group $G$ satisfying the condition of the main theorem is residually finite (resp., residually $p$-finite) if, and only if, the groups $A$ and $B$ are residually finite (resp., residually $p$-finite).*

*Proof.* The residual finiteness of $A$ and $B$, being equivalent to the finite separability of the trivial subgroup $E$ in this group, means that this subgroup does not belong to the family $\Delta(A) \cup \Delta(B)$, and since it is invariant in $G$, it is not conjugated with any subgroup of this family. Therefore, by Theorem 1.1(1), $E$ is finitely separable in $G$, i.e., $G$ is residually finite.

Similarly, the $p$-separability of $A$ and $B$ implies that $E$ does not belong to the family $\Delta_p(A) \cup \Delta_p(B)$. In addition, this subgroup is $p'$-isolated in the free factors, which means that they have no element of prime order not equal to $p$. By the torsion theorem for generalized free products (see, e.g., [5, Section IV, Theorem 1.6]) $G$ does not contain such elements too. Hence, $E$ is $p'$-isolated in $G$ and is $p$-separable in this group according to Theorem 1.1(2).

Thus, the sufficiency is proved while the necessity is clear. $\square$

We note that the criterion of the residual finiteness formulated in this corollary was proved by Boler and Evans [3].

## 2 Proof of the Main Theorem

Let $G = \langle A * B;\ H = K,\ \varphi \rangle$ be the free product of groups $A$ and $B$ with subgroups $H$ and $K$ amalgamated according to an isomorphism $\varphi$. Following [2] and [6] we shall call normal subgroups of finite index $R \leqslant A$ and $S \leqslant B$

(a) $(H, K, \varphi)$-*compatible* if $(R \cap H)\varphi = S \cap K$;

(b) $(H, K, \varphi, p)$-*compatible*, where $p$ is a fixed prime number, if there exist sequences of subgroups

$$R = R_0 \leqslant \dots \leqslant R_m = A \quad \text{and} \quad S = S_0 \leqslant \dots \leqslant S_n = B$$

such that:

(1) $R_i$ and $S_j$ are normal subgroups of $A$ and $B$, respectively $(0 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant n)$;

(2) $|R_{i+1}/R_i| = |S_{j+1}/S_j| = p$ $(0 \leqslant i \leqslant m-1,\ 0 \leqslant j \leqslant n-1)$;

(3) $\varphi$ maps the set $\{R_i \cap H,\ 0 \leqslant i \leqslant m\}$ onto the set $\{S_j \cap K,\ 0 \leqslant j \leqslant n\}$.

Let $\Omega$ be the family of all pairs of $(H, K, \varphi)$-compatible subgroups and $\Omega_p$ be the family of all pairs of $(H, K, \varphi, p)$-compatible subgroups. We shall denote by $\Omega(A)$, $\Omega_p(A)$, $\Omega(B)$, $\Omega_p(B)$ the projections of these families onto $A$ and $B$.

Now we want to prove that if $H$ and $K$ are retracts of the free factors, then $\Omega(A)$ and $\Omega(B)$ coincide with the families $\Theta(A)$ and $\Theta(B)$ of all normal subgroups of finite index of $A$ and $B$, respectively, and $\Omega_p(A)$ and $\Omega_p(B)$ coincide with the families $\Theta_p(A)$ and $\Theta_p(B)$ of all normal subgroups of finite $p$-index of $A$ and $B$, respectively. For this purpose we need the following.

**Proposition 2.1.** *Let $Y$ be a retract of a group $X$ and $F$ be a normal subgroup of $X$ such that $X = YF$ and $Y \cap F = 1$. Let $N$ be a normal subgroup of $Y$. Then $NF$ is a normal subgroup of $X$, $NF \cap Y = N$ and $X/NF \cong Y/N$. In particular, $[X : NF] = [Y : N]$.*

*Proof.* Since $F$ is normal in $X$ and $N$ is normal in $Y$,

$$(NF)^X \subseteq N^X F = (N^Y)^F F \subseteq N^F F \subseteq NF.$$

The triviality of the intersection of $Y$ and $F$ implies the following:

$$X/NF = YF/NF \cong Y/N(Y \cap F) = Y/N.$$

At last, considering an element $x \in NF \cap Y$ and writing it in the form $x = yf$, where $y \in N$ and $f \in F$, we get $f = y^{-1}x \in Y$. But $Y \cap F = 1$, so $f = 1$ and $x = y \in N$. Thus, $NF \cap Y \subseteq N$, and since the inverse inclusion is clear, $NF \cap Y = N$, as claimed. □

Returning to the proof of the main theorem, we consider an arbitrary normal subgroup $R$ of finite index of $A$. The subgroup $P = R \cap H$ is normal in $H$, so $Q = P\varphi$ is a normal subgroup of finite index of $K$.

Since $K$ is a retract of $B$, there exists a normal subgroup $F \leqslant B$ satisfying the conditions $B = KF$ and $K \cap F = 1$. By Proposition 2.1 $S = QF$ is a normal subgroup of finite index of $B$ and $S \cap K = Q = (R \cap H)\varphi$. Thus, $(R, S) \in \Omega$, and so $R \in \Omega(A)$.

Now suppose that $R$ has $p$-index in $A$. It is well known that every finite $p$-group possesses a normal series with the factors of order $p$. Let

$$1 = R_0/R \leqslant \cdots \leqslant R_m/R = A/R$$

be such a series of the factor-group $A/R$. Then $R = R_0 \leqslant \cdots \leqslant R_m = A$ is a sequence of normal subgroups of $A$, and since

$$R_{i+1}/R_i \cong (R_{i+1}/R)/(R_i/R),$$

its factors have the order $p$ too.

Let us put $P_i = R_i \cap H$, $Q_i = P_i\varphi$ and $S_i = Q_iF$, where $0 \leqslant i \leqslant m$ and $F$ is the subgroup of $B$ defined earlier. Then $P_i$ and $Q_i$ are normal and have finite index in $H$ and $K$, respectively. Therefore, by Proposition 2.1 these $S_i$ are normal and have finite index in $B$. Moreover,

$$S_{i+1}/S_i = Q_{i+1}F/Q_iF \cong Q_{i+1}/Q_i(Q_{i+1} \cap F).$$

But $Q_{i+1} \leqslant K$ and $K \cap F = 1$, so $S_{i+1}/S_i \cong Q_{i+1}/Q_i$ and $|S_{i+1}/S_i| = |Q_{i+1}/Q_i|$.

Further, we note that

$$\begin{aligned} Q_{i+1}/Q_i &\cong P_{i+1}/P_i = (R_{i+1} \cap H)/(R_i \cap H) \\ &= (R_{i+1} \cap H)/(R_i \cap H)(R_{i+1} \cap H \cap R_i) \\ &\cong (R_{i+1} \cap H)R_i/(R_i \cap H)R_i = (R_{i+1} \cap H)R_i/R_i \leqslant R_{i+1}/R_i. \end{aligned}$$

So the order of the factor-group $Q_{i+1}/Q_i$, and hence, the order of the factor-group $S_{i+1}/S_i$, divides the order of the factor-group $R_{i+1}/R_i$, which is equal to $p$. Since $p$ is a prime number, it follows that either $S_{i+1} = S_i$ or $|S_{i+1}/S_i| = p$. If we remove repeating members from the sequence $S = S_0 \leqslant \cdots \leqslant S_n = B$, then the second equality will always take place. At last, by construction, $S_i \cap K = Q_i = (R_i \cap H)\varphi$. Hence, $R \in \Omega_p(A)$.

Thus, all normal subgroups of finite index of $A$ are contained in $\Omega(A)$ and all normal subgroups of finite $p$-index are contained in $\Omega_p(A)$. Since the inverse inclusions are clear, we get the required equalities $\Theta(A) = \Omega(A)$ and $\Theta_p(A) = \Omega_p(A)$. The arguments for the group $B$ are just the same.

We shall say further that a subgroup $Y$ of a group $X$ is *separable by a family* $\Psi$ *of normal subgroups of* $X$ if $\bigcap_{N \in \Psi} YN = Y$. Recall (see [2]) that $\Psi$ is a $Y$*-filtration* if the subgroups $Y$ and $\{1\}$ are separable by it.

Let us denote by $\Lambda(A)$ and $\Lambda(B)$ the families of all cyclic subgroups of $A$ and $B$ which are not separable by $\Omega(A)$ and $\Omega(B)$, respectively, and by $\Lambda_p(A)$ and $\Lambda_p(B)$ the families of all $p'$-isolated cyclic subgroups of $A$ and $B$ which are not separable by $\Omega_p(A)$ and $\Omega_p(B)$, respectively. The following general statements take place.

**Proposition 2.2.** [7, Theorem 1.2] *Let the family $\Omega(A)$ be an $H$-filtration and the family $\Omega(B)$ be a $K$-filtration. Then a cyclic subgroup of $G$ is finitely separable if it is not conjugate with any subgroup of the family $\Lambda(A) \cup \Lambda(B)$.*

**Proposition 2.3.** [7, Theorem 1.6] *Let the family $\Omega_p(A)$ be an $H$-filtration and the family $\Omega_p(B)$ be a $K$-filtration. Then a $p'$-isolated cyclic subgroup of $G$ is $p$-separable if it is not conjugate with any subgroup of the family $\Lambda_p(A) \cup \Lambda_p(B)$.*

As can be easily shown, the finite separability (resp., $p$-separability) of a subgroup in a group is equivalent to its separability by the family of all normal subgroups of finite index (resp., finite $p$-index) of this group. Since, in our case, $\Omega(A)$ and $\Omega(B)$ exactly coincide with the families $\Theta(A)$ and $\Theta(B)$ of all normal subgroups of finite index of $A$ and $B$, respectively, the condition of Proposition 2.2 turns out to be equivalent to simultaneous fulfillment of the following two statements:

(1) $A$ and $B$ are residually finite;

(2) $H$ and $K$ are finitely separable in the free factors.

In just the same way the condition of Proposition 2.3 is equivalent to simultaneous fulfillment of the following two statements:

($1_p$) $A$ and $B$ are residually $p$-finite;

($2_p$) $H$ and $K$ are $p$-separable in the free factors.

In addition, the equalities

$$\Theta(A) = \Omega(A), \quad \Theta(B) = \Omega(B), \quad \Theta_p(A) = \Omega_p(A), \quad \Theta_p(B) = \Omega_p(B)$$

imply

$$\Lambda(A) = \Delta(A), \quad \Lambda(B) = \Delta(B), \quad \Lambda_p(A) = \Delta_p(A), \quad \Lambda_p(B) = \Delta_p(B).$$

So we can reduce both statements of Theorem 1.1 to Propositions 2.2 and 2.3 if we prove that the conditions (2) and ($2_p$) follow from the conditions (1) and ($1_p$), respectively.

**Proposition 2.4.** *A retract of a residually finite group is finitely separable in this group. A retract of a residually $p$-finite group is $p$-separable in this group.*

*Proof.* We shall prove both statements simultaneously.

Let $Y$ be a retract of a residually finite (resp., residually $p$-finite) group $X$ and $F$ be a normal subgroup of $X$ such that $X = YF$ and $Y \cap F = 1$. Also let $x \in X \setminus Y$ be an arbitrary element. To prove that $Y$ is finitely separable (resp., $p$-separable) we only need to find a normal subgroup $L$ of $X$ having finite index (resp., finite $p$-index) in this group and such that $x \notin YL$.

Write $x$ in the form $x = yf$, where $y \in Y$ and $f \in F$. Since $x \notin Y$, $f \neq 1$. So, using the residual finiteness (resp., the residual $p$-finiteness) of $X$, we can find a normal subgroup $N$ of finite index (resp., finite $p$-index) of this group such that $f \notin N$. Let us put $U = N \cap F$, $V = N \cap Y$, $L = VU$ and $M = VF$. By Proposition 2.1 $M$ is normal in $X$, and since

$$G/M \cong Y/V = Y/(N \cap Y) \cong YN/N \leqslant G/N,$$

it has finite index (resp., finite $p$-index) in this group. $L$ possesses the same properties: it follows from the easily verifying equality

$$L = VU = V(F \cap N) = VF \cap N = M \cap N$$

and from the inclusion

$$M/L = M/(M \cap N) \cong MN/N \leqslant G/N.$$

Now suppose that $x \in YL$. Since $YL = YVU = YU$, we can write $x$ in the form $x = zu$, where $z \in Y$ and $u \in U$. Then $yf = zu$ and $z^{-1}y = uf^{-1}$. But $z^{-1}y \in Y$, $uf^{-1} \in F$ and $Y \cap F = 1$. Therefore, $f = u \in N$, what contradicts the choice of $N$.

Thus, $x \notin YL$ and $L$ is a required subgroup. This ends the proof of this proposition and the main theorem. $\square$

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