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A Necessary Condition for The Residual Nilpotence of HNN-Extensions

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Abstract—Let G be a multiple HNN-extension of a group A, and let all its associated subgroups be properly contained in some locally nilpotent subgroup of A. We prove that if G is residually nilpotent, then all the associated subgroups are p'-isolated in A for some prime p. Moreover, if q is a prime such that G is residually a q'-torsion-free nilpotent group, then $p = q$.

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INTRODUCTION

Recall that given a property $\mathcal{P}$ of groups, a group G is said to be *residually* $\mathcal{P}$ if for every non-trivial $g \in G$ there exists a homomorphism σ of G onto a group P with property $\mathcal{P}$ such that $g\sigma$ is non-trivial. In this paper, we are concerned with the study of residual nilpotence and, in particular, residual p-finiteness (where p is a prime). For the definition of HNN-extension and other related terms we refer the reader to [1, Ch. IV].

The property of residual nilpotence has been studied for a long time: it is enough to recall the classical results of Magnus [2] on the residual nilpotence of free groups and of Gruenberg [3] on the residual p-finiteness of finitely generated nilpotent and free polynilpotent groups. However, the number of results concerning the residual nilpotence of HNN-extensions is so small that we can list them all.

Apparently, the first paper in this area belongs to Raptis and Varsos [4]. They give a characterization for the residual nilpotence of an HNN-extension of a finitely generated abelian group provided either this HNN-extension is ascending (i. e. its base group serves as one of the associated subgroups) or its associated subgroups coincide (in this section, we deal only with HNN-extensions with one stable letter). In [5], this characterization is generalized by a criterion for the residual nilpotence of an arbitrary HNN-extension of a finitely generated abelian group. The last paper also contains a characterization for the residual nilpotence and the residual p-finiteness of an arbitrary HNN-extension of a finite group (in [6], these criteria are extended to arbitrary graphs of finite groups).

These are all the results which concern the property of being residually an arbitrary nilpotent group. Other papers deal only with residual p-finiteness. First of all, let us mention two another criteria for an HNN-extension of a finite group to be residually a finite p-group, which are proved in [7] (see also [8]) and [9] respectively. In [7, 10], and [11], Moldavanskii applies the so-called "filtrational" approach of Baumslag [12] to the study of the residual p-finiteness of HNN-extensions (in the more recent paper [13], it is also done for arbitrary graphs of groups). As a result, he gives a characterization for the residual p-finiteness of

- Baumslag–Solitar groups [14] (which are all possible HNN-extensions of an infinite cyclic group),

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- Brunner's groups [15] (which are certain HNN-extensions of Baumslag—Solitar groups),
- an HNN-extension of a finitely generated nilpotent group with finite associated subgroups, and
- an HNN-extension of an arbitrary group with finitely generated central associated subgroups.

At last, in [16], a criterion is proved for a splitting extension of a finitely generated abelian group to be residually a finite p-group, and this result can serve as the first step in the study of the residual p-finiteness of ascending HNN-extensions of finitely generated abelian groups. Now, recall that given a prime p, a subgroup Y of a group X is said to be *p'-isolated* in X if for every element $x \in X$ and for any prime $q \neq p$ the inclusion $x^q \in Y$ implies that $x \in Y$. Each of the statements mentioned above asserts, in particular, that if an HNN-extension satisfying the conditions of this statement is residually nilpotent, then its associated subgroups are p'-isolated in the base group for some prime p. The objective of this paper is to show that this condition is necessary in much more general situations.

1. MAIN RESULTS

Let A be a group, and let $\{H_i, H_{-i} \mid i \in \mathcal{I}\}$ be a family of subgroups of A. If for every $i \in \mathcal{I}$ there exists an isomorphism $\varphi_i\colon H_i \to H_{-i}$, then we can consider the multiple HNN-extension

$$G = \langle A, t_i;\ t_i^{-1} H_i t_i = H_{-i},\ \varphi_i,\ i \in \mathcal{I}\rangle$$

of A with the stable letters t_i and the subgroups H_i, H_{-i} associated according to the isomorphisms φ_i, $i \in \mathcal{I}$ (we assume that this notation is fixed throughout the paper). The following theorem is our main result.

Theorem. *Let there exist a locally nilpotent subgroup N of A such that $H_{\varepsilon i}$ is properly contained in N for all $i \in \mathcal{I}$, $\varepsilon = \pm 1$.*

1. If G is residually nilpotent, then there exists a prime p such that $H_{\varepsilon i}$ is p'-isolated in A for all $i \in \mathcal{I}$, $\varepsilon = \pm 1$.

2. If there exists a prime p such that G is residually a p'-torsion-free nilpotent group (in other words, G is residually 'a nilpotent group whose periodic part is a p-group'), then $H_{\varepsilon i}$ is p'-isolated in A for all $i \in \mathcal{I}$, $\varepsilon = \pm 1$.

The corollary given below follows immediately from the second statement of Theorem.

Corollary. *Let all $H_{\varepsilon i}$ satisfy the same conditions as in Theorem. If G is residually a finite p-group for some prime p, then $H_{\varepsilon i}$ is p'-isolated in A for all $i \in \mathcal{I}$, $\varepsilon = \pm 1$. If G is residually torsion-free nilpotent, then $H_{\varepsilon i}$ is isolated in A for all $i \in \mathcal{I}$, $\varepsilon = \pm 1$.*

It should be pointed out that Azarov and Ivanova [17] proved an analogue of the first statement of Theorem for a free product of arbitrary many groups with one amalgamated subgroup. Integrating these two results, we get a generalization of the necessary condition for the residual nilpotence of an arbitrary graph of finitely generated abelian groups given in [6, Proposition 14]. The theorem formulated also generalizes the necessary condition for the residual nilpotence of an HNN-extension of a finitely generated abelian group proved in [4, Proposition 2.1]; this fact follows from Proposition 1 given below.

Proposition 1. *If H_i and H_{-i} are p'-isolated in A for some $i \in \mathcal{I}$ and for some prime p, then all subgroups $M_i^{(j)}$ defined by the rule $M_i^{(0)} = H_i \cap H_{-i}, M_i^{(j+1)} = M_i^{(j)} \cap M_i^{(j)}\varphi_i \cap M_i^{(j)}\varphi_i^{-1} (j \geq 0)$ are also p'-isolated in A.*

Proof. We use induction on j. It is easy to see that the intersection of p'-isolated subgroups is again a p'-isolated subgroup. Therefore, $M_i^{(0)}$ is p'-isolated in A.

Suppose that $M_i^{(j)}$ is p'-isolated in A for some $j \geq 0$. It follows, in particular, that $M_i^{(j)}$ is p'-isolated both in H_i and in H_{-i}. From the first statement we deduce that $M_i^{(j)}\varphi_i$ is p'-isolated in $H_{-i} = H_i\varphi_i$ and hence the intersection $M_i^{(j)} \cap M_i^{(j)}\varphi_i$ is also p'-isolated in H_{-i}. Since H_{-i} is p'-isolated in A, it is easy to see that $M_i^{(j)} \cap M_i^{(j)}\varphi_i$ turns out to be p'-isolated in A. Arguing as above, we conclude that $M_i^{(j)} \cap M_i^{(j)}\varphi_i^{-1}$ is p'-isolated in A. Thus, the intersection

$$(M_i^{(j)} \cap M_i^{(j)}\varphi_i) \cap (M_i^{(j)} \cap M_i^{(j)}\varphi_i^{-1}) = M_i^{(j)} \cap M_i^{(j)}\varphi_i \cap M_i^{(j)}\varphi_i^{-1}$$

is also p'-isolated in A. □

It follows from the first statement of Theorem and Proposition 1 that if G satisfies the condition of Theorem and is residually nilpotent, then there exists a prime p such that the subgroup $M_i = \bigcap_{j=0}^{\infty} M_i^{(j)}$ is p'-isolated in A for every $i \in \mathcal{I}$. If, in addition, A is finitely generated abelian and D_i is the isolator of M_i in A, then the quotient group D_i/M_i turns out to be a finite p-group, as Proposition 2.1 from [4] claimes.

As a discussion of possible generalizations of the theorem formulated, we give two examples showing that the following conditions of this theorem are essential: all $H_{\varepsilon i}$ ($i \in \mathcal{I}, \varepsilon = \pm 1$) are *properly* contained in N; N is *locally nilpotent*.

Example 1. The Baumslag–Solitar group $\mathrm{BS}(1,6) = \langle a, b;\ a^{-1}ba = b^6\rangle$ is an *ascending* HNN-extension of the infinite cyclic group generated by b. It is residually a finite 5-group by Theorem 3 from [7], but the associated subgroup generated by b^6 is not p'-isolated in the base group for every prime p.

Example 2. The group $X = \langle t, a, b;\ t^{-1}at = b^6\rangle$ is the HNN-extension of the *free* group $\langle a, b\rangle$ by t with the cyclic associated subgroups $\langle a\rangle$ and $\langle b^6\rangle$. Simultaneously, it is the free product of the free group $\langle t, a\rangle$ and the infinite cyclic subgroup $\langle b\rangle$ with the cyclic amalgamated subgroups $\langle t^{-1}at\rangle$ and $\langle b^6\rangle$. It is easy to see that $\langle t^{-1}at\rangle$ is isolated in $\langle t, a\rangle$. Therefore, X is residually torsion-free nilpotent by [18]. However, as well as above, the subgroup $\langle b^6\rangle$ is not p'-isolated in the base group $\langle a, b\rangle$ for every prime p.

2. PROOF OF THEOREM

We begin with the following auxiliary statement.

Proposition 2. *If X is a residually nilpotent group and Y is a finitely generated subgroup of X, then for every non-trivial element $y \in Y$ there exists a homomorphism of Y onto a finite group P of prime power order mapping y to a non-trivial element. If p is a prime such that X is residually a p'-torsion-free nilpotent group, then we always can take a finite p-group as P.*

Proof. Let $y \in Y \setminus \{1\}$. Since X is residually a nilpotent group, there exists a homomorphism ρ of this group onto a nilpotent group mapping y to a non-trivial element. Denoting by σ the restriction of ρ onto Y, we see that $Y\sigma$ is a finitely generated nilpotent group.

If the order of $y\sigma$ is infinite, then it does not belong to the periodic part $\tau(Y\sigma)$ of $Y\sigma$ and hence $(y\sigma)\tau(Y\sigma)$ is a non-trivial element of $Y\sigma/\tau(Y\sigma)$. The quotient group $Y\sigma/\tau(Y\sigma)$ is torsion-free, so it is residually a finite p-group for every prime p by the theorem of Gruenberg [3, Theorem 2.1].

If the order r of $y\sigma$ is finite and we take a prime divisor of r as p, then $y \notin \tau_{p'}(Y\sigma)$ (where $\tau_{p'}(Y\sigma)$ denotes the product of all Sylow subgroups of $\tau(Y\sigma)$ whose orders are not p-numbers) and therefore $(y\sigma)\tau_{p'}(Y\sigma)$ is a non-trivial element of $Y\sigma/\tau_{p'}(Y\sigma)$. Since $Y\sigma/\tau_{p'}(Y\sigma)$ is p'-torsion-free, the same theorem of Gruenberg implies that this quotient group is residually a finite p-group. Thus, in both cases there exists a homomorphism θ of $Y\sigma$ onto a group P of prime power order mapping $y\sigma$ to a non-trivial element. Moreover, if the periodic part of $Y\sigma$ is a p-group for some prime p, then we can take a finite p-group as P. Therefore, the composition of σ and θ is the desired homomorphism. □

If $a \in A$, $i \in \mathcal{I}$, $\varepsilon \in \{1, -1\}$, then we denote by $a_{i,\varepsilon}$ the element $t_i^{-\varepsilon}at_i^{\varepsilon}$. Let $a, b \in A$, $i \in \mathcal{I}$, $\varepsilon \in \{1, -1\}$. We put $u_{i,\varepsilon}^{(1)}(a, b) = [a_{i,\varepsilon}, b]$ and by induction $u_{i,\varepsilon}^{(k)}(a, b) = [a_{i,\varepsilon}, u_{i,\varepsilon}^{(k-1)}(a, b)]$, $k = 2, 3, \ldots$ If $c, d \in A$, $j \in \mathcal{I}$, $\delta \in \{1, -1\}$, we put also $v_{i,\varepsilon;j,\delta}^{(k)}(a, b; c, d) = [u_{i,\varepsilon}^{(k)}(a, b), t_j^{-3}u_{j,\delta}^{(k)}(c, d)t_j^3]$ for all $k \geq 1$.

Proposition 3. *If $a \notin H_{\varepsilon i}$, $b \notin H_{-\varepsilon i}$, $c \notin H_{\delta j}$, $d \notin H_{-\delta j}$, then for every $k \geq 1$*

1. $u_{i,\varepsilon}^{(k)}(a, b)$ is a product of the form $a_{i,\varepsilon}^{-1}b^{-1}t_i^{-\varepsilon}\ldots t_i^{\varepsilon}b$, which is reduced and has a length of 2^{k+1};

2. $u_{j,\delta}^{(k)}(c, d)$ is a product of the form $c_{j,\delta}^{-1}d^{-1}t_j^{-\delta}\ldots t_j^{\delta}d$, which is reduced and has a length of 2^{k+1};

3. $v^{(k)}_{i,\varepsilon;j,\delta}(a,b;c,d)$ *is not equal to* 1.

Proof. To verify (1) we use induction on k. For $k=1$ we have $u^{(1)}_{i,\varepsilon}(a,b)=a^{-1}_{i,\varepsilon}b^{-1}a_{i,\varepsilon}b=t_i^{-\varepsilon}a^{-1}t_i^{\varepsilon}b^{-1}t_i^{-\varepsilon}at_i^{\varepsilon}b$. Since $a\notin H_{\varepsilon i}$, $b\notin H_{-\varepsilon i}$, the last product is a reduced form of length 4. Therefore, $u^{(1)}_{i,\varepsilon}(a,b)$ possesses the required properties. Suppose now that (1) is true for some $k\geq 1$. We have

$$u^{(k+1)}_{i,\varepsilon}(a,b)=a^{-1}_{i,\varepsilon}u^{(k)}_{i,\varepsilon}(a,b)^{-1}a_{i,\varepsilon}u^{(k)}_{i,\varepsilon}(a,b)=a^{-1}_{i,\varepsilon}(b^{-1}t_i^{-\varepsilon}\dots t_i^{\varepsilon}ba_{i,\varepsilon})a_{i,\varepsilon}(a^{-1}_{i,\varepsilon}b^{-1}t_i^{-\varepsilon}\dots t_i^{\varepsilon}b).$$

Since $a\notin H_{\varepsilon i}$, $b\notin H_{-\varepsilon i}$, the element $t_i^{\varepsilon}ba^{\pm1}_{i,\varepsilon}b^{-1}t_i^{-\varepsilon}=t_i^{\varepsilon}bt_i^{-\varepsilon}a^{\pm1}t_i^{\varepsilon}b^{-1}t_i^{-\varepsilon}$ is reduced. Hence, the form of $u^{(k+1)}_{i,\varepsilon}(a,b)$ given above is reduced and, as is easily seen, it has a length of 2^{k+2}. The same argument can be used to verify (2). Let us prove (3).

From (1) and (2) we have

$$\begin{aligned}v^{(k)}_{i,\varepsilon;j,\delta}(a,b;c,d)&=u^{(k)}_{i,\varepsilon}(a,b)^{-1}t_j^{-3}u^{(k)}_{j,\delta}(c,d)^{-1}t_j^{3}u^{(k)}_{i,\varepsilon}(a,b)t_j^{-3}u^{(k)}_{j,\delta}(c,d)t_j^{3}\\&=(b^{-1}t_i^{-\varepsilon}\dots t_i^{\varepsilon}ba_{i,\varepsilon})t_j^{-3}(d^{-1}t_j^{-\delta}\dots t_j^{\delta}dc_{j,\delta})t_j^{3}(a^{-1}_{i,\varepsilon}b^{-1}t_i^{-\varepsilon}\dots t_i^{\varepsilon}b)t_j^{-3}(c^{-1}_{j,\delta}d^{-1}t_j^{-\delta}\dots t_j^{\delta}d)t_j^{3}.\end{aligned}$$

It follows from the conditions $a\notin H_{\varepsilon i}$, $b\notin H_{-\varepsilon i}$, $c\notin H_{\delta j}$, $d\notin H_{-\delta j}$ ($\varepsilon,\delta\in\{1,-1\}$) that the elements

$$a_{i,\varepsilon}t_j^{-3}d^{-1}t_j^{-\delta}=t_i^{-\varepsilon}at_i^{\varepsilon}t_j^{-3}d^{-1}t_j^{-\delta},\quad c_{j,\delta}t_j^{3}a^{-1}_{i,\varepsilon}=t_j^{-\delta}ct_j^{\delta+3}t_i^{-\varepsilon}a^{-1}t_i^{\varepsilon},$$
$$t_i^{\varepsilon}bt_j^{-3}c^{-1}_{j,\delta}=t_i^{\varepsilon}bt_j^{-\delta-3}c^{-1}t_j^{\delta},\quad t_j^{\delta}dt_j^{3}$$

are reduced. Therefore, $v^{(k)}_{i,\varepsilon;j,\delta}(a,b;c,d)$ possesses a reduced form of non-zero length and hence is not equal to 1 by Britton's Lemma (see, e. g., [1, p. 181]). □

Proof of Theorem. We begin with the first statement of Theorem following the idea from [17].

If $H_{\varepsilon i}$ is not isolated in A for some $i\in\mathcal{I}$, $\varepsilon\in\{1,-1\}$, then there exist an element $a\in A\setminus H_{\varepsilon i}$ and a prime p such that $a^p\in H_{\varepsilon i}$. Suppose that $H_{\delta j}$ is not p'-isolated in A for some $j\in\mathcal{I}$, $\delta\in\{1,-1\}$. It follows that we can find a prime $q\neq p$ and an element $c\in A\setminus H_{\delta j}$ such that $c^q\in H_{\delta j}$. Since $H_{-\varepsilon i}\neq N\neq H_{-\delta j}$, some elements $b\in N\setminus H_{-\varepsilon i}$ and $d\in N\setminus H_{-\delta j}$ can also be chosen.

Let $F=\mathrm{sgp}\{a,b,c,d,t_i,t_j\}$, $B=\mathrm{sgp}\{a^p\varphi_i^{\varepsilon},b,c^q\varphi_j^{\delta},d\}$. Since B is a finitely generated subgroup of N, it is nilpotent. Denote by n some fixed number that is greater than the nilpotency class of B.

By Proposition 3, $v^{(n)}_{i,\varepsilon;j,\delta}(a,b;c,d)$ is not equal to 1. By Proposition 2, it follows that there exists a homomorphism σ of F onto a finite group P of prime power order mapping $v^{(n)}_{i,\varepsilon;j,\delta}(a,b;c,d)$ to a non-trivial element.

Since at least one of p and q does not divide the order of P, we have $a\sigma=(a\sigma)^{pk}$ or $c\sigma=(c\sigma)^{ql}$ for suitable whole numbers k and l. Therefore, $v^{(n)}_{i,\varepsilon;j,\delta}(a,b;c,d)\sigma=v^{(n)}_{i,\varepsilon;j,\delta}(a^{pk},b;c,d)\sigma$ or $v^{(n)}_{i,\varepsilon;j,\delta}(a,b;c,d)\sigma=v^{(n)}_{i,\varepsilon;j,\delta}(a,b;c^{ql},d)\sigma$. It follows from the inclusions $a^p\in H_{\varepsilon i}$ and $c^q\in H_{\delta j}$ that $a^{pk}_{i,\varepsilon}=a^{pk}\varphi_i^{\varepsilon}$ and $c^{ql}_{j,\delta}=c^{ql}\varphi_j^{\delta}$. Therefore, $a^{pk}_{i,\varepsilon}$, $c^{ql}_{j,\delta}$ belong to B and $u^{(n)}_{i,\varepsilon}(a^{pk},b)$, $u^{(n)}_{j,\delta}(c^{ql},d)$ are simple commutators of elements of B of weight n. By the choice of n, the last two elements are equal to 1 and hence

$$v^{(n)}_{i,\varepsilon;j,\delta}(a^{pk},b;c,d)=1=v^{(n)}_{i,\varepsilon;j,\delta}(a,b;c^{ql},d).$$

Thus, $v^{(n)}_{i,\varepsilon;j,\delta}(a,b;c,d)\sigma=1$, which contradicts the choice of σ. This contradiction means that $H_{\delta j}$ is p'-isolated in A for all $j\in\mathcal{I}$, $\delta\in\{1,-1\}$ and p is required.

The proof of the second statement follows the above argument almost verbatim.

If $H_{\delta j}$ is not p'-isolated in A for some $j\in\mathcal{I}$, $\delta\in\{1,-1\}$, then there exist a prime $q\neq p$ and an element $c\in A\setminus H_{\delta j}$ such that $c^q\in H_{\delta j}$. As above, we take an arbitrary element $d\in N\setminus H_{-\delta j}$, put $F=\mathrm{sgp}\{c,d,t_j\}$, $B=\mathrm{sgp}\{c^q\varphi_j^{\delta},d\}$ and choose a number n that is greater than the nilpotency class of B. By Proposition 3 and Britton's Lemma, $u^{(n)}_{j,\delta}(c,d)\neq 1$. Therefore, by Proposition 2, there exists a homomorphism σ of F onto a finite p-group P mapping $u_{j,\delta}^{(n)}(c,d)$ to a non-trivial element. However, it follows from the inequality $q \neq p$ that $c\sigma = (c\sigma)^{ql}$ for a suitable whole number l and hence $u_{j,\delta}^{(n)}(c,d)\sigma = u_{j,\delta}^{(n)}(c^{ql},d)\sigma = 1$. This contradiction completes the proof. □

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