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A CHARACTERIZATION OF ROOT CLASSES OF GROUPS

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We prove that a class of groups is root in a sense of K. W. Gruenberg if, and only if, it is closed under subgroups and Cartesian wreath products. Using this result we obtain a condition which is sufficient for the generalized free product of two nilpotent groups to be residual solvable.

Key Words: Generalized free product; Nilpotent group; Residual solvability; Root class of groups.

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Recall that a class of groups $\mathcal{C}$ is called root if it satisfies the following conditions:

1) An arbitrary subgroup of a $\mathcal{C}$-group is also a $\mathcal{C}$-group;
2) The direct product of any two $\mathcal{C}$-groups belongs to $\mathcal{C}$;
3) The Gruenberg condition: for any group X and for any subnormal sequence $Z \leqslant Y \leqslant X$ with factors in $\mathcal{C}$, there exists a normal subgroup T of X such that $T \leqslant Z$ and $X/T \in \mathcal{C}$.

It is easy to see that the classes of all finite groups, of all finite p-groups, and of all solvable groups are root. The class of all nilpotent groups fails to be root because it does not satisfy the Gruenberg condition.

The notion of the root class was introduced by K. W. Gruenberg [4] in connection with studying of residual properties of solvable groups. Recall that a group X is said to be residually a $\mathcal{C}$-group for some class of groups $\mathcal{C}$ if, for any $x \in X \setminus 1$, there exists a homomorphism ψ of X onto a $\mathcal{C}$-group such that $x\psi \neq 1$.

D. N. Azarov and D. Tieudjo [1] proved that every free group is residually a $\mathcal{C}$-group for any nontrivial (i.e., containing at least one non-unit group) root class of groups. It follows from this statement and the results of K. W. Gruenberg [4] that, for every nontrivial root class of groups $\mathcal{C}$, the free product of an arbitrary number of residually $\mathcal{C}$-groups is itself residually a $\mathcal{C}$-group. The property of being "residually a $\mathcal{C}$-group", where $\mathcal{C}$ is an arbitrary root class of groups, was studied in [2] and [11] in relation to generalized free products and HNN-extensions.

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The results of these papers show that many conditions known to be necessary or sufficient for a given group to be residually a $\mathcal{C}$-group for some concrete root class $\mathcal{C}$ remain true as well in the case when $\mathcal{C}$ is an arbitrary root class of groups.

The definition given by K. W. Gruenberg does not allow to describe all root classes of groups clearly. The main aim of this paper is to give another, more simple, characterization of root classes.

It is easy to see that the second condition in the definition of the root class follows from the first and the third and, thus, is excessive. As for the Gruenberg condition, it can be replaced by a more clear criterion as the next theorem shows.

Theorem 1. *Let $\mathcal{C}$ be a class of groups closed under taking subgroups. Then the following statements are equivalent*:

1. *$\mathcal{C}$ satisfies the Gruenberg condition* (*and, hence, is root*),
2. *$\mathcal{C}$ is closed under Cartesian wreath products,*
3. *$\mathcal{C}$ is closed under extensions and, for any two groups $X, Y \in \mathcal{C}$, contains the Cartesian product $\prod_{y \in Y} X_y$, where X_y is an isomorphic copy of X for every $y \in Y$.*

The next three statements follow immediately from Theorem 1.

Corollary 1 ([3]). *If a class of groups consists of the only finite groups, then it is root if, and only if, it is closed under subgroups and extensions.*

Corollary 2. *The intersection of any two root classes of groups is again a root class of groups.*

Corollary 3. *If $\mathcal{C}$ is a root class of groups, then the class $\mathcal{C}_0$ of all torsion-free $\mathcal{C}$-groups is root too.*

The theorem given below serves as an example of application of Theorem 1 to studying of residual properties of generalized free products of groups.

Theorem 2. *Let $\mathcal{C}$ be a nontrivial root class of groups closed under taking quotient groups. Let also*

$$G = \langle A * B;\ H = K,\ \varphi \rangle$$

be the free product of nilpotent $\mathcal{C}$-groups A and B with subgroups $H \leqslant A$ and $K \leqslant B$ amalgamated according to an isomorphism $\varphi : H \to K$. Suppose that A and B possess such central series

$$1 = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_n = A,$$
$$1 = B_0 \leqslant B_1 \leqslant \cdots \leqslant B_n = B,$$

that $(A_i \cap H)\varphi = B_i \cap K$ for every $i \in \{1, 2, \ldots, n\}$. Then there exists a homomorphism of G onto a solvable $\mathcal{C}$-group which is injective on A and B. In particular, G is residually a $\mathcal{CS}$-group, where $\mathcal{CS}$ is the class of all solvable $\mathcal{C}$-groups.

We note that Theorem 2 strengthens and generalizes Theorem 8 from [6], which states that G is poly-(residually solvable), i.e., possesses a finite normal series with residually solvable factors.

Proof of Theorem 1. $1 \Rightarrow 2$. Let X, Y be arbitrary $\mathcal{C}$-groups, and let B be the Cartesian product of isomorphic copies of X indexed by the elements of Y (i.e., B is the set of all functions mapping Y into X with the coordinate-wise multiplication). Let also $W = X \wr Y$ be the Cartesian wreath product of X and Y. We need to show that $W \in \mathcal{C}$.

Recall that W is the set $Y \cdot B$ with the operation defined by the rule $y_1 b_1 y_2 b_2 = y_1 y_2 b^{y_2} b_2$, where $b^{y_2} \in B$ is the function mapping y to $y_2 y$ for every $y \in Y$. From this definition it follows that B is normal in W and $W/B \cong Y$.

Let $A = \{b \in B \mid b(1) = 1\}$. Then A is normal in B and $B/A \cong X$. Since $W/B \cong Y$, then $A \leqslant B \leqslant W$ is a subnormal sequence with $\mathcal{C}$-factors, and, by the Gruenberg condition, there exists a normal subgroup T of W such that $T \leqslant A$ and $W/T \in \mathcal{C}$.

As T is normal in W, it is contained in the subgroup

$$A^y = \{b \in B \mid b(y) = 1\}$$

for any $y \in Y$. But $\bigcap_{y \in Y} A^y = 1$; hence, $T = 1$ and $W \in \mathcal{C}$ as required.

$2 \Rightarrow 3$. Let X, Y be arbitrary $\mathcal{C}$-groups, $W = X \wr Y$, and $B = \prod_{y \in Y} X_y$, where X_y is an isomorphic copy of X for every $y \in Y$. Then $W \in \mathcal{C}$, $B \leqslant W$, and $B \in \mathcal{C}$ since $\mathcal{C}$ is closed under subgroups. Further, if Z is an extension of X by Y, then Z embeds in W [7] and, therefore, belongs to $\mathcal{C}$.

$3 \Rightarrow 1$. Let X be a group, and let $Z \leqslant Y \leqslant X$ be a subnormal sequence with $\mathcal{C}$-factors. We put $T = \bigcap_{s \in S} Z^s$, where S is some system of all cosets representatives of Y in X, and show that T is required.

It is obvious that T is a normal subgroup of X lying in Z. The quotient group Y/T embeds in the Cartesian product P of the quotient groups Y/Z^s, $s \in S$, by the theorem of Remak. Each of groups Y/Z^s is isomorphic to the $\mathcal{C}$-group Y/Z. Therefore, $P \in \mathcal{C}$, and $Y/T \in \mathcal{C}$ since $\mathcal{C}$ is closed under subgroups. Thus, $Y/T \in \mathcal{C}$, $X/Y \in \mathcal{C}$, and $X/T \in \mathcal{C}$ because $\mathcal{C}$ is closed under extensions. □

Proof of Theorem 2. We will use an induction on n.

First of all, let us recall that G is the quotient group of the ordinary free product of A and B by the normal closure of the set $\{h(h\varphi)^{-1} \mid h \in H\}$. Taking in this definition the direct product of A and B instead of the free product $A * B$, we get the generalized direct product

$$P = \langle A \times B;\ H = K,\ \varphi\rangle.$$

It is well known (see, e.g., [8, Theorem 4.3]) that there exist canonical monomorphisms of A and B into G, and so A and B can be considered as subgroups of G. The same statement is true for P provided H and K lie in the centers of A and B respectively [10].

Thus, if $n = 1$, there exist natural inclusions $\alpha : A \to P$ and $\beta : B \to P$. Since $h\alpha = (h\varphi)\beta$ for any $h \in H$, these inclusions can be continued to a homomorphism of G onto P, and this homomorphism is required. Indeed, $\mathcal{C}$ is a root class, hence $A \times B \in \mathcal{C}$. It remains to note that $\mathcal{C}$ is closed under quotient groups and so $P \in \mathcal{C}$.

Let now $n > 1$, and let

$$\bar{\varphi} : HA_1/A_1 \to KB_1/B_1$$

be a map such that $(hA_1)\bar{\varphi} = (h\varphi)B_1$ for every $h \in H$. It follows from the equality $(A_1 \cap H)\varphi = B_1 \cap K$ that $\bar{\varphi}$ is a correctly defined isomorphism of subgroups. Therefore, we can consider the generalized free product

$$\overline{G} = \langle \overline{A} * \overline{B};\ \overline{H} = \overline{K},\ \bar{\varphi} \rangle,$$

where $\overline{A} = A/A_1$, $\overline{B} = B/B_1$, $\overline{H} = HA_1/A_1$, and $\overline{K} = KB_1/B_1$.

Since $\mathcal{C}$ is closed under quotient groups, $\overline{A}, \overline{B} \in \mathcal{C}$. It is easy to see also that the series

$$1 = A_1/A_1 \leqslant A_2/A_1 \leqslant \cdots \leqslant A_n/A_1 = \overline{A},$$
$$1 = B_1/B_1 \leqslant B_2/B_1 \leqslant \cdots \leqslant B_n/B_1 = \overline{B}$$

are $\bar{\varphi}$-compatible. Hence, $\overline{G}$ satisfies the conditions of the theorem and, by induction hypothesis, there exists a homomorphism of this group onto a solvable $\mathcal{C}$-group Y which is injective on $\overline{A}$ and $\overline{B}$.

Since $\mathcal{C}$ is closed under subgroups, $A_1, B_1 \in \mathcal{C}$. Therefore, the group

$$G_1 = \langle A_1 * B_1;\ H_1 = K_1,\ \varphi_1 \rangle,$$

where $H_1 = H \cap A_1$, $K_1 = K \cap B_1$, and $\varphi_1 = \varphi|_{H_1}$, satisfies the conditions of the theorem too. Again by induction hypothesis, there exists a homomorphism of this group onto a solvable $\mathcal{C}$-group X which is injective on A_1 and B_1.

Now we use Lemma 2 from [5]. By this lemma, there exists a homomorphism ρ of G onto $X \wr Y$ which is injective on A and B. $X \wr Y$ is a solvable group and, by Theorem 1, belongs to $\mathcal{C}$. Hence, ρ is the required homomorphism.

By the condition of the theorem, $\mathcal{C}$ contains at least one nontrivial group. All cyclic subgroups of this group and their quotient groups belong to $\mathcal{C}$ too. Hence, $\mathcal{C}$ includes at least one cyclic group of prime order, say p. It is well known that every finite p-group possesses a normal series with the factors of order p. Taking into account that $\mathcal{C}$ is closed under extensions, we conclude that all finite p-groups are contained in $\mathcal{C}$.

Let $N = \ker \rho$. Since $N \cap A = N \cap B = 1$, then, by the theorem of Neumann [9], N is free. As it is known, free groups are residually p-finite for any prime p. Hence, N is residually a $\mathcal{CS}$-group.

Thus, G is an extension of the residually $\mathcal{CS}$-group N by the $\mathcal{CS}$-group G/N. By Corollary 2, the class of all solvable $\mathcal{C}$-groups is root. Therefore, G is a residually $\mathcal{CS}$-group by Lemma 1.5 from [4]. □

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