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On the cyclic subgroup separability of the free product of two groups with commuting subgroups

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Let G be the free product of groups A and B with commuting subgroups $H \leq A$ and $K \leq B$, and let $\mathcal{C}$ be the class of all finite groups or the class of all finite p-groups. We derive the description of all $\mathcal{C}$-separable cyclic subgroups of G provided this group is residually a $\mathcal{C}$-group. We prove, in particular, that if A, B are finitely generated nilpotent groups and H, K are p'-isolated in the free factors, then all p'-isolated cyclic subgroups of G are separable in the class of all finite p-groups. The same statement is true provided A, B are free and H, K are p'-isolated and cyclic.

Keywords: Free product of two groups with commuting subgroups; subgroup separability; residual p-finiteness.

Mathematics Subject Classification: 20E26, 20E06

0. Introduction

Let π be a set of prime numbers, and let $\mathcal{F}_\pi$ be the class of all finite π-groups (recall that a finite group is said to be a π-*group* if, and only if, all prime divisors of its order belong to π). It is the main aim of this paper to investigate the $\mathcal{F}_\pi$-separability of cyclic subgroups of free products of two groups with commuting subgroups.

Recall (see [9, Sec. 4.2]) that *the free product of two groups A and B with commuting subgroups $H \leq A$ and $K \leq B$*,

$$G = \langle A * B; [H, K] = 1\rangle,$$

is the quotient group of the ordinary free product $A * B$ by the mutual commutant $[H, K]$ of H and K. Obviously, G is intermediate between the free and direct products of A and B and is isomorphic to them if $H = K = 1$ or $H = A$, $K = B$, respectively.

The construction of free product with commuting subgroups was firstly introduced in [9] and arises in many different contexts. In particular, certain one-relator groups and graph products can be described in terms of this construction. At the same time, any free product with commuting subgroups itself is an iterated generalized free product of two groups (see the second section of the paper for more details), and this fact lets us apply to it the appropriate results and research methods.

Certain algorithmic problems are investigated in respect to this construction in [2, 4, 10]. Loginova [7, 8] researches its residual properties and obtains a condition which is necessary and sufficient for G to be residually an $\mathcal{F}_\pi$-group provided π either coincides with the set of all prime numbers or is an one-element set. Also, she proves that if all cyclic subgroups of A and B are separable in the class of all finite groups, then G possesses the same property. At last, Tieudjo and Moldavanskii [13–16] consider in detail the special case of the given construction when A and B are cyclic.

This paper generalizes the results of Loginova and Tieudjo mentioned above. We derive the description of all $\mathcal{F}_\pi$-separable cyclic subgroups of G provided π either coincides with the set of all prime numbers or is an one-element set and not necessarily all cyclic subgroups of the free factors are $\mathcal{F}_\pi$-separable. We use ordinary methods of combinatorial group theory. However, it should be poined out that the study of the property of $\mathcal{F}_\pi$-separability has certain features in the case when π does not coincide with the set of all prime numbers.

It is easy to see (see Proposition 1.3 below) that the π'-isolatedness of a subgroup is necessary for the $\mathcal{F}_\pi$-separability of this subgroup. If π coincides with the set of all prime numbers, then every subgroup turns out to be π'-isolated, and, therefore, the requirement of the π'-isolatedness of it does not mean anything. Otherwise the necessary condition formulated above becomes essential. As the elementary example we can take an infinite cyclic group which surely contains subgroups which are not π'-isolated. Thus, if we study the property of $\mathcal{F}_\pi$-separability, it makes sense to consider only π'-isolated subgroups and to look for conditions under which all of them are $\mathcal{F}_\pi$-separable.

This feature forces us to introduce amendments (sometimes not quite obvious) in some customary arguments. For example, to prove the ordinary finite separability of a cyclic subgroup C of a generalized free product P we usually find a homomorphism of P onto a generalized free product of two finite groups such that the image of C does not contain the image of a given element $x \in P$. In case of $\mathcal{F}_\pi$-separability we must at first sight construct a homomorphism ψ of P onto a generalized free product of two finite π-groups such that the image of C is π'-isolated and does not contain $x\psi$. However, it can be impossible to find a homomorphism satisfying the first claim (see [11], the remark at the end of Sec. 3). Therefore, we must look for a homomorphism ψ such that $x\psi$ does not belong to a π'-isolated cyclic subgroup containing $C\psi$.

Section 1 of the paper is devoted to the discussion of the relationship between the notions of subgroup separability, subgroup isolatedness and quasi-regularity.

Section 2 contains the proof of the main theorem. Some corollaries and applications of this result are given in Sec. 3 of the paper where the $\mathcal{F}_\pi$-separability of cyclic subgroups of direct products of two groups is also studied.

1. Separability, Isolatedness and Quasi-Regularity

Let X be a group, and let Ω be a family of normal subgroups of X. We shall say that a subgroup Y of X *is separated by* Ω if

$$\bigcap_{N \in \Omega} YN = Y.$$

For each subgroup Y, we call the subgroup

$$\bigcap_{N \in \Omega} YN$$

the Ω-*closure* of Y and denote it by $\Omega\text{-}Cl(Y)$. It is easy to see that the Ω-closure of Y is the least subgroup which contains Y and is separated by Ω.

Let further $\Omega_\pi(X)$ denote the family of all normal subgroups of finite π-index of X (i.e. the family of all such subgroups that the quotient groups by them belong to $\mathcal{F}_\pi$). If a subgroup Y is separated by $\Omega_\pi(X)$, it is called $\mathcal{F}_\pi$*-separable* in X. A group X is said to be *residually an* $\mathcal{F}_\pi$*-group* if its trivial subgroup is $\mathcal{F}_\pi$-separable. If π coincides with the set of all prime numbers and $\mathcal{F}_\pi$ is the class of all finite groups, respectively, we get the well-known notions of (*finite*) *subgroup separability* and *residual finiteness.*

We shall say also that a group X is $\mathcal{F}_\pi$*-quasi-regular* with respect to its subgroup Y if, for each subgroup $M \in \Omega_\pi(Y)$, there exists a subgroup $N \in \Omega_\pi(X)$ such that $N \cap Y \leq M$. The notion of quasi-regularity is closely tied to separability as the next statement shows.

Proposition 1.1. *Let X be a group, and let Y be its $\mathcal{F}_\pi$-separable subgroup. Then X is $\mathcal{F}_\pi$-quasi-regular with respect to Y if, and only if, all subgroups of $\Omega_\pi(Y)$ are $\mathcal{F}_\pi$-separable in X.*

Proof. *Necessity.* Let M be a subgroup of $\Omega_\pi(Y)$, and let $x \in X$ be an arbitrary element which does not belong to M. If $x \notin Y$, we can use the $\mathcal{F}_\pi$-separability of Y and find a subgroup $N \in \Omega_\pi(X)$ such that $x \notin YN$ and, hence, $x \notin MN$.

Let now $x \in Y$. Since X is $\mathcal{F}_\pi$-quasi-regular with respect to Y, there exists a subgroup $N \in \Omega_\pi(X)$ satisfying the condition $N \cap Y \leq M$. It is easy to see that in this case $MN \cap Y = M$ and so again $x \notin MN$.

Thus, M is $\mathcal{F}_\pi$-separable in X.

Sufficiency. Let again M be an arbitrary subgroup of $\Omega_\pi(Y)$, and let

$$1 = y_1, y_2, \ldots, y_n$$

be a set of representatives of all cosets of M in Y. Since M is $\mathcal{F}_\pi$-separable in X, we can find a subgroup $N \in \Omega_\pi(X)$ such that $y_2, \ldots, y_n \notin MN$.

If we suppose now that the intersection $N \cap Y$ is not contained in M, choose an element $g \in (N \cap Y) \backslash M$, and write it in the form $g = xy_i$ for a suitable $i \in \{2, \ldots, n\}$ and $x \in M$, then we get $xy_i \in N$. But it follows from here that $y_i \in MN$ that contradicts the choice of N. Thus, $N \cap Y \leq M$, and X is $\mathcal{F}_\pi$-quasi-regular with respect to Y. □

We call a subgroup Y of a group X *subnormal* in this group if there exists a sequence of subgroups

$$Y = Y_0 \leq Y_1 \leq \cdots \leq Y_n = X$$

every term of which is normal in the next.

Proposition 1.2. *If Y is a subnormal subgroup of finite π-index of a group X, then the $\mathcal{F}_\pi$-quasi-regularity of X with respect to Y holds.*

Proof. Let M be an arbitrary subgroup of $\Omega_\pi(Y)$. Since Y is a subnormal subgroup of finite π-index of X, then M possesses the same properties. Hence, there exists a series

$$M = M_0 \leq M_1 \leq \cdots \leq M_n = X$$

such that $M_i \in \Omega_\pi(M_{i+1})$, $i = 0, 1, \ldots, n-1$. Let us show that there is a subgroup $N \in \Omega_\pi(X)$ such that $N \leq M$ and, therefore, $N \cap Y \leq M$.

We shall use induction on n.

For $n = 1$, we have $M \in \Omega_\pi(X)$ and so can simply put $N = M$.

Let now $n > 1$ and

$$N_1 = \bigcap_{x \in M_2} M^x.$$

It is obvious that N_1 is normal in M_2. Consider a set of representatives of all right cosets of M in M_2: $\{x_1, x_2, \ldots, x_q\}$.

Since any element $x \in M_2$ can be written in the form $x = m_x x_i$ for a suitable $i \in \{1, \ldots, q\}$ and $m_x \in M$, then $M^x = M^{x_i}$ and

$$N_1 = \bigcap_{i=1}^{q} M^{x_i}.$$

Since M_1 is normal in M_2, it is invariant under the conjugation by any element x_i. From this it follows that all subgroups M^{x_i} belong to $\Omega_\pi(M_1)$. Hence the quotient group M_1/N_1 is isomorphic to a subgroup of the direct product of the finite π-groups M_1/M^{x_i} and so is itself a finite π-group.

Thus, N_1 belongs to $\Omega_\pi(M_2)$, and we can apply the induction hypothesis to it. In accordance with this hypothesis there exists a subgroup $N \in \Omega_\pi(X)$ such that $N \leq N_1$. Since $N_1 \leq M$, N satisfies the desired property, and the proposition is proved. □

As usual, we denote by π' the complement of a set π in the set of all prime numbers. Recall that a subgroup Y of a group X is said to be π'-*isolated* in X if, for each element $x \in X$ and for each number $q \in \pi'$, the inclusion $x^q \in Y$ implies $x \in Y$.

Proposition 1.3. *Every $\mathcal{F}_\pi$-separable subgroup is π'-isolated. In particular, if a group is residually an $\mathcal{F}_\pi$-group, then it is π'-torsion-free.*

Proof. Let us suppose that a subgroup Y of a group X is not π'-isolated and $x \in X$ is an element such that $x \notin Y$ but $x^q \in Y$ for some number $q \in \pi'$. Let also N be an arbitrary subgroup of $\Omega_\pi(X)$, and let n be the order of x modulo this subgroup.

Since n is a π-number, there exists a natural number m such that $qm \equiv 1 \pmod{n}$ and, hence, $x \equiv x^{qm} \pmod{N}$. Then $x \in YN$ and, because N has selected arbitrarily, Y is not $\mathcal{F}_\pi$-separable in X. □

Let us call the least π'-isolated subgroup of a group X containing a subgroup Y the *π'-isolator* of Y in X and denote it by $\mathcal{I}_{\pi'}(X, Y)$.

Proposition 1.4. *If X is residually an $\mathcal{F}_\pi$-group, then the π'-isolator of an arbitrary abelian subgroup Y of X is an abelian subgroup and coincides with the set of all π'-roots which can be extracted from the elements of Y in X.*

Proof. Since the $\Omega_\pi(X)$-closure of Y is $\mathcal{F}_\pi$-separable, then it is a π'-isolated subgroup of X containing Y. Therefore,

$$\mathcal{I}_{\pi'}(X, Y) \leq \Omega_\pi(X)\text{-}Cl(Y).$$

Suppose that the commutator of some elements $x_1, x_2 \in \mathcal{I}_{\pi'}(X, Y)$ is not equal to 1. Then, since X is residually an $\mathcal{F}_\pi$-group, there exists a subgroup $N \in \Omega_\pi(X)$ such that $[x_1, x_2] \notin N$. From the other hand, $x_1, x_2 \in \Omega_\pi(X)\text{-}Cl(Y)$ in view of the inclusion mentioned above, and, in particular, $x_1, x_2 \in YN$. So the elements x_1N, x_2N belong to the abelian subgroup YN/N of X/N and $[x_1N, x_2N] = 1$. It follows from here that $[x_1, x_2] \in N$, and we get a contradiction proving that $\mathcal{I}_{\pi'}(X, Y)$ is an abelian group.

Let now $u, v \in X$ be arbitrary elements such that $u^q, v^r \in Y$ for some π'-numbers q and r. Then $u, v \in \mathcal{I}_{\pi'}(X, Y)$ and $(uv)^{qr} = u^{qr}v^{qr} \in Y$. Therefore, the set of all π'-roots which can be extracted from the elements of Y in X is a subgroup and, hence, coincides with $\mathcal{I}_{\pi'}(X, Y)$. □

Proposition 1.5. *If X is residually an $\mathcal{F}_\pi$-group, then the π'-isolator of an arbitrary locally cyclic subgroup Y of X is a locally cyclic subgroup.*

Lemma 1.6. *If $x, y \in X$ and $y^q \in \langle x \rangle$ for some π'-number q, then the subgroup $\langle x, y \rangle$ is cyclic.*

Proof. First of all let us observe that $\mathcal{I}_{\pi'}(X, \langle x \rangle)$ is an abelian subgroup by the previous proposition and $y \in \mathcal{I}_{\pi'}(X, \langle x \rangle)$. Therefore, $[x, y] = 1$.

Let q be the least positive π'-number such that $y^q \in \langle x \rangle$. Let also $y^q = x^k$, $d = (k, q)$, $q = dq_1$ and $k = dk_1$. Then d is a π'-number and $(y^{-q_1}x^{k_1})^d = 1$. But X is residually an $\mathcal{F}_\pi$-group, so it is π'-torsion-free by Proposition 1.3. Therefore, $y^{-q_1}x^{k_1} = 1$ and $y^{q_1} \in \langle x \rangle$.

If $d > 1$, the last inclusion contradicts the choice of q. So $(k, q) = 1$. From this it follows that $ku + qv = 1$ for certain whole numbers u, v and

$$x = x^{ku+qv} = y^{qu}x^{qv} = (y^u x^v)^q, \quad y = y^{ku+qv} = y^{ku}x^{kv} = (y^u x^v)^k.$$

Thus $\langle x, y \rangle = \langle y^u x^v \rangle$. □

Proof of Proposition. Let u, v be arbitrary elements of $\mathcal{I}_{\pi'}(X, Y)$. By Proposition 1.4, $\mathcal{I}_{\pi'}(X, Y)$ coincides with the set of all π'-roots which can be extracted from the elements of Y in X. So $u^q = y_1 \in Y$, and $v^r = y_2 \in Y$ for some π'-numbers q and r.

Since Y is locally cyclic, the subgroup $\langle y_1, y_2 \rangle$ is cyclic and is generated by an element $y \in Y$. By lemma, the subgroup $\langle y, u \rangle$ is also cyclic and is generated by an element $w \in \mathcal{I}_{\pi'}(X, Y)$. If we apply now lemma to w and v, then we get that $\langle w, v \rangle = \langle t \rangle$ for some element $t \in \mathcal{I}_{\pi'}(X, Y)$. Thus, $u, v \in \langle t \rangle$, and the proposition is proved. □

2. The Main Theorem

To formulate the main result of this paper we need some auxiliary designations.

If X is a group, then we denote by $\Delta_\pi(X)$ the family of all π'-isolated cyclic subgroups of this group which are not $\mathcal{F}_\pi$-separable in it. By Proposition 1.3, the family of all $\mathcal{F}_\pi$-separable cyclic subgroups of X certainly does not contain all cyclic subgroups which are not π'-isolated in X. The equality $\Delta_\pi(X) = \varnothing$ means that all other cyclic subgroups of X belong to this family.

Let

$$G = \langle A * B; [H, K] = 1 \rangle$$

be the free product of groups A and B with commuting subgroups $H \leq A$ and $K \leq B$ (these notations will be fixed till the end of the paper). We put

$$\Theta_\pi(HK) = \{(X \cap H)(Y \cap K) \mid X \in \Omega_\pi(A), Y \in \Omega_\pi(B)\}$$

and denote by $\Lambda_\pi(HK)$ the family of all cyclic subgroups which lie and are π'-isolated in HK but are not separated by $\Theta_\pi(HK)$.

Theorem 2.1. *Let π be a set of prime numbers that coincides with the set of all prime numbers or is one-element, and let at least one of the following equivalent statements holds*:

(1) *A and B are residually $\mathcal{F}_\pi$-groups, H and K are $\mathcal{F}_\pi$-separable in the factors*;
(2) *G is residually an $\mathcal{F}_\pi$-group.*

Then a π'-isolated cyclic subgroup of G is $\mathcal{F}_\pi$-separable in this group if, and only if, it is not conjugated with any subgroup of the family

$$\Delta_\pi(A) \cup \Delta_\pi(B) \cup \Lambda_\pi(HK).$$

Proof. First of all we note that the equivalence of conditions 1 and 2 was stated in [7]. So, it is necessary to prove the criterion of the $\mathcal{F}_\pi$-separability of a cyclic subgroup of G only. We begin with the necessity of the condition.

If a cyclic subgroup C belongs to $\Delta_\pi(A)$, then it is not $\mathcal{F}_\pi$-separable in A and there exists an element $a \in A\backslash C$ such that $a \in CX$ for any subgroup $X \in \Omega_\pi(A)$. But then $a \in CL$ for every subgroup $L \in \Omega_\pi(G)$ since $L \cap A \in \Omega_\pi(A)$. Therefore, C is not $\mathcal{F}_\pi$-separable in G. It is obvious that any subgroup conjugated with C is also not $\mathcal{F}_\pi$-separable in G.

It can be proved in precisely the same way that any subgroup of $\Delta_\pi(B)$ or $\Lambda_\pi(HK)$ is not $\mathcal{F}_\pi$-separable in G. It needs only to note that, if $L \in \Omega_\pi(G)$, then

$$(L \cap H)(L \cap K) \in \Theta_\pi(HK).$$

Before proving the sufficiency we give some facts about the structure and the properties of free products with commuting subgroups.

Recall that *the free product of groups M and N with subgroups $U \leq M$ and $V \leq N$ amalgamated according to an isomorphism $\varphi : U \to V$,*

$$T = \langle M * N; U = V, \varphi\rangle,$$

is the quotient group of the ordinary free product $M * N$ of M and N by the normal closure of the set $\{u(u\varphi)^{-1} \mid u \in U\}$.

It is known that the subgroups of T generated by the generators of M and N are isomorphic to these groups and, therefore, may be identified with them. Herewith, U and V turn out to be coincident and this fact lets us write down T in the form

$$T = \langle M * N; U\rangle$$

considering φ as given.

We also need one more simply checked property of elements of generalized free products of two groups. Recall that *the length of an element* of T is the length of any reduced form of this element.

Proposition 2.2 ([12, Proposition 2.1.6]). *For every two elements $x, y \in T$, if one of them has an even length and $y = x^q$ for some positive number q, then the other element has an even length too and $l(y) = l(x)q$.*

It is easy to see [7] that the structure of G can be described in terms of the construction of free product with amalgamated subgroup as follows.

Let $U = HK$, $M = \langle A, U\rangle$ and $N = \langle B, U\rangle$. Then U is the direct product $H \times K$ of H and K, M is the free product $\langle A * U; H\rangle$ of A and U with H amalgamated, N is the free product $\langle B * U; K\rangle$ of B and U with K amalgamated, and G is the free product $\langle M * N; U\rangle$ of M and N with U amalgamated.

If Y is a subgroup of a group X, we put

$$\Omega_\pi(X, Y) = \{L \cap Y \mid L \in \Omega_\pi(X)\}$$

and denote by $\Delta_\pi(X, Y)$ the family of all cyclic subgroups which lie and are π'-isolated in Y but are not separated by $\Omega_\pi(X, Y)$. The following statements take place.

Proposition 2.3 ([11, Theorems 1.2 and 1.6]). *Let π be a set of prime numbers which coincides with the set of all prime numbers or is one-element, and let $T = \langle M * N; U\rangle$ be the free product of groups M and N with an amalgamated subgroup U. If $\{1\}$ and U are separated by both $\Omega_\pi(T, M)$ and $\Omega_\pi(T, N)$, then a π'-isolated cyclic subgroup of T is $\mathcal{F}_\pi$-separable in this group if, and only if, it is not conjugated with any subgroup of $\Delta_\pi(T, M) \cup \Delta_\pi(T, N)$.*

Proposition 2.4 ([7, Lemmas 1 and 3]). *Let π be a set of prime numbers which coincides with the set of all prime numbers or is one-element. If A, B are residually $\mathcal{F}_\pi$-groups and H, K are $\mathcal{F}_\pi$-separable in the free factors, then $\{1\}$ and U are separated by both $\Omega_\pi(G, M)$ and $\Omega_\pi(G, N)$.*

It also follows from the proof of [7, Lemmas 1 and 3] that the statement below holds.

Proposition 2.5. *Let π be a set of prime numbers which coincides with the set of all prime numbers or is one-element, $X \in \Omega_\pi(A)$, $Y \in \Omega_\pi(B)$, and*

$$Q = (X \cap H)(Y \cap K).$$

Then any subgroup $R \in \Omega_\pi(M)$ such that $R \cap A = X$ and $R \cap U = Q$ belongs to $\Omega_\pi(G, M)$. The same statement holds for the subgroups of $\Omega_\pi(N)$.

Now we can prove the sufficiency of the condition. Due to Propositions 2.3 and 2.4, we need to show only that every subgroup of $\Delta_\pi(G, M)$ is conjugated with some subgroup of $\Delta_\pi(A) \cup \Lambda_\pi(U)$ and every subgroup of $\Delta_\pi(G, N)$ does with some subgroup of $\Delta_\pi(B) \cup \Lambda_\pi(U)$. We consider M, the arguments for N are the same.

Let $X \in \Omega_\pi(A)$ and $Y \in \Omega_\pi(B)$ be arbitrary subgroups, and let $Q = (X \cap H) \cdot (Y \cap K)$. Then $X \cap H = Q \cap H$, and, hence, the function $\varphi_{X,Q} : HX/X \to HQ/Q$ which maps an element hX, $h \in H$, to hQ is a correctly defined isomorphism of subgroups. Therefore, we can construct the group

$$M_{X,Q} = \langle A/X * U/Q; HX/X = HQ/Q, \varphi_{X,Q}\rangle$$

and extend the natural homomorphisms of A onto A/X and of U onto U/Q to the homomorphism $\rho_{X,Q}$ of M onto $M_{X,Q}$.

Consider the map σ of generators of $M_{X,Q}$ defined by the rules

$$(aX)\sigma = aX \ \ (a \in A), \quad (hQ)\sigma = hX \ \ (h \in H), \quad (kQ)\sigma = 1 \ \ (k \in K)$$

and extend it to the map τ of words. It is easy to see that τ maps all defining relations of $M_{X,Q}$ to 1 and, hence, defines a surjective homomorphism of $M_{X,Q}$ onto A/X.

Let $Z = \ker \tau$. Since τ acts injectively on A/X, then Z meets HQ/Q trivially. Therefore, by the theorem of Karras and Solitar [5], Z is the free product of a free group and groups which are isomorphic to KQ/Q. Since $KQ/Q \in \mathcal{F}_\pi$, all factors of this free product are residually $\mathcal{F}_\pi$-groups. Thus, Z is also residually an $\mathcal{F}_\pi$-group [3, Theorem 4.1].

Since $M_{X,Q}/Z \cong A/X$ and $A/X \in \mathcal{F}_\pi$, then $M_{X,Q}$ is an extension of residually an $\mathcal{F}_\pi$-group by an $\mathcal{F}_\pi$-group. Therefore, $M_{X,Q}$ is residually an $\mathcal{F}_\pi$-group [3, Lemma 1.5], and all π'-isolated cyclic subgroups of $M_{X,Q}$ are $\mathcal{F}_\pi$-separable [1, Lemma 3], [11, Proposition 3.1].

Let us note that, if C is a cyclic subgroup of M, and we want to prove that C is separated by $\Omega_\pi(G, M)$, then it is sufficient to find, for every element $g \in M \backslash C$, a pair of subgroups $X \in \Omega_\pi(A)$, $Y \in \Omega_\pi(B)$ such that $g\rho_{X,Q}$ does not belong to some π'-isolated cyclic subgroup $D_{X,Q}$ of $M_{X,Q}$ containing $C\rho_{X,Q}$ (where, as early, $Q = (X \cap H)(Y \cap K)$).

Indeed, as it was just proved, all π'-isolated cyclic subgroups of $M_{X,Q}$ are $\mathcal{F}_\pi$-separable. So, if X and Y possess the properties mentioned above, then there exists a subgroup $R_{X,Q} \in \Omega_\pi(M_{X,Q})$ such that $g\rho_{X,Q} \notin D_{X,Q}R_{X,Q}$ and, hence, $g\rho_{X,Q} \notin C\rho_{X,Q}R_{X,Q}$.

Further, since $M_{X,Q}$ is residually an $\mathcal{F}_\pi$-group and the quotient groups A/X and U/Q are finite, we can choose $R_{X,Q}$ so that

$$R_{X,Q} \cap A/X = R_{X,Q} \cap U/Q = 1.$$

Then the pre-image R of $R_{X,Q}$ under $\rho_{X,Q}$ satisfies the equalities $R \cap A = X$ and $R \cap U = Q$. It follows now from Proposition 2.5 that $R \in \Omega_\pi(G, M)$ and, at the same time, $g \notin CR$ as required.

So let $C = \langle c \rangle$ be a π'-isolated cyclic subgroup of M which is not conjugated with any subgroup of $\Delta_\pi(A) \cup \Lambda_\pi(U)$. We show that C is separated by $\Omega_\pi(G, M)$.

Let $g \in M$ be an arbitrary element not belonging to C, and let

$$g = g_1 g_2 \cdots g_m, \quad c = c_1 c_2 \cdots c_n$$

be reduced forms of g and c which are considered as the elements of the generalized free product M. Applying, if necessary, a suitable inner automorphism of G defining by an element of M we can assume further that c is cyclically reduced.

Let, at first, $n = 1$, and let, for definiteness, $c \in A$ (the case when $c \in U$ is considered in the same way).

Note that, if $X \in \Omega_\pi(A)$, $Y \in \Omega_\pi(B)$ and $Q = (X \cap H)(Y \cap K)$, then the free factors of $M_{X,Q}$ are finite π-groups and, hence, all their cyclic subgroups are π'-isolated. So, it is sufficient to find a pair of subgroups $X \in \Omega_\pi(A)$, $Y \in \Omega_\pi(B)$ such that $g\rho_{X,Q} \notin C\rho_{X,Q}$.

By the condition, C is separated by $\Omega_\pi(A)$. Therefore, if $g \in A$, then there exists a subgroup $X \in \Omega_\pi(A)$ such that $g \notin CX$ and, hence, $g\rho_{X,Q} \notin C\rho_{X,Q}$ regardless of the choice of Y.

Let $g \notin A$. Then, if $m = 1$, $g = g_1 \in U \backslash H$. If $m > 1$, every syllable g_i of the reduced form of g belongs to one of the free factors and does not belong to the amalgamated subgroup.

By the condition of the theorem, H is separated by $\Omega_\pi(A)$.

If $u \in U \backslash H$ and $u = hk$ for suitable elements $h \in H$, $k \in K$, then $k \neq 1$ and, since B is residually an $\mathcal{F}_\pi$-group, there exists a subgroup $Y \in \Omega_\pi(B)$ such that $k \notin Y$. Then

$$u \notin H(X \cap H)(Y \cap K) = H(Y \cap K)$$

regardless of the choice of a subgroup $X \in \Omega_\pi(A)$. Thus, H is separated by $\Theta_\pi(U)$.

It follows from this remark that, for every i ($1 \leq i \leq m$), one can find a pair of subgroups $X_i \in \Omega_\pi(A)$, $Y_i \in \Omega_\pi(B)$ such that $g_i \notin HX_i$ if $g_i \in A$ and $g_i \notin H(Y_i \cap K)$ if $g_i \in U$. Let us put

$$X = \bigcap_{i=1}^{m} X_i, \quad Y = \bigcap_{i=1}^{m} Y_i.$$

It is obvious that $X \in \Omega_\pi(A)$, $Y \in \Omega_\pi(B)$ and the presentation of $g\rho_{X,Q}$ as the product

$$g\rho_{X,Q} = g_1\rho_{X,Q} g_2\rho_{X,Q} \cdots g_m\rho_{X,Q}$$

is still a reduced form in $M_{X,Q}$. Therefore, the length $l(g\rho_{X,Q})$ of $g\rho_{X,Q}$ equals the length of g, and, if $m = 1$, then $g\rho_{X,Q} \in U\rho_{X,Q} \backslash H\rho_{X,Q}$. Thus, in this case $g\rho_{X,Q}$ does not belong to $C\rho_{X,Q}$ too.

Let now $n \geqslant 2$. As above, we find a pair of subgroups $X \in \Omega_\pi(A)$, $Y \in \Omega_\pi(B)$ such that $l(g\rho_{X,Q}) = l(g)$ and $l(c\rho_{X,Q}) = l(c)$. Note that $c\rho_{X,Q}$ is still a cyclically reduced element.

For any pair of subgroups $V \in \Omega_\pi(A)$, $W \in \Omega_\pi(B)$ such that $V \leq X$ and $W \leq Y$, the equality $l(c\rho_{V,P}) = l(c)$ holds, where $P = (V \cap H)(W \cap K)$. So the powers of the roots which can be extracted from $c\rho_{V,P}$ are bounded as a whole by Proposition 2.2. It follows now from Proposition 1.5 that $\mathcal{I}_{\pi'}(M_{V,P}, C\rho_{V,P})$ is a cyclic subgroup. We show that V and W can be chosen in such a way that $g\rho_{V,P}$ does not belong to $\mathcal{I}_{\pi'}(M_{V,P}, C\rho_{V,P})$.

Write n in the form $n = qt$, where q is a π-number, t is a π'-number if π does not coincide with the set of all prime numbers and $t = 1$ otherwise.

Case 1. n does not divide mt.

Since n does not divide mt, then, by Proposition 2.2, $(g\rho_{X,Q})^t \notin C\rho_{X,Q}$. Show that $g\rho_{X,Q} \notin \mathcal{I}_{\pi'}(M_{X,Q}, C\rho_{X,Q})$.

Let $d_{X,Q}$ denote a generator of $\mathcal{I}_{\pi'}(M_{X,Q}, C\rho_{X,Q})$, and let $(d_{X,Q})^z = c\rho_{X,Q}$. It follows from Proposition 2.2 that $z|n$. But z is a π'-number, so it divides t and, therefore,

$$(\mathcal{I}_{\pi'}(M_{X,Q}, C\rho_{X,Q}))^t \leq C\rho_{X,Q}.$$

Thus, supposing that $g\rho_{X,Q} \in \mathcal{I}_{\pi'}(M_{X,Q}, C\rho_{X,Q})$ we get the inclusion $(g\rho_{X,Q})^t \in C\rho_{X,Q}$, which contradicts the relation stated above.

Case 2. $mt = nk$ for certain positive k.

Since C is π'-isolated in M and $g \notin C$, then $g^t \neq c^{\pm k}$. Arguing as above we find subgroups $R \in \Omega_\pi(A)$, $S \in \Omega_\pi(B)$ such that

$$(g^{-t}c^k)\rho_{R,L} \neq 1 \neq (g^{-t}c^{-k})\rho_{R,L},$$

where $L = (R \cap H)(S \cap K)$.

Let us put $V = X \cap R$, $W = Y \cap S$ and $P = (V \cap H)(W \cap K)$. Then $(g\rho_{V,P})^t \neq (c\rho_{V,P})^{\pm k}$, and, since

$$l(g\rho_{V,P}) = l(g) = m, \quad l(c\rho_{V,P}) = l(c) = n,$$

we have $(g\rho_{V,P})^t \notin C\rho_{V,P}$. As well as in the case discussed above it follows that $g\rho_{V,P} \notin \mathcal{I}_{\pi'}(M_{V,P}, C\rho_{V,P})$, and the proof is finished. □

3. Some Corollaries

Unfortunately, to describe the family $\Lambda_\pi(HK)$ is difficult in general case, and below we try to make it at least in some special cases.

Proposition 3.1. *If A is $\mathcal{F}_\pi$-quasi-regular with respect to H and B is $\mathcal{F}_\pi$-quasi-regular with respect to K, then $\Lambda_\pi(HK) = \Delta_\pi(HK)$.*

Proof. Indeed, if $L \in \Omega_\pi(HK)$, then we can use the $\mathcal{F}_\pi$-quasi-regularity and find such subgroups $X \in \Omega_\pi(A)$, $Y \in \Omega_\pi(B)$ that $X \cap H \leq L \cap H$, $Y \cap K \leq L \cap K$ and, hence, $(X \cap H)(Y \cap K) \leq L$. Therefore, if a subgroup lies in HK and is separated by $\Omega_\pi(HK)$, then it is separated by $\Theta_\pi(HK)$ too. Since the inverse statement is obvious, $\Lambda_\pi(HK) = \Delta_\pi(HK)$ as required. □

The next corollary follows directly from Propositions 3.1 and 1.2.

Corollary 3.2. *Let H, K be subnormal in A, B and have finite π-indexes in them. Then $\Lambda_\pi(HK) = \Delta_\pi(HK)$.*

However, the structure of the set $\Delta_\pi(HK)$ is also not simple. And so we continue the research of the separability of the cyclic subgroups lying in HK and prove the following.

Proposition 3.3. *Let A and B be residually $\mathcal{F}_\pi$-groups, and let C be a π'-isolated cyclic subgroup of HK generated by an element $c = hk$, where $h \in H$, $k \in K$. If h has an infinite order and all subgroups of $\Omega_\pi(\mathcal{I}_{\pi'}(A, \langle h \rangle))$ are $\mathcal{F}_\pi$-separable in A, then C is separated by $\Theta_\pi(HK)$.*

Proof. Let $g \in HK \backslash C$ be an arbitrary element. We need to find subgroups $X \in \Omega_\pi(A)$, $Y \in \Omega_\pi(B)$ such that $g \notin C(X \cap H)(Y \cap K)$.

Write g in the form $g = ab$ for suitable elements $a \in H$, $b \in K$.

If $a \notin \Omega_\pi(A)$-$Cl(\langle h \rangle)$, then there exists a subgroup $X \in \Omega_\pi(A)$ such that $a \notin \langle h \rangle X$. Then $ab \notin C(X \cap H)K$ since HK is the direct product of H and K, and we can take B as Y. The case when $b \notin \Omega_\pi(B)\text{-}Cl(\langle k\rangle)$ is considered in the same way.

Let now $a \in \Omega_\pi(A)\text{-}Cl(\langle h\rangle)$ and $b \in \Omega_\pi(B)\text{-}Cl(\langle k\rangle)$.

Since A is residually an $\mathcal{F}_\pi$-group, then $\mathcal{I}_{\pi'}(A, \langle h\rangle)$ is locally cyclic according to Proposition 1.5. By the condition, this subgroup is $\mathcal{F}_\pi$-separable in A, hence, $\Omega_\pi(A)\text{-}Cl(\langle h\rangle) = \mathcal{I}_{\pi'}(A, \langle h\rangle)$ and $a \in \mathcal{I}_{\pi'}(A, \langle h\rangle)$. Denote by u a generator of the subgroup $\langle a, h\rangle$ and consider two cases.

Case 1. $\langle k\rangle$ is finite.

Since B is residually an $\mathcal{F}_\pi$-group, its every finite subgroup is $\mathcal{F}_\pi$-separable. Therefore,

$$b \in \Omega_\pi(B)\text{-}Cl(\langle k\rangle) = \langle k\rangle$$

and C is a π'-isolated cyclic subgroup of the finitely generated abelian group $\langle u, k\rangle$.

It is known [7] that all π'-isolated subgroups of a finitely generated nilpotent group are $\mathcal{F}_\pi$-separable. So there exists a subgroup $L \in \Omega_\pi(\langle u, k\rangle)$ such that $g \notin CL$.

Note that, for any π-number l, $\mathcal{I}_{\pi'}(A, \langle h\rangle)^l \cap \langle u\rangle \leq \langle u^l\rangle$.

Indeed, let $v \in \mathcal{I}_{\pi'}(A, \langle h\rangle)$ and $v^l \in \langle u\rangle$. By Proposition 1.4, $\mathcal{I}_{\pi'}(A, \langle h\rangle)$ coincides with the set of all π'-roots which can be extracted from $\langle h\rangle$ in A. So there exists a π'-number m such that $v^m \in \langle h\rangle \leq \langle u\rangle$. Since l and m are relatively prime, the equality $l\alpha + m\beta = 1$ holds for suitable integers α and β. From this it follows that $v = v^{l\alpha+m\beta} \in \langle u\rangle$ and $v^l \in \langle u^l\rangle$.

Let now $l = [\langle u\rangle : L \cap \langle u\rangle]$. The quotient group $\mathcal{I}_{\pi'}(A, \langle h\rangle)/\mathcal{I}_{\pi'}(A, \langle h\rangle)^l$ is locally cyclic, and the orders of all its elements divide l. Therefore, $\mathcal{I}_{\pi'}(A, \langle h\rangle)/\mathcal{I}_{\pi'}(A, \langle h\rangle)^l$ is a cyclic $\mathcal{F}_\pi$-group and so $\mathcal{I}_{\pi'}(A, \langle h\rangle)^l \in \Omega_\pi(\mathcal{I}_{\pi'}(A, \langle h\rangle))$.

By Proposition 1.1, A is $\mathcal{F}_\pi$-quasi-regular with respect to $\mathcal{I}_{\pi'}(A, \langle h\rangle)$. We use this fact and choose a subgroup $X \in \Omega_\pi(A)$ so that

$$X \cap \mathcal{I}_{\pi'}(A, \langle h\rangle) \leq \mathcal{I}_{\pi'}(A, \langle h\rangle)^l.$$

Since B is residually an $\mathcal{F}_\pi$-group, there exists also a subgroup $Y \in \Omega_\pi(B)$ satisfying the condition $Y \cap \langle k\rangle = 1$. Suppose that $g \in C(X \cap H)(Y \cap K)$ and $g = c^n xy$ for suitable elements $x \in X \cap H$, $y \in Y \cap K$ and a number n.

Since $g, c \in \langle u, k\rangle$ and $a, h \in \mathcal{I}_{\pi'}(A, \langle h\rangle)$, then

$$\begin{aligned} x \in X \cap H \cap \langle u, k\rangle \cap \mathcal{I}_{\pi'}(A, \langle h\rangle) &= X \cap \mathcal{I}_{\pi'}(A, \langle h\rangle) \cap \langle u\rangle \\ &\leq \mathcal{I}_{\pi'}(A, \langle h\rangle)^l \cap \langle u\rangle \leq \langle u^l\rangle = L \cap \langle u\rangle \end{aligned}$$

and

$$y \in Y \cap K \cap \langle u, k\rangle = Y \cap \langle k\rangle = 1,$$

whence $g \in CL$ which contradicts the choice of L. Thus,

$$g \notin C(X \cap H)(Y \cap K)$$

as required.

Case 2. $\langle k\rangle$ is infinite.

Let $a = u^l$ and $h = u^q$. Since $\mathcal{I}_{\pi'}(A, \langle h\rangle)$ coincides with the set of all π'-roots which can be extracted from $\langle h\rangle$ in A, we can assume that q is a π'-number.

Suppose that $b^q = k^l$. Then $(ab)^q = u^{lq}b^q = h^l k^l = (hk)^l$. But C is π'-isolated, so $q = 1$ and $g \in C$ which is impossible. Therefore, $b^q k^{-l} \neq 1$ and, since B is residually an $\mathcal{F}_\pi$-group, there exists a subgroup $Z \in \Omega_\pi(B)$ which does not contain this element.

Consider the group

$$G_Z = \langle A * B/Z; [H, KZ/Z] = 1\rangle.$$

It is obvious that the natural homomorphism of B onto B/Z can be extended to the surjective homomorphism $\rho_Z : G \to G_Z$. Let $D = \mathcal{I}_{\pi'}(\langle u, kZ\rangle, C\rho_Z)$.

Since D lies in the finitely generated abelian group, it is cyclic and is generated by an element d. Write it in the form $d = x \cdot yZ$ for suitable elements $x \in H$, $yZ \in KZ/Z$.

Let s be a π'-number such that $d^s \in C\rho_Z$. Then $x^s \in \langle h\rangle$ and so $x \in \mathcal{I}_{\pi'}(A, \langle h\rangle)$. From this it follows that $\mathcal{I}_{\pi'}(A, \langle x\rangle) = \mathcal{I}_{\pi'}(A, \langle h\rangle)$.

Suppose that $g\rho_Z \in D$. Then $(g\rho_Z)^r = (c\rho_Z)^m$ for some numbers r, m, and we can assume without loss of generality that r is a π'-number.

Since $b \in \Omega_\pi(B)\text{-}Cl(\langle k\rangle)$, then $b \in \langle k\rangle Z$ and $b \equiv k^t \pmod Z$ for a suitable number t. We have

$$(u^q \cdot kZ)^m = (h \cdot kZ)^m = (c\rho_Z)^m = (g\rho_Z)^r = (a \cdot bZ)^r = (u^l \cdot (kZ)^t)^r$$

whence $m \equiv tr \pmod n$, where n is the order of kZ, and $qm = lr$ due to infinity of the orders of h and u.

Since Z has a finite π-index in B, n is a π-number. Therefore, the congruencies $qtr \equiv qm \equiv lr \pmod n$ imply that $qt \equiv l \pmod n$. But in this case

$$1 \equiv k^{tq-l} \equiv b^q k^{-l} \pmod Z$$

which contradicts the choice of Z.

Thus, $g\rho_Z \notin D$, and, while we do not assert that D is π'-isolated in $H \cdot KZ/Z$, we can, however, repeat for G_Z the same arguments as in Case 1. As a result, we can find in this group, subgroups $X \in \Omega_\pi(A)$, $Y/Z \in \Omega_\pi(B/Z)$ such that

$$g\rho_Z \notin D(X \cap H)(Y/Z \cap KZ/Z).$$

It is obvious that X and Y are required, and the proposition is proved. □

Note that, if $H = A$ and $K = B$, then the relations

$$\Theta_\pi(HK) \subseteq \Omega_\pi(HK), \quad \Lambda_\pi(HK) = \Delta_\pi(HK)$$

hold. So Proposition 3.3 and the corollary given below can be used also to describe all $\mathcal{F}_\pi$-separable cyclic subgroups of an arbitrary direct product of two groups.

Corollary 3.4. *Let the following statements hold*:

(1) *A and B are residually $\mathcal{F}_\pi$-groups,*
(2) *H and K are π'-isolated in the free factors,*
(3) *all cyclic subgroups which lie in H and are π'-isolated in A are $\mathcal{F}_\pi$-separable in A,*
(4) *all cyclic subgroups which lie in K and are π'-isolated in B are $\mathcal{F}_\pi$-separable in B,*
(5) *H and K do not contain locally cyclic subgroups which are not cyclic unless π coincides with the set of all prime numbers.*

Then $\Lambda_\pi(HK) = \varnothing$.

Proof. Let C be a π'-isolated cyclic subgroup of HK generated by an element $c = hk$, where $h \in H$, $k \in K$.

Let C be finite, and let $g = ab$, $a \in H$, $b \in K$, be an arbitrary element which does not belong to C. Since A and B are residually $\mathcal{F}_\pi$-finite, there exist subgroups $X \in \Omega_\pi(A)$, $Y \in \Omega_\pi(B)$ such that

$$X \cap \langle h \rangle = Y \cap \langle k \rangle = 1,$$

$a^{-1}h^m \notin X$ whenever $a^{-1}h^m \neq 1$, and $b^{-1}k^n \notin Y$ whenever $b^{-1}k^n \neq 1$. It is obvious that $g \notin C(X \cap H)(Y \cap K)$ and so C is separated by $\Theta_\pi(HK)$.

If C is infinite, then at least one of the elements h and k has an infinite order. Let, for definiteness, h be such an element.

Since H is π'-isolated in A, then $\mathcal{I}_{\pi'}(A, \langle h \rangle) \leq H$. If π coincides with the set of all prime numbers, then any subgroup is π'-isolated and so $\mathcal{I}_{\pi'}(A, \langle h \rangle) = \langle h \rangle$. Otherwise, by condition, H does not contain locally cyclic subgroups, and, therefore, $\mathcal{I}_{\pi'}(A, \langle h \rangle)$ is cyclic too. But, again by condition, any cyclic subgroup which lies in H and is π'-isolated in A is $\mathcal{F}_\pi$-separable in A. Hence, $\mathcal{I}_{\pi'}(A, \langle h \rangle)$ and all its subgroups of finite π-index are $\mathcal{F}_\pi$-separable in A.

Thus, we can use Proposition 3.3 in this case, which states that C is separated by $\Theta_\pi(HK)$. □

The corollary proved in a combination with the main theorem lets us generalize the result of Loginova [8] that, if G is residually finite and all cyclic subgroups of A and B are finitely separable in these groups, then all cyclic subgroups of G are finitely separable too.

Theorem 3.5. *Let π be a set of prime numbers which coincides with the set of all prime numbers or is one-element. Let all cyclic subgroups which lie in H and are π'-isolated in A be $\mathcal{F}_\pi$-separable in A, and let all cyclic subgroups which lie in K and are π'-isolated in B be $\mathcal{F}_\pi$-separable in B. Let also H and K do not contain locally cyclic subgroups which are not cyclic unless π coincides with the set of all prime numbers. If G is residually an $\mathcal{F}_\pi$-group, then a π'-isolated cyclic subgroup of* *this group is $\mathcal{F}_\pi$-separable in it if, and only if, it is not conjugated with any subgroup of $\Delta_\pi(A) \cup \Delta_\pi(B)$.*

It is not difficult to show (see e.g. [6]) that every π'-isolated cyclic subgroup of an arbitrary free group is $\mathcal{F}_\pi$-separable regardless of the choice of a set π. As it was already noted during the proof of Proposition 3.3, the stronger assertion holds for finitely generated nilpotent groups: all π'-isolated subgroups of such groups are $\mathcal{F}_\pi$-separable. So the next two statements follow from Theorem 2.1 and Corollary 3.4.

Theorem 3.6. *Let π be a set of prime numbers which coincides with the set of all prime numbers or is one-element. Let A and B be free groups, and let H and K be cyclic subgroups which are π'-isolated in the free factors. Then all π'-isolated cyclic subgroups of G are $\mathcal{F}_\pi$-separable.*

Theorem 3.7. *Let π be a set of prime numbers which coincides with the set of all prime numbers or is one-element, and let A and B be finitely generated nilpotent groups. If G is residually an $\mathcal{F}_\pi$-group, then all its π'-isolated cyclic subgroups are $\mathcal{F}_\pi$-separable.*

Note that the conditions of Theorem 3.6 are true for the group

$$G_{mn} = \langle a, b; [a^m, b^n] = 1 \rangle$$

with a suitable choice of numbers m and n. This group is investigated in [13, 14], where the finite separability of all its cyclic subgroups is proved.

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