CERTAIN RESIDUAL PROPERTIES OF GENERALIZED BAUMSLAG–SOLITAR GROUPS

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Abstract. Let G be a generalized Baumslag–Solitar group and $\mathcal{C}$ be a class of groups containing at least one non-unit group and closed under taking subgroups, extensions, and Cartesian products of the form $\prod_{y \in Y} X_y$, where $X, Y \in \mathcal{C}$ and X_y is an isomorphic copy of X for every $y \in Y$. We give a criterion for G to be residually a $\mathcal{C}$-group provided $\mathcal{C}$ consists only of periodic groups. We also prove that G is residually a torsion-free $\mathcal{C}$-group if $\mathcal{C}$ contains at least one non-periodic group and is closed under taking homomorphic images. These statements generalize and strengthen some known results. Using the first of them, we provide criteria for a GBS-group to be a) residually nilpotent; b) residually torsion-free nilpotent; c) residually free.

Introduction

A group is called a *generalized Baumslag–Solitar group*, or a *GBS-group*, if it is the fundamental group of a graph of groups with infinite cyclic vertex and edge groups. GBS-groups have been actively studied in recent years [1, 5–10, 16, 17], and many of these investigations are devoted to establishing a connection between the algebraic properties of GBS-groups and the structure of the graphs defining them. The aim of this paper is to describe certain residual properties of a GBS-group in terms of the associated graph of groups. We strengthen some known results (for example, on the residual finiteness and the residual p-finiteness of GBS-groups) and give a criterion for the residual nilpotence of a GBS-group.

Let $\mathcal{C}$ be a class of groups. A group G is said to be *residually a* $\mathcal{C}$*-group* if, for any non-unit element $g \in G$, there exists a homomorphism σ of G onto a group of $\mathcal{C}$ such that $g\sigma \neq 1$. The most commonly considered situation is when $\mathcal{C}$ is the class of all finite groups, all finite p-groups (where p is a prime number), all nilpotent groups or all solvable groups. In these cases G is called *residually finite*, *residually* p*-finite*, *residually nilpotent* or *residually solvable* respectively.

We say that a class $\mathcal{C}$ of groups is *root* if it contains at least one non-unit group and is closed under taking subgroups, extensions, and Cartesian products of the form $\prod_{y \in Y} X_y$, where $X, Y \in \mathcal{C}$ and X_y is an isomorphic copy of X for every $y \in Y$. The notion of a root class was introduced by Gruenberg [12], and the above definition is equivalent to that given in [12]; see [25] for details.

The classes of all finite groups, all finite p-groups, all periodic groups of finite exponent, all solvable groups, and all torsion-free groups can serve as examples of root classes. It is also easy to see that the intersection of any number of root classes is again a root class. At the same time, the classes of all nilpotent groups, all torsion-free nilpotent groups, and all finite nilpotent groups are not root because they are not closed under taking extensions.

The main goal of this paper is to get necessary and sufficient conditions for a GBS-group to be residually a $\mathcal{C}$-group, where $\mathcal{C}$ is an arbitrary, not any specific root class. The sense of studying residually $\mathcal{C}$-groups, where $\mathcal{C}$ is an arbitrary class of groups, is to get many

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results at once using the same reasoning. This approach was originally proposed in [12,21] and turned out to be very fruitful in the study of free constructions of groups in the case when $\mathcal{C}$ was a root class; see, e. g. [2, 25–31].

For a root class $\mathcal{C}$ and a GBS-group G, we give a criterion for G to be residually a $\mathcal{C}$-group if $\mathcal{C}$ consists of only periodic groups (Theorem 3) and a sufficient condition for G to be residually a $\mathcal{C}$-group if $\mathcal{C}$ contains at least one non-periodic group (Theorem 4). Using the first of these results, we prove criteria for a GBS-group to be a) residually nilpotent (Theorem 5); b) residually torsion-free nilpotent and residually free (Theorem 6). All the proofs use only the classical methods of combinatorial group theory and the basic concepts of graph theory.

1. Statement of results

First, we formulate a number of known statements on the residual properties of (ordinary) Baumslag–Solitar groups since they complement the results obtained in this paper and are used in the proofs of some of them.

Recall that a *Baumslag–Solitar group* is a group with the presentation

$$\mathrm{BS}(m,n) = \langle a, b;\ a^{-1}b^m a = b^n\rangle,$$

where m and n are non-zero integers. Since $\mathrm{BS}(m,n)$, $\mathrm{BS}(n,m)$, and $\mathrm{BS}(-m,-n)$ are pairwise isomorphic, we can assume without loss of generality that $|n| \geqslant m > 0$.

We also recall that if ρ is a set of primes, then a *ρ-number* is an integer, all prime divisors of which belong to ρ, and a *ρ-group* is a periodic group, the orders of all elements of which are ρ-numbers. If ρ consists of one number p, then we write "p-" instead of "$\{p\}$-".

For a class of groups $\mathcal{C}$ consisting only of periodic groups, let $\rho(\mathcal{C})$ denote the set of primes defined as follows: $p \in \rho(\mathcal{C})$ if and only if there exists a $\mathcal{C}$-group X such that p divides the order of some element of X.

Theorem 1. [30] *Let $\mathcal{C}$ be a root class of groups consisting only of periodic groups and closed under taking quotient groups, $\rho(\mathcal{C})$ be the set of primes defined above. Then the following statements hold.*

1. *If $1 < m < |n|$, then $\mathrm{BS}(m,n)$ is not residually a $\mathcal{C}$-group.*
2. *$\mathrm{BS}(m,m)$ is residually a $\mathcal{C}$-group if and only if m is a $\rho(\mathcal{C})$-number.*
3. *$\mathrm{BS}(m,-m)$ is residually a $\mathcal{C}$-group if and only if m is a $\rho(\mathcal{C})$-number and $2 \in \rho(\mathcal{C})$.*
4. *$\mathrm{BS}(1,n)$, $|n| \neq 1$, is residually a $\mathcal{C}$-group if and only if there exists $p \in \rho(\mathcal{C})$ not dividing n and such that the order of the image $n + p\mathbb{Z}$ of n in the multiplicative group of $\mathbb{Z}_p$ is a $\rho(\mathcal{C})$-number.*

We note that, in fact, Theorem 1 is true for an arbitrary root class consisting of periodic groups (see Proposition 2.3 below).

Theorem 2. [22] *$\mathrm{BS}(m,n)$ is residually nilpotent if and only if either $m = 1$ and $n \neq 2$, or $m > 1$ and $n = \varepsilon m$ for some $\varepsilon = \pm 1$.*

Now we turn to generalized Baumslag–Solitar groups. For a detailed description of the fundamental groups of graphs of groups and GBS-groups, we refer the reader to Sections 3 and 4. Here we just recall that each GBS-group can be defined by a *graph with labels* (which are non-zero integers associated with the edges of the graph). This graph is called *reduced* if each edge that is not a loop has labels different from ± 1. It is easy to show that any GBS-group can be defined by a reduced labeled graph (see Section 4 for details).

A GBS group is called *elementary* if it is isomorphic to $\mathbb{Z}$, $\mathrm{BS}(1,1) \cong \mathbb{Z}\times\mathbb{Z}$ or $\mathrm{BS}(1,-1)$ [17, p. 6]. It is known that a GBS-group is solvable if it is elementary or isomorphic to $\mathrm{BS}(1,q)$, where $q \neq \pm 1$ [6].

Until the end of this section, let Γ be a non-empty finite connected graph, $\mathcal{L}(\Gamma)$ be a labeled graph over Γ, G be the GBS-group defined by $\mathcal{L}(\Gamma)$, and $\Delta\colon G \to \mathbb{Q}^*$ be the *modular homomorphism* of G (defined if G is not elementary; see Section 4). The theorems and corollaries formulated below are the main results of this paper.

Theorem 3. *Let $\mathcal{C}$ be a root class of groups consisting only of periodic groups and $\rho(\mathcal{C})$ be the set of primes defined above. Suppose also that G is not solvable and $\mathcal{L}(\Gamma)$ is reduced.*

1. *If $\operatorname{Im}\Delta = \{1\}$, then G is residually a $\mathcal{C}$-group if and only if all the labels of $\mathcal{L}(\Gamma)$ are $\rho(\mathcal{C})$-numbers.*
2. *If $\operatorname{Im}\Delta = \{1, -1\}$, then G is residually a $\mathcal{C}$-group if and only if all the labels of $\mathcal{L}(\Gamma)$ are $\rho(\mathcal{C})$-numbers and $2 \in \rho(\mathcal{C})$.*
3. *If $\operatorname{Im}\Delta \not\subseteq \{1, -1\}$, then G is not residually a $\mathcal{C}$-group.*

Corollary 1. *The following statements are equivalent.*

1. *G is residually finite.*
2. *G is residually finite solvable.*
3. *Either G is solvable, or it is not and $\operatorname{Im}\Delta \subseteq \{1, -1\}$.*

Corollary 2. *Let G be not solvable, $\mathcal{L}(\Gamma)$ be reduced, and ρ be a non-empty set of primes. The following statements are equivalent.*

1. *G is residually a finite ρ-group.*
2. *G is residually a finite solvable ρ-group.*
3. *G is residually a periodic ρ-group of finite exponent.*
4. *G is residually a periodic solvable ρ-group of finite exponent.*
5. *$\operatorname{Im}\Delta \subseteq \{1, -1\}$, all the labels of $\mathcal{L}(\Gamma)$ are ρ-numbers, and if $-1 \in \operatorname{Im}\Delta$, then $2 \in \rho$.*

Theorem 4. *Let $\mathcal{C}$ be a root class of groups containing at least one non-periodic group.*

1. *If G is elementary, then it is a torsion-free $\mathcal{C}$-group.*
2. *Let G be not elementary and Q be the subring of $\mathbb{Q}$ generated by $\operatorname{Im}\Delta$. If the additive group of Q belongs to $\mathcal{C}$, then G is residually a torsion-free $\mathcal{C}$-group. In particular, if $\operatorname{Im}\Delta \subseteq \{1, -1\}$ or $\mathcal{C}$ is closed under taking quotient groups, then G is residually a torsion-free $\mathcal{C}$-group.*

Corollary 3. *An arbitrary GBS-group is residually a torsion-free solvable group.*

The largest cyclic normal subgroup of G is called the *cyclic radical* of this group and is denoted by $C(G)$. The cyclic radical exists if G is not isomorphic to $\mathrm{BS}(1, 1)$ or $\mathrm{BS}(1, -1)$ [7, p. 1808].

Theorem 5. *Let G be not solvable and $\mathcal{L}(\Gamma)$ be reduced.*

1. *If $\operatorname{Im}\Delta = \{1\}$, then G is residually nilpotent if and only if it is residually a finite p-group for some prime number p.*
2. *If $\operatorname{Im}\Delta = \{1, -1\}$, then the following statements are equivalent:*

   a) G is residually nilpotent;

   b) G is residually a finite nilpotent $\{2, p\}$-group for some prime number p (which can be equal to 2);

   c) all the labels of $\mathcal{L}(\Gamma)$ are p-numbers for some prime number p, and if $p \neq 2$, then every elliptic element that is conjugate to its inverse belongs to $C(G)$.
3. *If $\operatorname{Im}\Delta \not\subseteq \{1, -1\}$, then G is not residually nilpotent.*

The definition of elliptic element can be found in Section 4. We note that if $\mathcal{L}(\Gamma)$ is reduced, all its labels are p-numbers for some prime number $p \neq 2$, and $\operatorname{Im}\Delta = \{1, -1\}$, then there is an algorithm that checks whether every elliptic element that is conjugate to its inverse belongs to $C(G)$; this algorithm is given at the end of Section 6.

Theorem 6. *Let G be not cyclic. The following statements are equivalent.*
1. *G is residually torsion-free nilpotent.*
2. *G is residually free.*
3. *G is isomorphic to the direct product of a free group and an infinite cyclic group.*

Thus, Theorems 1 and 3 (in combination with Proposition 2.3) give a criterion for G to be residually a $\mathcal{C}$-group, where $\mathcal{C}$ is a root class of groups consisting only of periodic groups, while Theorems 2 and 5 do a criterion for the residual nilpotence of G thereby answering [3, Question 4]. We note that Corollaries 1, 2, and 3 strengthen and generalize Corollary 7.7 of [17], Theorem 1 of [10], and Corollary 3 of [23] respectively. The rest of the paper is devoted to the proofs of the formulated statements.

2. Some auxiliary statements

Throughout this section, if $\mathcal{C}$ is a class of groups consisting only of periodic groups, then $\rho(\mathcal{C})$ denotes the set of primes defined above.

Proposition 2.1. *Let $\mathcal{C}$ be a class of groups consisting only of periodic groups and closed under taking subgroups and extensions. Then any finite solvable $\rho(\mathcal{C})$-group belongs to $\mathcal{C}$.*

Proof. Let X be a finite solvable $\rho(\mathcal{C})$-group. Then there is a polycyclic series $\mathcal{S}$ in X such that the orders of all its factors belong to $\rho(\mathcal{C})$. Let p be the order of some factor F. By the definition of $\rho(\mathcal{C})$, p divides the order of an element of some $\mathcal{C}$-group, and so this group contains an element, say x, of order p. Then F is isomorphic to the cyclic subgroup $\langle x \rangle$ generated by x, and since $\mathcal{C}$ is closed under taking subgroups, $\langle x \rangle \in \mathcal{C}$. Thus, all the factors of $\mathcal{S}$ are $\mathcal{C}$-groups, and $X \in \mathcal{C}$ because $\mathcal{C}$ is closed under taking extensions. □

Proposition 2.2. [27, Proposition 17] *Let $\mathcal{C}$ be a root class of groups consisting only of periodic groups. Then any $\mathcal{C}$-group is of finite exponent.*

Proposition 2.3. *Theorem 1 is valid for any root class $\mathcal{C}$ consisting of periodic groups.*

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote the class of finite solvable $\rho(\mathcal{C})$-groups and the class of periodic $\rho(\mathcal{C})$-groups of finite exponent respectively. It follows from Propositions 2.1 and 2.2 that $\mathcal{C}_1 \subseteq \mathcal{C} \subseteq \mathcal{C}_2$. One can easily verify that $\mathcal{C}_1$ and $\mathcal{C}_2$ are root classes closed under taking quotient groups. It is also obvious that $\rho(\mathcal{C}_1) = \rho(\mathcal{C}) = \rho(\mathcal{C}_2)$. Therefore, if G is residually a $\mathcal{C}$-group, then it is residually a $\mathcal{C}_2$-group and satisfies the necessary conditions from Theorem 1 (depending only on m, n, and $\rho(\mathcal{C})$). In the same way, if the sufficient conditions from Theorem 1 are satisfied (which also depend only on m, n, and $\rho(\mathcal{C})$), then G is residually a $\mathcal{C}_1$-group and hence is residually a $\mathcal{C}$-group. □

Proposition 2.4. *Let $\mathcal{C}$ be an arbitrary root class of groups. Then the following statements are true.*
1. *Every free group is residually a $\mathcal{C}$-group.*
2. *The direct product of any two residually $\mathcal{C}$-groups is residually a $\mathcal{C}$-group.*
3. *Any extension of a residually $\mathcal{C}$-group by a $\mathcal{C}$-group is residually a $\mathcal{C}$-group.*

Proof. Statements 1 and 3 follow from [2, Theorem 1] and [12, Lemma 1.5] respectively. Statement 2 is verified directly. □

As usual, by a *group of prime power order* we mean a finite group whose order is a power of some prime number.

Proposition 2.5. *Let X be a finitely generated group. If X is residually nilpotent, then it is residually a group of prime power order. If X is residually torsion-free nilpotent, then it is residually a finite p-group for every prime number p.*

Proof. Since X is finitely generated, it is residually a finitely generated nilpotent group or residually a finitely generated torsion-free nilpotent group. Therefore, the required statement follows from [12, Theorem 2.1]. □

Proposition 2.6. *Let X be a group, x and y be elements of X such that $x^{-1}yx = y^{-1}$. Let also p be a prime number and ψ be a homomorphism of X onto a finite p-group. If $p \neq 2$, then $y\psi = 1$.*

Proof. Let $\gamma_i(X)$ denote the i-th member of the lower central series of X, and let r be the order of $y\psi$. It is easy to verify that $y^{2^i} \in \gamma_{i+1}(X)$ for every $i \geqslant 0$. Since $X\psi$ is nilpotent, it follows that $y^{2^i}\psi = 1$ for some $i \geqslant 0$. If $p \neq 2$, then $1 = (r, 2^i) = r\alpha + 2^i\beta$ for suitable integers α, β and hence $y\psi = (y\psi)^{r\alpha+2^i\beta} = 1$ as required. □

3. The fundamental group of a graph of groups

Let Γ be a non-empty undirected graph with a vertex set V and an edge set E (loops and multiple edges are allowed). To turn Γ into a *graph of groups*, we denote the vertices of Γ that are the ends of an edge $e \in E$ by $e(1)$, $e(-1)$ and assign to each vertex $v \in V$ some group G_v, to each edge $e \in E$ a group H_e and injective homomorphisms $\varphi_{+e}\colon H_e \to G_{e(1)}$, $\varphi_{-e}\colon H_e \to G_{e(-1)}$. We denote the resulting graph of groups by $\mathcal{G}(\Gamma)$, the subgroups $H_e\varphi_{+e}$ and $H_e\varphi_{-e}$ ($e \in E$) by H_{+e} and H_{-e}. We also call G_v ($v \in V$), H_e ($e \in E$), and $H_{\varepsilon e}$ ($e \in E$, $\varepsilon = \pm 1$) *vertex groups*, *edge groups*, and *edge subgroups* respectively. All designations introduced in this paragraph are assumed to be fixed until the end of the section.

It should be noted that an edge e of $\mathcal{G}(\Gamma)$ is associated with two different homomorphisms φ_{+e}, φ_{-e} even in the case when e is a loop, i. e. $e(1) = e(-1)$. Therefore, we can consider $\mathcal{G}(\Gamma)$ as a directed graph assuming that φ_{+e} corresponds to the origin while φ_{-e} does to the terminus of e.

Let F be a maximal forest in Γ and E_F be the set of edges of Γ that belong to F. The *fundamental group* of $\mathcal{G}(\Gamma)$ is the group $\pi_1(\mathcal{G}(\Gamma))$ whose generators are the generators of G_v ($v \in V$) and symbols t_e ($e \in E \setminus E_F$) and whose defining relations are the relations of G_v ($v \in V$) and all possible relations of the form

$$h_e\varphi_{+e} = h_e\varphi_{-e} \quad (e \in E_F,\ h_e \in H_e),$$
$$t_e^{-1}(h_e\varphi_{+e})t_e = h_e\varphi_{-e} \quad (e \in E \setminus E_F,\ h_e \in H_e),$$

where $h_e\varphi_{\varepsilon e}$ ($\varepsilon = \pm 1$) is the word in the generators of $G_{e(\varepsilon)}$ defining the image of h_e under $\varphi_{\varepsilon e}$ [24, § 5.1].

Obviously, the presentation of $\pi_1(\mathcal{G}(\Gamma))$ depends on the choice of F. It is known, however, that all the groups with the presentations corresponding to different maximal forests of Γ are isomorphic [24, § 5.1]. This allows us to talk about the fundamental group of a graph of groups without mentioning a specific maximal forest. It is also known that, for each vertex $v \in V$, the identity mapping of the generators of G_v to $\pi_1(\mathcal{G}(\Gamma))$ defines an injective homomorphism [24, § 5.2] and so G_v can be considered as a subgroup of $\pi_1(\mathcal{G}(\Gamma))$. This easily implies

Proposition 3.1. *Let Γ' be an arbitrary connected subgraph of Γ, T' be a maximal subtree of Γ', and $\mathcal{G}(\Gamma')$ be the graph of groups whose vertices and edges correspond to the same groups and homomorphisms as in $\mathcal{G}(\Gamma)$. Then there exists a maximal forest F in Γ such that $F \cap \Gamma' = T'$. If the presentations of $\pi_1(\mathcal{G}(\Gamma))$ and $\pi_1(\mathcal{G}(\Gamma'))$ correspond to the indicated forest F and T', then the identity mapping of the generators of $\pi_1(\mathcal{G}(\Gamma'))$ to $\pi_1(\mathcal{G}(\Gamma))$ defines an injective homomorphism.*

The next statement is a special case of [29, Proposition 13].

Proposition 3.2. *Let Γ be finite and N be a normal subgroup of $\pi_1(\mathcal{G}(\Gamma))$ that meets each subgroup G_v $(v \in V)$ trivially. Then N is free.*

As usual, we say that a group possesses some property *locally* if each of its finitely generated subgroups possesses this property.

Proposition 3.3. [15, Theorem 1] *Let Γ be connected, every G_v $(v \in V)$ locally satisfy a non-trivial identity, and, for each $e \in E$, $[G_{e(1)} : H_{+e}] \neq 1 \neq [G_{e(-1)} : H_{-e}]$, $[G_{e(1)} : H_{+e}] \cdot [G_{e(-1)} : H_{-e}] > 4$. If $\pi_1(\mathcal{G}(\Gamma))$ is locally residually nilpotent, then there exists a prime number p such that, for any $e \in E$, $\varepsilon = \pm 1$, $H_{\varepsilon e}$ is p'-isolated in $G_{e(\varepsilon)}$ (i. e., for each $g \in G_{e(\varepsilon)}$ and for each prime number q, it follows from $g^q \in H_{\varepsilon e}$ and $p \neq q$ that $g \in H_{\varepsilon e}$).*

Let Γ consist of two vertices and an edge e connecting them. Recall that in this case $\pi_1(\mathcal{G}(\Gamma))$ is said to be the *free product of $G_{e(1)}$ and $G_{e(-1)}$ with H_{+e} and H_{-e} amalgamated*. The groups $G_{e(1)}$ and $G_{e(-1)}$ are called the *free factors* of this free product (the terminology used here and below and concerning free products with amalgamated subgroups and HNN-extensions follows the monographs [18, 20]). The presentation of an element $g \in \pi_1(\mathcal{G}(\Gamma))$ in the form $g = g_1 \dots g_n$, $n \geqslant 1$, is said to be *reduced* if every multiplier g_i belongs to one of the groups $G_{e(1)}$, $G_{e(-1)}$ and no two neighboring multipliers g_i, g_{i+1} lie simultaneously in $G_{e(1)}$ or $G_{e(-1)}$. The number n is called the *length* of this reduced form. The normal form theorem for generalized free products (see, e. g. [20, Theorem 4.4]) implies that if an element $g \in \pi_1(\mathcal{G}(\Gamma))$ has at least one reduced form of length greater than 1, then it does not belong to any of the free factors $G_{e(1)}$, $G_{e(-1)}$ and, in particular, differs from 1.

If Γ has only one vertex v and at least one loop, then $\pi_1(\mathcal{G}(\Gamma))$ is said to be the *HNN-extension of G_v with the stable letters t_e $(e \in E)$*. The group G_v is called the *base group* of this HNN-extension. In this case, by a *reduced form* of an element $g \in \pi_1(\mathcal{G}(\Gamma))$ we mean the product $g = g_0 t_{e_{i_1}}^{\varepsilon_1} g_1 \dots t_{e_{i_n}}^{\varepsilon_n} g_n$, where $n \geqslant 0$, $g_0, g_1, \dots, g_n \in G_v$, $e_{i_1}, \dots, e_{i_n} \in E$, $\varepsilon_1, \dots, \varepsilon_n \in \{1, -1\}$, and, for each $k \in \{1, \dots, n-1\}$, if $i_k = i_{k+1}$ and $\varepsilon_k = -\varepsilon_{k+1}$, then $g_k \notin H_{-\varepsilon_k e_{i_k}}$. As above, n is called the *length* of this reduced form. It is known [4] that if an element $g \in \pi_1(\mathcal{G}(\Gamma))$ has at least one reduced form of length greater than 0, then it does not belong to the base group G_v and, in particular, differs from 1.

Proposition 3.4. *Let $P(m,n) = \langle x, y;\ x^m = y^n \rangle$, $1 < |m|, |n|$, $\mathcal{C}$ be an arbitrary class of groups consisting only of periodic groups, and $\rho(\mathcal{C})$ be the set of primes defined in Section 1. If $P(m,n)$ is residually a $\mathcal{C}$-group, then m and n are $\rho(\mathcal{C})$-numbers.*

Proof. Suppose that m is not a $\rho(\mathcal{C})$-number, i. e. there exists a prime number $p \notin \rho(\mathcal{C})$ such that $p \mid m$. Let $k = m/p$ and $z = [x^k, y]$.

Obviously, $P(m,n)$ is the free product of the infinite cyclic groups $\langle x \rangle$ and $\langle y \rangle$ with the subgroups $\langle x^m \rangle$ and $\langle y^n \rangle$ amalgamated. Since $|k| < |m|$ and $1 < |n|$, then $x^k \notin \langle x^m \rangle$ and $y \notin \langle y^n \rangle$. Therefore, z has a reduced form of length 4 and hence differs from 1.

Let ψ be an arbitrary homomorphism of $P(m,n)$ onto a $\mathcal{C}$-group. Then the order q of $x\psi$ is finite and is a $\rho(\mathcal{C})$-number. Since $p \notin \rho(\mathcal{C})$, then $1 = (p,q) = \alpha p + \beta q$ for some integers α, β and $x^k\psi = (x^k\psi)^{\alpha p + \beta q} = (x^k\psi)^{\alpha p} = (x^m\psi)^\alpha = (y^n\psi)^\alpha$. Therefore, $z\psi = 1$. Since ψ is chosen arbitrarily, it follows that $P(m,n)$ is not residually a $\mathcal{C}$-group.

Similar arguments prove that if $P(m,n)$ is residually a $\mathcal{C}$-group, then n is a $\rho(\mathcal{C})$-number. □

If all the vertex and edge groups of $\mathcal{G}(\Gamma)$ are infinite cyclic and their generators g_v $(v \in V)$ and h_e $(e \in E)$ are fixed, then the homomorphism $\varphi_{\varepsilon e}$ $(e \in E, \varepsilon = \pm 1)$ is uniquely defined by a number $\lambda(\varepsilon e) \in \mathbb{Z} \setminus \{0\}$ such that $g_{e(\varepsilon)}^{\lambda(\varepsilon e)} = h_e \varphi_{\varepsilon e}$. Therefore, instead of $\mathcal{G}(\Gamma)$, we can consider a *labeled graph* $\mathcal{L}(\Gamma)$, which is obtained from Γ by associating each edge $e \in E$ with non-zero integers $\lambda(+e)$ and $\lambda(-e)$.

If all the vertex and edge groups of $\mathcal{G}(\Gamma)$ are finite cyclic, then $\mathcal{G}(\Gamma)$ can be replaced by a graph $\mathcal{M}(\Gamma)$, in which labels are assigned not only to the edges, but also to the vertices: the label $\mu(v)$ at a vertex v means that the vertex group G_v is of order $\mu(v)$. Of course, for each edge $e \in E$, the equality $|\mu(e(1))/\lambda(+e)| = |\mu(e(-1))/\lambda(-e)|$ must hold. We need such graphs in our proofs.

We call the group defined by $\mathcal{L}(\Gamma)$ ($\mathcal{M}(\Gamma)$) the *fundamental group of the labeled graph* $\mathcal{L}(\Gamma)$ ($\mathcal{M}(\Gamma)$) and denote it by $\pi_1(\mathcal{L}(\Gamma))$ (respectively $\pi_1(\mathcal{M}(\Gamma))$). In order to avoid ambiguity when specifying the presentation of this group, $\mathcal{L}(\Gamma)$ (and $\mathcal{M}(\Gamma)$) must be considered directed. In each of these graphs, the ends of an edge e are, as before, denoted by $e(1)$ and $e(-1)$.

4. GBS-groups and their properties

It follows from the previous section that each GBS-group can be defined by a labeled graph $\mathcal{L}(\Gamma)$ for some finite connected graph Γ and vice versa, each labeled graph $\mathcal{L}(\Gamma)$ over a non-empty finite connected graph Γ defines some GBS-group. Until the end of the paper, we assume that $\Gamma = (V, E)$ is an arbitrary non-empty finite connected graph with a vertex set V and an edge set E, $\mathcal{L}(\Gamma)$ is a graph with labels $\lambda(\varepsilon e)$ ($e \in E$, $\varepsilon = \pm 1$), and G is the corresponding GBS-group with the vertex groups $G_v = \langle g_v \rangle$ ($v \in V$) and the edge subgroups $H_{\varepsilon e} = \langle g_{e(\varepsilon)}^{\lambda(\varepsilon e)} \rangle$ ($e \in E$, $\varepsilon = \pm 1$). If Γ' is a subgraph of Γ, then by $\mathcal{L}(\Gamma')$ we denote the labeled graph, the edges of which are associated with the same labels as in $\mathcal{L}(\Gamma)$.

As mentioned above, the graph $\mathcal{L}(\Gamma)$ is called *reduced* if, for each $e \in E$, $\varepsilon = \pm 1$, the equality $|\lambda(\varepsilon e)| = 1$ implies that e is a loop [11, p. 224]. Suppose that $\mathcal{L}(\Gamma)$ is not reduced. Then it contains an edge e such that $e(1) \neq e(-1)$ and $|\lambda(\varepsilon e)| = 1$ for some $\varepsilon = \pm 1$. Let us choose a maximal subtree of Γ containing e. Then $g_{e(\varepsilon)} = g_{e(-\varepsilon)}^{\lambda(\varepsilon e)\lambda(-\varepsilon e)}$ in G and hence the generator $g_{e(\varepsilon)}$ can be excluded from the presentation of G. In $\mathcal{L}(\Gamma)$, this operation corresponds to the contraction of e with preliminary multiplication of all the labels around the vertex $e(\varepsilon)$ by $\lambda(\varepsilon e)\lambda(-\varepsilon e)$. Such a transformation of $\mathcal{L}(\Gamma)$ is called an *elementary collapse* (see [16, p. 480]). Since Γ is finite, then $\mathcal{L}(\Gamma)$ can always be reduced by performing a finite number of elementary collapses.

If we replace the generator of a certain vertex group with its inverse, then all the labels around the corresponding vertex change sign. Similarly, replacing the generator of a certain edge group with the inverse leads to a change in the signs of the labels at the ends of this edge. The listed changes of the generators induce isomorphisms of G, and the corresponding graph transformations are called *admissible changes of signs* [16, p. 479].

Let some maximal subtree T of Γ be fixed. It is easy to see that one can make all the labels at the ends of the edges of T positive by applying suitable admissible sign changes. We call the resulting graph $\mathcal{L}(\Gamma)$ T*-positive*.

An element $a \in G$ is said to be *elliptic* if it is conjugate to an element of some vertex group. If G is not elementary, then the ellipticity of an element does not depend on the choice of the graph $\mathcal{L}(\Gamma)$ defining G, the set of elliptic elements is invariant under automorphisms of G, and any two elliptic elements a, $b \in G$ are commensurable, i. e. $\langle a \rangle \cap \langle b \rangle \neq 1$ [16, Lemma 2.1, Corollary 2.2]. This allows us to define the mapping $\Delta\colon G \to \mathbb{Q}^*$ as follows.

Let $g \in G$ be an arbitrary element. Take a non-unit elliptic element a. Then the element $g^{-1}ag$ is also elliptic and hence there exist numbers m and n such that $g^{-1}a^m g = a^n$. We put $\Delta(g) = n/m$.

This definition does not depend on the choice of a, m, and n [14]. The constructed mapping Δ is called the *modular homomorphism* of G. The notation Δ is used below without special explanations.

Proposition 4.1. [17, Propositions 7.5, 7.11] *Let G be non-solvable and $n/m \in \operatorname{Im}\Delta\setminus\{1\}$ be a rational number written in lowest terms. Then G contains a subgroup isomorphic to $\mathrm{BS}(m,n)$.*

Proposition 4.2. [17, Lemma 7.6] *If G is non-solvable and contains a subgroup isomorphic to $\mathrm{BS}(1,n)$, $|n| \neq 1$, then it contains a subgroup isomorphic to $\mathrm{BS}(q,qn)$, where q is some prime number.*

Proposition 4.3. *Let Γ be a tree and $\mathcal{I}$ be a non-empty finite set of indices, which is the disjoint union of the set $\big\{(e,\varepsilon) \mid e \in E,\ \varepsilon = \pm 1\big\}$ and some set $\mathcal{J}$. Let also $\Sigma = \{H_i \mid i \in \mathcal{I}\}$ be a family of subgroups of G_v $(v \in V)$ and $\nu\colon \mathcal{I} \to V$ be a function such that, for any $i \in \mathcal{I}$, $1 \neq H_i \leqslant G_{\nu(i)}$ and if $i = (e,\varepsilon)$ for some $e \in E$, $\varepsilon = \pm 1$, then $H_i = H_{e\varepsilon}$ and $\nu(i) = e(\varepsilon)$. Finally, let $K = \bigcap_{i\in\mathcal{I}} H_i$ and $\chi(i) = [G_{\nu(i)} : H_i]$. Then the following statements hold.*

1. *$K \leqslant \bigcap_{v\in V} G_v$ and therefore the numbers $\mu(v) = [G_v : K]$ $(v \in V)$ are defined.*
2. *$K \neq 1$ and therefore all the numbers $\mu(v)$ $(v \in V)$ are finite.*
3. *The least common multiple μ of $\mu(v)$ $(v \in V)$ divides $\prod_{i\in\mathcal{I}} \chi(i)$.*

Proof. 1. We note that $V = \{\nu(i) \mid i \in \mathcal{I}\}$. Indeed, if Γ consists of one vertex v, then $v = \nu(i)$ for all $i \in \mathcal{I}$ and the desired equality holds because $\mathcal{I}$ is non-empty. Otherwise, each vertex is incident to at least one edge, and so, for any $v \in V$, there exist $e \in E$, $\varepsilon = \pm 1$ such that $v = e(\varepsilon) = \nu(i)$, where $i = (e,\varepsilon) \in \mathcal{I}$. Hence,

$$K = \bigcap_{i\in\mathcal{I}} H_i \leqslant \bigcap_{i\in\mathcal{I}} G_{\nu(i)} = \bigcap_{v\in V} G_v,$$

as required.

2. We put $H = \bigcap_{v\in V} G_v$ and use induction on the number of vertices in Γ to show that H is an infinite cyclic subgroup. If Γ contains only one vertex v, then $H = G_v$ and the required statement is obvious. Therefore, we further assume that Γ contains more than one vertex and so $E \neq \varnothing$.

Let $f \in E$ be an arbitrary edge and $\Gamma - f$ be the graph that is obtained from Γ by removing f. Since Γ is a tree, $\Gamma - f$ has exactly two connected components. For every $\varepsilon = \pm 1$, denote by Γ_ε the connected component of $\Gamma - f$ that contains $f(\varepsilon)$ and by V_ε the vertex set of Γ_ε. By the inductive hypothesis, the subgroup $H_\varepsilon = \bigcap_{v\in V_\varepsilon} G_v$ is infinite cyclic, and hence $H_\varepsilon \cap H_{\varepsilon f} \neq 1$ as the intersection of two non-trivial subgroups of $G_{f(\varepsilon)}$.

By Proposition 3.1, the free product of the groups $G_{f(1)}$ and $G_{f(-1)}$ with the subgroups H_{+f} and H_{-f} amalgamated is embedded into G by means of the identity mapping of the generators. Therefore, the equalities $H_{+f} = G_{f(1)} \cap G_{f(-1)} = H_{-f}$ hold in G [20, Theorem 4.4.3], and

$$H = H_1 \cap H_{-1} = (H_1 \cap G_{f(1)}) \cap (H_{-1} \cap G_{f(-1)}) = (H_1 \cap H_{+f}) \cap (H_{-1} \cap H_{-f})$$

is the intersection of two non-trivial subgroups of the infinite cyclic group $H_{+f} = H_{-f}$.

Thus, $H \neq 1$, and so $H_i \cap H$ is an infinite cyclic subgroup of $G_{\nu(i)}$ for any $i \in \mathcal{I}$. As noted above, $V = \{\nu(i) \mid i \in \mathcal{I}\}$, hence $K = \bigcap_{i\in\mathcal{I}}(H_i \cap G_{\nu(i)}) = \bigcap_{i\in\mathcal{I}}(H_i \cap H)$. Since all the subgroups $H_i \cap H$ $(i \in \mathcal{I})$ lie in H and $\mathcal{I}$ is finite, it follows that $K \neq 1$.

3. We again use induction on the number of vertices in Γ. If Γ contains only one vertex v, then $\mu = \mu(v) = [G_v : K]$ and $\chi(i) = [G_v : H_i]$. Therefore, the required statement follows from the relation $[G_v : \bigcap_{i\in\mathcal{I}} H_i] \mid \prod_{i\in\mathcal{I}}[G_v : H_i]$, and we further assume that Γ has at least two vertices.

Let us choose an arbitrary edge $f \in E$ and denote by Γ_ε ($\varepsilon = \pm 1$) the connected component of the graph $\Gamma - f$ that contains $f(\varepsilon)$. Let also V_ε be the vertex set of Γ_ε, $\mathcal{I}_\varepsilon = \{i \mid i \in \mathcal{I},\ \nu(i) \in V_\varepsilon\}$, $K_\varepsilon = \bigcap_{i \in \mathcal{I}_\varepsilon} H_i$, and $p_\varepsilon = \prod_{i \in \mathcal{I}_\varepsilon} \chi(i)$. Then $\prod_{i \in \mathcal{I}} \chi(i) = p_1 p_{-1}$ and the equality $K = K_1 \cap K_{-1}$ holds in G.

By the definition of ν, the set of indices $\mathcal{I}_\varepsilon$ contains the pair (f, ε) and is therefore non-empty. It is also easy to see that Γ_ε, $\mathcal{I}_\varepsilon$, $\Sigma_\varepsilon = \{H_i \mid i \in \mathcal{I}_\varepsilon\}$, and $\nu_\varepsilon = \nu|_{\mathcal{I}_\varepsilon}$ satisfy all the conditions of the proposition. Hence, the numbers $\mu_\varepsilon(v) = [G_v : K_\varepsilon]$ ($v \in V_\varepsilon$) are defined and finite in view of Statements 1, 2, while their least common multiple μ_ε divides p_ε by the inductive hypothesis.

Since $K_1 \leqslant H_{+f}$, $K_{-1} \leqslant H_{-f}$, and $H_{+f} = G_{f(1)} \cap G_{f(-1)} = H_{-f}$, then $K_1 K_{-1} \leqslant G_{f(1)} \cap G_{f(-1)}$ and $[K_1 K_{-1} : K_\varepsilon] \mid [G_{f(\varepsilon)} : K_\varepsilon] = \mu_\varepsilon(f(\varepsilon)) \mid \mu_\varepsilon$ for every $\varepsilon = \pm 1$. Therefore, for any $\varepsilon = \pm 1$, $v \in V_\varepsilon$, we have

$$\mu(v) = [G_v : K] = [G_v : K_\varepsilon][K_\varepsilon : K_\varepsilon \cap K_{-\varepsilon}] = [G_v : K_\varepsilon][K_\varepsilon K_{-\varepsilon} : K_{-\varepsilon}] \mid \mu_\varepsilon \mu_{-\varepsilon} \mid p_1 p_{-1}.$$

Hence, the least common multiple of the numbers $\mu(v)$ ($v \in V = V_1 \cup V_{-1}$) also divides the product $p_1 p_{-1}$. □

Proposition 4.4. *Let G be non-elementary, T be a maximal subtree of Γ, E_T be the edge set of T, and $K = \bigcap_{e \in E,\, \varepsilon = \pm 1} H_{\varepsilon e}$. Then the following statements hold.*

1. *$K \leqslant \bigcap_{v \in V} G_v$ and therefore the numbers $\mu(v) = [G_v : K]$ ($v \in V$) are defined.*
2. *$K \neq 1$ and therefore all the numbers $\mu(v)$ ($v \in V$) are finite.*
3. *The least common multiple μ of $\mu(v)$ ($v \in V$) divides $\prod_{e \in E,\, \varepsilon = \pm 1} \lambda(\varepsilon e)$.*
4. *If $\mathcal{L}(\Gamma)$ is T-positive, then $g_v^{\mu(v)} = g_w^{\mu(w)}$ for any $v, w \in V$ and $\lambda(+e)/\mu(e(1)) = \lambda(-e)/\mu(e(-1))$ for any $e \in E_T$.*
5. *If $\operatorname{Im} \Delta \subseteq \{1, -1\}$, then K is normal in G and the centralizer of K in G coincides with $\Delta^{-1}(1)$.*
6. *If $\operatorname{Im} \Delta \subseteq \{1, -1\}$ and τ is a homomorphism of G such that $\ker \tau \cap G_v = K$ for all $v \in V$, then $\ker \tau$ is an extension of K by a free group.*
7. *If $\operatorname{Im} \Delta \subseteq \{1, -1\}$ and $\mathcal{L}(\Gamma)$ is reduced, then $C(G) = K$.*

Proof. 1, 2, 3. Since G is an HNN-extension of the tree product $P = \pi_1(\mathcal{L}(T))$ and all the groups G_v ($v \in V$) are contained in P, then Statements 1–3 are obtained by applying Proposition 4.3 to the tree T, the group P, the set

$$\mathcal{I} = \big\{(e, \varepsilon) \mid e \in E_T,\ \varepsilon = \pm 1\big\} \cup \big\{(e, \varepsilon) \mid e \in E \setminus E_T,\ \varepsilon = \pm 1\big\},$$

the family $\Sigma = \{H_{e\varepsilon} \mid e \in E,\ \varepsilon = \pm 1\}$, and the function $\nu\colon \mathcal{I} \to V$ such that $\nu(e, \varepsilon) = e(\varepsilon)$. It should only be noted that E and $\mathcal{I}$ are non-empty because G is not elementary. Besides, the relation $\mu \mid \prod_{e \in E,\, \varepsilon = \pm 1} |\lambda(\varepsilon e)|$, which follows from Statement 3 of Proposition 4.3, is equivalent to the required one.

4. We use induction on the length of the path connecting v and w in T. If $v = w$, the statement is obvious, so we assume that this path contains an edge e connecting v with some vertex u (which may coincide with w). Let also, for definiteness, $e(1) = v$ and $e(-1) = u$.

By the inductive hypothesis, $g_u^{\mu(u)} = g_w^{\mu(w)}$. Since $e \in E_T$, the equalities $g_v^{\lambda(+e)} = g_u^{\lambda(-e)}$ and $H_{+e} = H_{-e}$ hold in G. It follows that $[H_{+e} : K] = [H_{-e} : K] = k$ for some $k \geqslant 1$. Since $\mathcal{L}(\Gamma)$ is T-positive, then $[G_v : H_{+e}] = \lambda(+e)$ and $[G_u : H_{-e}] = \lambda(-e)$. Therefore,

$$\mu(v) = [G_v : K] = [G_v : H_{+e}][H_{+e} : K] = \lambda(+e)k,$$
$$\mu(u) = [G_u : K] = [G_u : H_{-e}][H_{-e} : K] = \lambda(-e)k,$$

and

$$g_v^{\mu(v)} = g_v^{\lambda(+e)k} = g_u^{\lambda(-e)k} = g_u^{\mu(u)} = g_w^{\mu(w)}.$$

Since the vertices v and w are chosen arbitrarily, the above equalities also imply that $\lambda(+e)/\mu(e(1)) = \lambda(-e)/\mu(e(-1))$ for any $e \in E_T$.

5. Let x be a generator of K. It is obvious that $[x, g_v] = 1$ for every $v \in V$. The element x is elliptic, so if $e \in E \setminus E_T$, then there exists a number $n \geqslant 1$ such that $t_e^{-1} x^n t_e = x^{\Delta(t_e)n}$. Since $\operatorname{Im}\Delta \subseteq \{1, -1\}$, $t_e^{-1} x t_e \in H_{-e}$, $x^{\Delta(t_e)} \in H_{-e}$, and H_{-e} is infinite cyclic, the last equality implies that $t_e^{-1} x t_e = x^{\Delta(t_e)}$. Therefore, K is normal in G.

If $g \in G$ is an arbitrary element, then $g^{-1}xg \in K$ because K is normal. It follows that $g^{-1}xg = x^{\Delta(g)}$, and hence $[g, x] = 1$ if and only if $\Delta(g) = 1$.

6. Consider the quotient group $\overline{G} = G/K$ and the labeled graph $\mathcal{M}(\Gamma)$ that is obtained from $\mathcal{L}(\Gamma)$ by assigning to each $v \in V$ the label $\mu(v) = [G_v : K]$. It is easy to see that $\overline{G}$ is isomorphic to $\pi_1(\mathcal{M}(\Gamma))$ and the vertex groups under this isomorphism correspond to the quotient groups G_v/K $(v \in V)$. Since $K \leqslant \ker\tau$, the mapping $\bar{\tau}\colon \overline{G} \to \operatorname{Im}\tau$ taking the coset gK $(g \in G)$ to $g\tau$ is well defined and is a surjective homomorphism. It follows from the equalities $\ker\tau \cap G_v = K$ $(v \in V)$ that $\ker\bar{\tau} \cap G_v/K = 1$ for all $v \in V$. Therefore, by Proposition 3.2, $\ker\bar{\tau}$ is a free group. It also easily follows from the definition of $\bar{\tau}$ that the preimage of $\ker\bar{\tau}$ under the natural homomorphism $G \to \overline{G}$ coincides with $\ker\tau$. Thus, $\ker\tau$ is an extension of K by the free group $\ker\bar{\tau}$.

7. Let $H = \bigcap_{v\in V} G_v$. By [7, Lemma 5], $C(G) \leqslant H$. Let us show that, for every element $h \in H \setminus K$, there exists an element $g \in G$ such that $g^{-1}hg \notin H$. This will mean that K is the largest subgroup of H normal in G and therefore $K = C(G)$.

Let $h \in H \setminus K$ be an arbitrary element. Then $h \notin H_{\varepsilon e}$ for some $e \in E \setminus E_T$, $\varepsilon = \pm 1$. Indeed, it is obvious if Γ contains only one vertex and so $E_T = \varnothing$. Suppose that $E_T \neq \varnothing$. Then every vertex of Γ is incident to some $e \in E_T$. For any $e \in E_T$, the equalities $H_{+e} = G_{e(1)} \cap G_{e(-1)} = H_{-e}$ hold in G, as already noted in the proof of Proposition 4.3. Therefore, $H = \bigcap_{e\in E_T,\, \varepsilon=\pm1} H_{\varepsilon e}$, and h possesses the desired property.

Let us consider G as an HNN-extension with the stable letter t_e. Then the element $t_e^{-\varepsilon} h t_e^{\varepsilon}$ has a reduced form of length 2 and hence cannot belong to H contained in the base group of this HNN-extension. Thus, t_e^{ε} is the required element. □

5. Proofs of Theorems 3, 4 and Corollaries 1–3

Proposition 5.1. *Let G be non-elementary, T be a maximal subtree of Γ, and $\mathcal{L}(\Gamma)$ be T-positive. Let also $K = \bigcap_{e\in E,\, \varepsilon=\pm1} H_{\varepsilon e}$ and μ be the least common multiple of $\mu(v) = [G_v : K]$ $(v \in V)$. Finally, let Q be the subring of $\mathbb{Q}$ generated by $\operatorname{Im}\Delta$, Q^+ be the additive group of Q, A be a free abelian group with the basis $\{a_q \mid q \in \operatorname{Im}\Delta\}$, and X be the splitting extension of Q^+ by A such that the automorphism $\widehat{a_q}|_{Q^+}$ acts as multiplication by q. Then the mapping of the generators of G to X given by the rule*

$$g_v \mapsto \mu/\mu(v) \quad (v \in V), \qquad t_e \mapsto a_{\Delta(t_e)} \quad (e \in E \setminus E_T),$$

defines a homomorphism of G into X.

Proof. We extend the indicated mapping of the generators to the mapping of words σ and show that the latter takes all the defining relations of G into the equalities valid in X.

If e is an edge of T, then $\lambda(+e)/\mu(e(1)) = \lambda(-e)/\mu(e(-1))$ by Proposition 4.4 and

$$g_{e(1)}^{\lambda(+e)}\sigma = \lambda(+e)\mu/\mu(e(1)) = \lambda(-e)\mu/\mu(e(-1)) = g_{e(-1)}^{\lambda(-e)}\sigma.$$

Let $e \in E$ be an edge that does not belong to T. By Proposition 4.4, the equality $g_{e(1)}^{\mu(e(1))} = g_{e(-1)}^{\mu(e(-1))}$ holds in G; we denote this element by g for brevity. Since g is elliptic and

$$t_e^{-1} g^{\lambda(+e)\mu/\mu(e(1))} t_e = t_e^{-1} g_{e(1)}^{\lambda(+e)\mu} t_e = g_{e(-1)}^{\lambda(-e)\mu} = g^{\lambda(-e)\mu/\mu(e(-1))},$$

then $\Delta(t_e) = \big(\lambda(-e)\mu/\mu(e(-1))\big)/\big(\lambda(+e)\mu/\mu(e(1))\big)$. This implies that

$$\Big(t_e^{-1} g_{e(1)}^{\lambda(+e)} t_e\Big)\sigma = \big(\lambda(+e)\mu/\mu(e(1))\big)\cdot\Delta(t_e) = \lambda(-e)\mu/\mu(e(-1)) = \Big(g_{e(-1)}^{\lambda(-e)}\Big)\sigma. \qquad \square$$

Proposition 5.2. *Let G be non-elementary, T be a maximal subtree of Γ, and $\mathcal{L}(\Gamma)$ be T-positive. Let also $K = \bigcap_{e\in E,\,\varepsilon=\pm1} H_{\varepsilon e}$ and μ be the least common multiple of $\mu(v) = [G_v : K]$ $(v \in V)$. If $\operatorname{Im}\Delta = \{1\}$, then G is an $(F\times\mathbb{Z})$-by-$\mathbb{Z}_\mu$-group, where F is some free group (i. e. G is isomorphic to an extension of $F\times\mathbb{Z}$ by $\mathbb{Z}_\mu$). If $\operatorname{Im}\Delta = \{1,-1\}$, then G is an $((F\times\mathbb{Z})$-by-$\mathbb{Z}_\mu)$-by-$\mathbb{Z}_2$-group.*

Proof. Let Q, X, and $\sigma\colon G \to X$ be the subring, the group, and the homomorphism from Proposition 5.1, E_T be the edge set of T. Since $\operatorname{Im}\Delta \subseteq \{1,-1\}$, then $Q = \mathbb{Z}$. Therefore, X has the presentation $\big\langle x, a_1;\ [x,a_1] = 1\big\rangle$ if $\operatorname{Im}\Delta = \{1\}$, or

$$\big\langle x, a_1, a_{-1};\ [x,a_1] = [a_1,a_{-1}] = 1,\ a_{-1}^{-1}xa_{-1} = x^{-1}\big\rangle$$

if $\operatorname{Im}\Delta = \{1,-1\}$ (here x denotes the generator of the additive group Q^+ of Q equal to 1).

Let Y be the group with the presentation $\big\langle x;\ x^\mu = 1\big\rangle$ if $\operatorname{Im}\Delta = \{1\}$, and

$$\big\langle x, a_{-1};\ x^\mu = 1,\ a_{-1}^2 = 1,\ a_{-1}^{-1}xa_{-1} = x^{-1}\big\rangle$$

if $\operatorname{Im}\Delta = \{1,-1\}$. Obviously, σ can be extended to a homomorphism τ of G into Y. Since μ is the least common multiple of $\mu(v)$ $(v \in V)$, the greatest common divisor of $\mu/\mu(v)$ $(v \in V)$ is equal to 1 and therefore $x \in \operatorname{Im}\tau$. If $\Delta(t_e) = 1$ for each edge $e \in E\setminus E_T$, then $\operatorname{Im}\Delta = \{1\}$. Hence, if $\operatorname{Im}\Delta = \{1,-1\}$, then there exists an edge $e \in E\setminus E_T$ such that $\Delta(t_e) = -1$ and so $t_e\tau = a_{-1}$. Therefore, $\operatorname{Im}\tau = Y$.

Since $G_v\tau = \langle x^{\mu/\mu(v)}\rangle$, then $\ker\tau \cap G_v = G_v^{\mu(v)} = K$ for every $v \in V$ and, by Proposition 4.4, $\ker\tau$ is an extension of K by a free group. It is well known that such an extension is splittable, i. e. $\ker\tau = KF$, where F is a free subgroup of G and $K\cap F = 1$. It remains to show that $[K,F] = 1$ and therefore $\ker\tau = K\times F$.

If $\operatorname{Im}\Delta = \{1\}$, then K is central in G by Proposition 4.4. Let $\operatorname{Im}\Delta = \{1,-1\}$ and $g \in \ker\tau$ be an arbitrary element. Because τ takes t_e $(e \in E\setminus E_T,\ \Delta(t_e) = -1)$ to a_{-1} and g_v $(v \in V)$, t_e $(e \in E\setminus E_T,\ \Delta(t_e) = 1)$ into $\langle x\rangle$, the number of occurrences of the generators of the first type and inverse to them in the record of g must be even. Since the conjugation of K by t_e $(e \in E\setminus E_T,\ \Delta(t_e) = -1)$ is an automorphism of this subgroup of order 2 and all the elements g_v $(v \in V)$, t_e $(e \in E\setminus E_T,\ \Delta(t_e) = 1)$ belong to the centralizer of K, then g also belongs to the centralizer of K. Hence, K is central in $\ker\tau$, and $[K,F] = 1$. $\square$

Proof of Theorem 3. According to Proposition 2.3, Theorem 1 is valid for the class $\mathcal{C}$ from the statement of Theorem 3. Therefore, we can further use it in the process of proof.

1, 2. *Necessity.* Let $e \in E$ be an arbitrary edge. If e is not a loop, then, by Proposition 3.1, G contains a subgroup isomorphic to

$$P\big(\lambda(+e),\lambda(-e)\big) = \Big\langle g_{e(1)}, g_{e(-1)};\ g_{e(1)}^{\lambda(+e)} = g_{e(-1)}^{\lambda(-e)}\Big\rangle.$$

Since the labeled graph defining G is reduced, then $1 < |\lambda(+e)|, |\lambda(-e)|$. Hence, it follows from Proposition 3.4 that $\lambda(+e)$ and $\lambda(-e)$ are $\rho(\mathcal{C})$-numbers. If e is a loop, then, again by Proposition 3.1, G contains a subgroup isomorphic to $\mathrm{BS}(\lambda(+e),\lambda(-e))$. Because $\operatorname{Im}\Delta \subseteq \{1,-1\}$, the equality $|\lambda(+e)| = |\lambda(-e)|$ holds. Therefore, $\lambda(+e)$ and $\lambda(-e)$ are $\rho(\mathcal{C})$-numbers as Theorem 1 states.

Let $\operatorname{Im}\Delta = \{1,-1\}$. Then, by Proposition 4.1, G contains a subgroup isomorphic to $\mathrm{BS}(1,-1)$, and Theorem 1 implies that $2 \in \rho(\mathcal{C})$.

Sufficiency. Choose some maximal subtree T in Γ and transform the labeled graph $\mathcal{L}(\Gamma)$ defining G to a T-positive form. Obviously, all the labels are still $\rho(\mathcal{C})$-numbers after this operation. Let $K = \bigcap_{e \in E,\, \varepsilon = \pm 1} H_{\varepsilon e}$ and μ be the least common multiple of $\mu(v) = [G_v : K]$ $(v \in V)$. By Proposition 4.4, μ is a $\rho(\mathcal{C})$-number, and, by Proposition 2.1, $\mathbb{Z}_\mu \in \mathcal{C}$. If $2 \in \rho(\mathcal{C})$, then, according to the same proposition, $\mathbb{Z}_2 \in \mathcal{C}$. Proposition 5.2 states that, for some free group F, G is an $(F \times \mathbb{Z})$-by-$\mathbb{Z}_\mu$-group if $\operatorname{Im}\Delta = \{1\}$, or an $((F \times \mathbb{Z})$-by-$\mathbb{Z}_\mu)$-by-$\mathbb{Z}_2$-group if $\operatorname{Im}\Delta = \{1, -1\}$. Hence, G is residually a $\mathcal{C}$-group by Proposition 2.4.

3. Since $\operatorname{Im}\Delta \not\subseteq \{1, -1\}$, Propositions 4.1 and 4.2 imply that G contains a subgroup isomorphic to $\mathrm{BS}(m, n)$, where $1 < m < |n|$. This subgroup is not residually a $\mathcal{C}$-group by Theorem 1. Therefore, G is also not residually a $\mathcal{C}$-group. □

Proof of Theorem 4. 1. Since $\mathcal{C}$ contains at least one non-periodic group and is closed under taking subgroups and extensions, it contains an infinite cyclic group and both of its extensions by means of an infinite cyclic group. Therefore, every elementary GBS-group is a torsion-free $\mathcal{C}$-group.

2. Choose some maximal subtree T in Γ and transform the labeled graph $\mathcal{L}(\Gamma)$ defining G to a T-positive form. Since the fundamental groups of the original and modified labeled graphs are isomorphic, the subring Q remains unchanged under the indicated transformation and therefore $Q^+ \in \mathcal{C}$.

Let A, X, and $\sigma\colon G \to X$ be the groups and the homomorphism from the statement of Proposition 5.1. By the definition, σ acts injectively on all the vertex groups, and Proposition 3.2 says that $\ker\sigma$ is a free group.

As noted above, an infinite cyclic group belongs to $\mathcal{C}$. By the definition of root class, $\mathcal{C}$ also contains the Cartesian product $P = \prod_{z \in \mathbb{Z}} C_z$, where C_z is an infinite cyclic group for each $z \in \mathbb{Z}$. Since A is isomorphic to a subgroup of P and $\mathcal{C}$ is closed under taking subgroups and extensions, then A, X, and $\operatorname{Im}\sigma$ belong to $\mathcal{C}$. Therefore, G is residually a $\mathcal{C}$-group by Proposition 2.4.

It remains to note that Q^+ is a homomorphic image of a free abelian group of countable rank, which is a subgroup of P. Therefore, if $\mathcal{C}$ is closed under taking quotient groups, then $Q^+ \in \mathcal{C}$. If $\operatorname{Im}\Delta \subseteq \{1, -1\}$, then Q^+ is an infinite cyclic group, that belongs to $\mathcal{C}$ as noted above. □

Proof of Corollaries 1–3. Let ρ be a non-empty set of primes. Using the definition of root class, it is easy to verify that the classes of finite ρ-groups, finite solvable ρ-groups, periodic ρ-groups of finite exponent, periodic solvable ρ-groups of finite exponent, and all solvable groups are root. Therefore, the implications $1 \Rightarrow 5$, $3 \Rightarrow 5$, $5 \Rightarrow 2$, and $5 \Rightarrow 4$ in Corollary 2 follow from Theorem 3, the implications $1 \Rightarrow 3$ and $3 \Rightarrow 2$ in Corollary 1 do from Theorems 1 and 3, and Corollary 3 follows from Theorem 4. The implications $4 \Rightarrow 3$, $2 \Rightarrow 1$ in Corollary 2 and $2 \Rightarrow 1$ in Corollary 1 are obvious. □

6. An algorithm for verifying the condition of Theorem 5

Let E^* be the set of paths in Γ. We define a function $\xi\colon E^* \to \{1, -1\}$ as follows. If $e \in E$, then $\xi(e) = \operatorname{sign}\lambda(+e)\lambda(-e)$. If $s = (e_1, e_2, \ldots, e_n)$ is a path in Γ, then $\xi(s) = \prod_{i=1}^{n} \xi(e_i)$. In particular, if the length of s is equal to 0, then $\xi(s) = 1$.

The algorithm given below assigns labels to the vertices of Γ. The label corresponding to a vertex v is denoted by $\zeta(v)$ and can be equal to ± 1. Initially, all the vertices are unlabeled.

Algorithm. 1. If the graph has no labeled vertices, take an arbitrary vertex v of Γ. Otherwise, choose some vertex v, which has no label and is adjacent to one of the labeled vertices.

2. If there is a loop e at v such that $\xi(e) = -1$, then the algorithm terminates without labeling v.

3. Let E_v be the set of edges of Γ, each of which connects v with some of the already labeled vertices, and let, for any $e \in E_v$, ε_e denote the number that is equal to ± 1 and satisfies the relation $e(\varepsilon_e) = v$.

3.1. If there exist $e_1,\ e_2 \in E_v$ such that $\xi(e_1)\zeta(e_1(-\varepsilon_{e_1})) \neq \xi(e_2)\zeta(e_2(-\varepsilon_{e_2}))$, then the algorithm terminates without labeling v.

3.2. Otherwise, we define $\zeta(v)$ as follows: $\zeta(v) = 1$ if $E_v = \varnothing$, and $\zeta(v) = \xi(e)\zeta(e(-\varepsilon_e))$ if $E_v \neq \varnothing$ and e is some edge from E_v (the independence of $\zeta(v)$ from the choice of e is ensured by Step 3.1).

4. If all the vertices of Γ are labeled, then the algorithm terminates; otherwise, it returns to Step 1.

Proposition 6.1. 1. *If $\xi(s) = 1$ for any closed path s in Γ, then the above algorithm terminates by labeling all the vertices of Γ for any sequence of vertex selection at Step* 1.

2. *If, for some sequence of vertex selection at Step* 1, *the above algorithm terminates by labeling all the vertices of Γ, then $\xi(s) = 1$ for any closed path s in Γ.*

Proof. First of all, we note that if v is some vertex of Γ labeled by the algorithm and u is the vertex that was labeled first, then Γ contains a path s from u to v consisting of vertices labeled no later than v, and besides $\zeta(v) = \xi(s)$. This is not difficult to show using induction on the number of steps of the algorithm.

1. Let us fix some sequence of vertex selection at Step 1 and consider an arbitrary vertex v from this sequence. If there is a loop e at v, then it is a closed path and therefore $\xi(e) = 1$. Let $E_v \neq \varnothing$, $e_1, e_2 \in E_v$ be arbitrary edges, $v_1 = e_1(-\varepsilon_{e_1})$, $v_2 = e_2(-\varepsilon_{e_2})$, and u be the first vertex labeled by the algorithm. Then there exist paths s_1, s_2 connecting v_1, v_2 with u and such that $\zeta(v_1) = \xi(s_1)$, $\zeta(v_2) = \xi(s_1)$. Let s be the path composed of the paths s_1, s_2 and the edges e_1, e_2. This path is closed, so $1 = \xi(s) = \xi(s_1)\xi(s_2)\xi(e_1)\xi(e_2)$ and $\xi(e_1)\zeta(v_1) = \xi(e_1)\xi(s_1) = \xi(e_2)\xi(s_2) = \xi(e_2)\zeta(v_2)$. Thus, the algorithm terminates neither at Step 2 nor at Step 3.1, and the vertex v is among the labeled ones. Since it is chosen arbitrarily, this means that the algorithm labels all the vertices of Γ.

2. Since the algorithm labels all the vertices of the graph without terminating at Step 2, then $\xi(e) = 1$ for every loop $e \in E$ and further we can consider only closed paths that do not contain loops. We argue by induction on the number n of iterations (Steps 1–4) required for the algorithm to label all the vertices of a path of the indicated form.

Let s be a closed path without loops, the vertices of which are labeled in n iterations. At each iteration of the algorithm, no more than one vertex is labeled. Therefore, if $n = 1$, then the length of s is equal to 0 and the equality $\xi(s) = 1$ is obvious. Further, we assume that s is of non-zero length (so $n > 1$) and ξ has the required value for every closed path whose vertices are labeled in at most $n - 1$ iterations.

Let v be the last labeled vertex of s. If necessary, we split s into the closed parts, each of which begins and ends at v, and assume that s passes through v only once. Then the fragment (v_1, e_1, v, e_2, v_2) of s is uniquely defined, where e_1, e_2 are edges (which may coincide) and v_1, v_2 are vertices (which may also coincide). Since there are no loops in s, the relations $v_1 \neq v \neq v_2$ and $e_1, e_2 \in E_v$ hold.

Let u be the first vertex labeled by the algorithm. Then there exist paths s_1, s_2 connecting v_1, v_2 with u, consisting of the vertices labeled no later than v_1, v_2 respectively, and such that $\zeta(v_1) = \xi(s_1)$, $\zeta(v_2) = \xi(s_2)$. Let s_0 denote the path obtained from s by removing v and e_1, e_2. Then the union s_3 of the paths s_0, s_1, s_2 is a closed path, all the vertices of which are labeled in at most $n - 1$ iterations, and, by the inductive hypothesis, $1 = \xi(s_3) = \xi(s_0)\xi(s_1)\xi(s_2)$. Since v is labeled by the algorithm, then the condition of Step 3.1 cannot be satisfied and so $\xi(e_1)\zeta(v_1) = \xi(e_2)\zeta(v_2)$. Hence, $\xi(e_1)\xi(e_2) = \zeta(v_1)\zeta(v_2) = \xi(s_1)\xi(s_2)$, and therefore $\xi(s) = \xi(s_0)\xi(e_1)\xi(e_2) = \xi(s_0)\xi(s_1)\xi(s_2) = 1$, as required. □

Proposition 6.2. *Let G be not solvable, $\mathcal{L}(\Gamma)$ be reduced, $\operatorname{Im}\Delta = \{1, -1\}$, and all the labels $\lambda(\varepsilon e)$ $(e \in E,\ \varepsilon = \pm 1)$ are p-numbers for some prime number $p \ne 2$. Let also*

$$E' = \{e \in E \mid H_{+e} \ne C(G) \ne H_{-e}\}.$$

1. *If every elliptic element of G that is conjugate to its inverse belongs to $C(G)$, then $\xi(s) = 1$ for every closed path s in Γ all of whose edges are contained in E'.*

2. *If $\xi(s) = 1$ for every closed path s in Γ all of whose edges are contained in E', then every elliptic element of G that is conjugate to its inverse belongs to $C(G)$ and the quotient group $G/C(G)$ is residually a finite p-group.*

Proof. We fix some maximal subtree T of Γ and begin with a few remarks concerning both Statement 1 and Statement 2.

By Proposition 4.4,

$$1 \ne C(G) = \bigcap_{\substack{e\in E,\\ \varepsilon=\pm1}} H_{\varepsilon e} \leqslant \bigcap_{v\in V} G_v$$

and the least common multiple μ of $\mu(v) = [G_v : C(G)]$ $(v \in V)$ divides the product $\prod_{e\in E,\,\varepsilon=\pm1} \lambda(\varepsilon e)$. Therefore, μ and all the indices $\mu(v)$ $(v \in V)$ are p-numbers.

If Γ contains one vertex v and $\mu = \mu(v) = 1$, then $E' = \varnothing$, every elliptic element of G belongs to $G_v = C(G)$, and $G/C(G)$ is a free group, that is residually a finite p-group by Proposition 2.4. Therefore, both Statement 1 and Statement 2 are true. If Γ has more than one vertex, then each of its vertices is incident to some edge that is not a loop. Since $\mathcal{L}(\Gamma)$ is reduced, it follows that every vertex group contains some proper edge subgroup and so $C(G) \ne G_v$ for all $v \in V$. Thus, further, we can assume that all $\mu(v)$ $(v \in V)$ are different from 1 and therefore are divisible by p.

Let $e \in E$ be an arbitrary edge. Then $g_{e(1)}^{\lambda(+e)} \sim_G g_{e(-1)}^{\lambda(-e)}$ and $H_{+e} \sim_G H_{-e}$. Since $C(G)$ is normal in G, it follows that $[H_{+e} : C(G)] = [H_{-e} : C(G)] = k_e$ for some p-number $k_e \geqslant 1$ and $|\lambda(+e)|k_e = \mu(e(1))$, $|\lambda(-e)|k_e = \mu(e(-1))$.

Let us now turn directly to the proof of Statements 1 and 2.

1. We put $g'_v = g_v^{\mu(v)/p}$ $(v \in V)$ and show that, for any edge $e \in E'$, $g'_{e(1)}$ and $(g'_{e(-1)})^{\xi(e)}$ are conjugate in G.

Indeed, let $e \in E'$ be an arbitrary edge. Then $k_e \ne 1$ and so $p \mid k_e$. The relation $g_{e(1)}^{\lambda(+e)} \sim_G g_{e(-1)}^{\lambda(-e)}$ implies that $g_{e(1)}^{|\lambda(+e)|} \sim_G g_{e(-1)}^{\xi(e)|\lambda(-e)|}$. Hence,

$$g'_{e(1)} = g_{e(1)}^{|\lambda(+e)|(k_e/p)} \sim_G g_{e(-1)}^{\xi(e)|\lambda(-e)|(k_e/p)} = \big(g'_{e(-1)}\big)^{\xi(e)}.$$

Thus, if s is a closed path in Γ, all the edges of which are contained in E', and $\xi(s) = -1$, then, for every vertex v of this path, $g'_v \sim_G (g'_v)^{\xi(s)} = (g'_v)^{-1}$. Since $g'_v \notin C(G)$, Statement 1 is proved.

2. Let E_T denote the edge set of T. To prove the residual p-finiteness of the quotient group $G/C(G)$, we define a mapping σ_0 of the generators g_v $(v \in V)$ and t_e $(e \in E \setminus E_T)$ of G to $\mathbb{Z}_\mu$ as follows.

Let $\Gamma' = (V, E')$ be the graph obtained from Γ by removing all the edges not included in E', and let $\Gamma'_i = (V_i, E'_i)$ be some connected component of Γ'. Choose an arbitrary vertex $v \in V_i$ and put $g_v\sigma_0 = \mu/\mu(v)$. If $w \in V_i$ is an arbitrary vertex and s is a path in Γ'_i connecting v and w, we put $g_w\sigma_0 = \xi(s)\mu/\mu(w)$. It follows from the condition of Statement 2 that, for any two paths s_1, s_2 connecting v and w in Γ'_i, the equality $\xi(s_1) = \xi(s_2)$ holds. Therefore, the above definition is correct. In a similar way, we define the action of σ_0 on the generators of the vertex groups contained in all other connected components of Γ'. Let us also put $t_e\sigma_0 = 0$ for all $e \in E \setminus E_T$.

We extend σ_0 to a mapping of words σ and show that the latter takes all the defining relations of G into the equalities valid in $\mathbb{Z}_\mu$.

Let $e \in E$ be an arbitrary edge. As shown earlier, $|\lambda(+e)|k_e = \mu(e(1))$ and $|\lambda(-e)|k_e = \mu(e(-1))$, where $k_e = [H_{+e} : C(G)] = [H_{-e} : C(G)]$. If $e \in E'$, v is the fixed vertex chosen above from the connected component of Γ', to which e belongs, and s_1, s_{-1} are some paths in Γ' connecting v with $e(1)$, $e(-1)$ respectively, then $\xi(s_1) = \xi(s_{-1})\xi(e)$. It follows that

$$\xi(s_1) \cdot \operatorname{sign} \lambda(+e) = \xi(s_{-1}) \cdot \operatorname{sign} \lambda(-e)$$

and

$$\begin{aligned} g_{e(1)}^{\lambda(+e)}\sigma &= \xi(s_1)\lambda(+e)\mu/\mu(e(1)) \\ &= \xi(s_1) \cdot \operatorname{sign} \lambda(+e) \cdot |\lambda(+e)|\mu/\mu(e(1)) \\ &= \xi(s_{-1}) \cdot \operatorname{sign} \lambda(-e) \cdot |\lambda(-e)|\mu/\mu(e(-1)) \\ &= \xi(s_{-1})\lambda(-e)\mu/\mu(e(-1)) \\ &= g_{e(-1)}^{\lambda(-e)}\sigma. \end{aligned}$$

If $e \notin E'$, then $k_e = 1$ and therefore

$$g_{e(1)}^{\lambda(+e)}\sigma = \varepsilon|\lambda(+e)|\mu/\mu(e(1)) = \varepsilon\mu \equiv \delta\mu = \delta|\lambda(-e)|\mu/\mu(e(-1)) = g_{e(-1)}^{\lambda(-e)}\sigma \pmod{\mu}$$

for some $\varepsilon, \delta = \pm 1$.

Thus, σ defines a homomorphism of G into the finite p-group $\mathbb{Z}_\mu$. It follows from the definition of σ that, for each $v \in V$, the order of $g_v\sigma$ is equal to $\mu(v)$ and therefore $\ker\sigma \cap G_v = C(G)$. Hence, according to Proposition 4.4, $\ker\sigma$ is an extension of $C(G)$ by a free group. This implies that the quotient group $G/C(G)$ is an extension of the indicated free group by a finite p-group. Such an extension is residually a finite p-group by Proposition 2.4.

Suppose now that x and y are elements of G such that $x^{-1}yx = y^{-1}$. Then

$$\big(xC(G)\big)^{-1}\big(yC(G)\big)\big(xC(G)\big) = \big(yC(G)\big)^{-1},$$

and the residual p-finiteness of $G/C(G)$ proved above together with Proposition 2.6 and the relation $p \neq 2$ imply that $yC(G) = 1$, i. e. $y \in C(G)$. Thus, Statement 2 is completely proved. □

An algorithm for verifying the condition of Statement 2-*c* of Theorem 5. Let G be not solvable, $\mathcal{L}(\Gamma)$ be reduced, and $\operatorname{Im}\Delta = \{1, -1\}$. Then $C(G) \leqslant \bigcap_{v\in V} G_v$ and there is an algorithm calculating the numbers $\mu(v) = [G_v : C(G)]$ $(v \in V)$ [7, § 5]. This allows us to find the graph Γ' which is obtained from Γ by removing all the edges not included in the set $E' = \{e \in E \mid H_{+e} \neq C(G) \neq H_{-e}\}$. By Propositions 6.1 and 6.2, to complete the verification of the condition of Statement 2-c, it remains to apply Algorithm given above to each connected component of Γ'.

7. Proof of Theorems 5 and 6

Proposition 7.1. *Let G be non-solvable and $\mathcal{L}(\Gamma)$ be reduced. If G is residually nilpotent, then all the labels $\lambda(\varepsilon e)$ $(e \in E,\ \varepsilon = \pm 1)$ are p-numbers for some prime number p.*

Proof. Since $\mathcal{L}(\Gamma)$ is reduced, then $|\lambda(+e)| \neq 1 \neq |\lambda(-e)|$ for each edge e that is not a loop. Let us show that these relations can be assumed to hold for all loops of $\mathcal{L}(\Gamma)$.

By Proposition 2.5, G is residually finite, and, by Theorem 3, $\operatorname{Im}\Delta \subseteq \{1, -1\}$. It follows that $\mathcal{L}(\Gamma)$ cannot contain a loop e such that $|\lambda(\varepsilon e)| = 1 \neq |\lambda(-\varepsilon e)|$ for some $\varepsilon = \pm 1$.

Let Γ' be the subgraph of Γ which is obtained from the latter by removing each loop e such that (in $\mathcal{L}(\Gamma)$) $|\lambda(+e)| = 1 = |\lambda(-e)|$, and let $G' = \pi_1(\mathcal{L}(\Gamma'))$. Then, by Proposition 3.1, G' is isomorphic to a subgroup of G and so is residually nilpotent. Since $|\lambda(+e)| \neq 1 \neq |\lambda(-e)|$ for every edge e of Γ' and 1 is a power of any number p, then we can consider Γ', $\mathcal{L}(\Gamma')$, and G' instead of Γ, $\mathcal{L}(\Gamma)$, and G respectively.

So, we assume that $|\lambda(\varepsilon e)| \neq 1$ for all $e \in E$, $\varepsilon = \pm 1$. If, for any edge $e \in E$, at least one of the numbers $|\lambda(+e)|$, $|\lambda(-e)|$ is greater than 2, then the required statement follows from Proposition 3.3. Therefore, it remains to show that if $|\lambda(+e)| = 2 = |\lambda(-e)|$ for some edge $e \in E$, then all the labels $\lambda(\varepsilon f)$ ($f \in E$, $\varepsilon = \pm 1$) are 2-numbers.

On the contrary, let $f \in E$ be an edge such that at least one of the numbers $\lambda(+f)$, $\lambda(-f)$ is divisible by a prime number $p \neq 2$. Let us show that Γ and $\mathcal{L}(\Gamma)$ can, if necessary, be modified so that a) Γ contains a simple chain, the first and last edges of which are e and f; b) the fundamental groups of the original and modified labeled graphs are isomorphic.

Indeed, if there is no chain of the indicated type in Γ, then at least one of the following statements holds: 1) e and f are not loops and connect identical pairs of vertices, i. e. $f(1) = e(\varepsilon)$ and $f(-1) = e(-\varepsilon)$ for some $\varepsilon = \pm 1$; 2) e is a loop; 3) f is a loop. In the first case, we add to Γ a new vertex v_f and an edge connecting this vertex with $f(-1)$; replace f with an edge connecting $f(1)$ and v_f; in $\mathcal{L}(\Gamma)$, assign the labels $(1, 1)$ to the first of the added edges, $\lambda(+f)$ (at $f(1)$) and $\lambda(-f)$ (at v_f) to the second. We perform exactly the same transformations in the third case and modify Γ and $\mathcal{L}(\Gamma)$ in a similar way in the second. In all cases, the original labeled graph is obtained from the modified one by an elementary collapse; therefore, their fundamental groups are isomorphic.

Let Ω be a simple chain in Γ that begins with e and ends with f. By Proposition 3.1, $\pi_1(\mathcal{L}(\Omega))$ is embedded in $\pi_1(\mathcal{L}(\Gamma))$ and so is residually nilpotent. For definiteness, let $e(1)$ and $f(-1)$ be the ends of the chain, and let $\varepsilon = \pm 1$ be a number such that $p \mid \lambda(\varepsilon f)$. Consider the elements

$$x_1 = \big[g_{e(1)}, g_{e(-1)}\big], \quad x_2 = \big[g_{f(-\varepsilon)}, g_{f(\varepsilon)}^{\lambda(\varepsilon f)/p}\big],$$
$$x = \big[x_1, x_2\big] = g_{e(-1)}^{-1} g_{e(1)}^{-1} g_{e(-1)} g_{e(1)} x_2^{-1} g_{e(1)}^{-1} g_{e(-1)}^{-1} g_{e(1)} g_{e(-1)} x_2.$$

Let Ω_1 be the chain obtained from Ω by removing $e(1)$ and e, Ω_2 be the chain obtained from Ω by removing $f(-1)$ and f, $F_1 = \pi_1(\mathcal{L}(\Omega_1))$, and $F_2 = \pi_1(\mathcal{L}(\Omega_2))$. Then $\pi_1(\mathcal{L}(\Omega))$ is the free product P_1 of the groups $G_{e(1)}$, F_1 with the subgroups H_{+e}, H_{-e} amalgamated and, at the same time, the free product P_2 of the groups F_2, $G_{f(-1)}$ with the subgroups H_{+f}, H_{-f} amalgamated. Since $|\lambda(-\varepsilon f)| \neq 1$ and $|\lambda(\varepsilon f)/p| < |\lambda(\varepsilon f)|$, then x_2 has a reduced form of length 4 in P_2 and hence does not belong to the free factor F_2 and its subgroup H_{-e}. It follows from this and the equalities $|\lambda(+e)| = 2 = |\lambda(-e)|$ that x has a reduced form of length at least 8 in P_1 and therefore is different from 1.

Let q be an arbitrary prime number and ψ be a homomorphism of $\pi_1(\mathcal{L}(\Omega))$ onto a finite q-group. If $q \neq 2$ and r is the order of $g_{e(1)}\psi$, then $(r, 2) = 1$. This equality and the inclusions $g_{e(1)}^2\psi \in H_{+e}\psi$, $g_{e(1)}^r\psi \in H_{+e}\psi$ imply that $g_{e(1)}\psi \in H_{+e}\psi = H_{-e}\psi$ and $x_1\psi = 1$. Similarly, if $q \neq p$ and s is the order of $g_{f(\varepsilon)}^{\lambda(\varepsilon f)/p}\psi$, then it follows from the inclusions

$$\big(g_{f(\varepsilon)}^{\lambda(\varepsilon f)/p}\big)^p \psi \in H_{\varepsilon f}\psi, \quad \big(g_{f(\varepsilon)}^{\lambda(\varepsilon f)/p}\big)^s \psi \in H_{\varepsilon f}\psi$$

that $x_2\psi = 1$. Thus, for each homomorphism of $\pi_1(\mathcal{L}(\Omega))$ onto a finite group of prime power order, the image of x turns out to be equal to 1. This contradicts the residual nilpotence of $\pi_1(\mathcal{L}(\Omega))$ by Proposition 2.5. □

Proof of Theorem 5. 1. If G is residually nilpotent, then, by Proposition 7.1, all the labels $\lambda(\varepsilon e)$ ($e \in E$, $\varepsilon = \pm 1$) are p-numbers for some prime number p. This fact, the equality $\operatorname{Im}\Delta = \{1\}$, and Theorem 3 imply the residual p-finiteness of G. Since every finite p-group is nilpotent, the inverse statement is obvious.

2. The implication $b \Rightarrow a$ is obvious.

$a \Rightarrow c$. By Proposition 7.1, all the labels $\lambda(\varepsilon e)$ ($e \in E$, $\varepsilon = \pm 1$) are p-numbers for some prime number p. Suppose that $p \neq 2$ and there exists an elliptic element a, which is conjugate to its inverse but does not belong to the cyclic radical of G.

Let T be some fixed maximal subtree of Γ and E_T be the edge set of T. Replacing, if necessary, a by its conjugate, we can assume that $a \in G_v$ for some $v \in V$. Since $C(G)$ is normal in G, then a still does not belong to $C(G)$ after the replacement. We put

$$E_1 = \big\{e \in E \;\big|\; |\lambda(+e)| = 1 = |\lambda(-e)|\big\},$$
$$E_2 = \big\{e \in E \;\big|\; |\lambda(+e)| \neq 1 \neq |\lambda(-e)|\big\}$$

and show that there exists an edge $e \in E_2$ such that $a \notin H_{\varepsilon e}$ for some $\varepsilon = \pm 1$.

Indeed, $C(G) = \bigcap_{e \in E,\, \varepsilon = \pm 1} H_{\varepsilon e}$ by Proposition 4.4. Since $\mathcal{L}(\Gamma)$ is reduced, every edge $e \in E$ that is not a loop belongs to E_2. In particular, $E_T \subseteq E_2$. If e is a loop, then $\Delta(t_e) = \lambda(-e)/\lambda(+e)$ and it follows from the equality $\operatorname{Im}\Delta = \{1, -1\}$ that either $e \in E_1$, or $e \in E_2$. If $E = E_1$, then Γ has only one vertex, $C(G)$ coincides with the corresponding vertex group and therefore contains all the elliptic elements of G, what contradicts the relation $a \notin C(G)$. Hence, either $\mathcal{L}(\Gamma)$ has one vertex and at least one loop $e \in E_2$, or it contains at least two vertices and then each vertex is incident to some edge $e \in E_T \subseteq E_2$. In both cases, $C(G) = \bigcap_{e \in E_2,\, \varepsilon = \pm 1} H_{\varepsilon e}$, and this implies the existence of the sought edge e.

We now consider two cases.

Case 1. $e \in E_T$.

It is easy to see that there is a simple chain Ω in T containing e and such that one of its ends coincides with v, while the other does with $e(\delta)$ for some $\delta = \pm 1$. By Proposition 3.1, $\pi_1(\mathcal{L}(\Omega))$ is embedded in G by means of the identity mapping of the generators.

Let Ω' be the chain obtained from Ω by removing e and $e(\delta)$. Then $\pi_1(\mathcal{L}(\Omega))$ is the free product of the groups $\pi_1(\mathcal{L}(\Omega'))$ and $G_{e(\delta)}$ with the subgroups $H_{-\delta e}$ and $H_{\delta e}$ amalgamated. Consider the elements

$$x_1 = \big[g_{e(\delta)}, g_{e(-\delta)}\big], \quad x_2 = \big[x_1, a\big] = g_{e(-\delta)}^{-1} g_{e(\delta)}^{-1} g_{e(-\delta)} g_{e(\delta)} a^{-1} g_{e(\delta)}^{-1} g_{e(-\delta)}^{-1} g_{e(\delta)} g_{e(-\delta)} a.$$

Since $|\lambda(+e)| \neq 1 \neq |\lambda(-e)|$, $a \notin H_{\varepsilon e}$, and the equality $H_{\varepsilon e} = H_{-\varepsilon e}$ holds in $\pi_1(\mathcal{L}(\Omega))$, then x_2 has a reduced form of length at least 8 in this group and therefore is different from 1.

Let q be an arbitrary prime number and ψ be a homomorphism of G onto a finite q-group. If $q \neq 2$, then $a\psi = 1$ by Proposition 2.6. Let $q = 2$ and r be the order of $g_{e(1)}\psi$. Since $\lambda(+e)$ is a p-number and $p \neq 2$, then $(r, \lambda(+e)) = 1$. It follows that $g_{e(1)}\psi \in H_{+e}\psi = H_{-e}\psi$ and $x_1\psi = 1$. Thus, for any value of q, the equality $x_2\psi = 1$ holds. By Proposition 2.5, this contradicts the residual nilpotence of G.

Case 2. $e \notin E_T$.

Let $x_1 = \big[t_e^{\varepsilon} g_{e(-\varepsilon)} t_e^{-\varepsilon}, g_{e(\varepsilon)}\big]$. It follows from the relations $|\lambda(+e)| \neq 1 \neq |\lambda(-e)|$, $a \notin H_{\varepsilon e}$ that the element

$$x_2 = \big[x_1, a\big] = g_{e(\varepsilon)}^{-1} t_e^{\varepsilon} g_{e(-\varepsilon)}^{-1} t_e^{-\varepsilon} g_{e(\varepsilon)} t_e^{\varepsilon} g_{e(-\varepsilon)} t_e^{-\varepsilon} a^{-1} t_e^{\varepsilon} g_{e(-\varepsilon)}^{-1} t_e^{-\varepsilon} g_{e(\varepsilon)}^{-1} t_e^{\varepsilon} g_{e(-\varepsilon)} t_e^{-\varepsilon} g_{e(\varepsilon)} a$$

has a reduced form of length 8 in the group G considered as an HNN-extension with the stable letter t_e. Therefore, $x_2 \neq 1$. However, as above, if ψ is a homomorphism of G onto a finite 2-group, then $g_{e(-\varepsilon)}\psi \in H_{-\varepsilon e}\psi$, whence $\big(t_e^{\varepsilon} g_{e(-\varepsilon)} t_e^{-\varepsilon}\big)\psi \in H_{\varepsilon e}\psi$ and so $x_1\psi = 1$. Thus, in Case 2, the image of x_2 is also equal to 1 for any homomorphism of G onto a group of prime power order, and we again get a contradiction with the residual nilpotence of G.

$c \Rightarrow b$. Choose some maximal subtree T in Γ and transform $\mathcal{L}(\Gamma)$ to a T-positive form. Since this operation consists only in replacing some of the generators g_v ($v \in V$) by their inverse, then, after it, the conditions of Statement 2-c remain valid.

If all the labels $\lambda(\varepsilon e)$ ($e \in E$, $\varepsilon = \pm 1$) are 2-numbers, then, by Propositions 4.4 and 5.2, G is an extension of the direct product of two free groups by a finite 2-group. Such an extension is residually a finite 2-group by Proposition 2.4. So, further, we assume that $p \neq 2$.

Let $g \in G$ be an arbitrary non-unit element. We show that there exists a homomorphism of G onto a finite p-group or a finite 2-group taking g to a non-unit element.

By Proposition 6.2, $G/C(G)$ is residually a finite p-group. Therefore, if $g \notin C(G)$, then the natural homomorphism of G onto $G/C(G)$ can be extended to the desired one. Hence, further, we can assume that $g \in C(G)$.

Let Q, X, and $\sigma\colon G \to X$ be the subring, the group, and the homomorphism from Proposition 5.1. Since $\operatorname{Im}\Delta = \{1, -1\}$, then $Q = \mathbb{Z}$ and X has the presentation

$$\big\langle x, a_1, a_{-1};\ [x, a_1] = [a_1, a_{-1}] = 1,\ a_{-1}^{-1} x a_{-1} = x^{-1} \big\rangle$$

(here, as above, x denotes the generator of the additive group Q^+ of Q equal to 1). By Proposition 4.4, $C(G) \leqslant \bigcap_{v \in V} G_v$. Hence, $g \in G_v$ for each $v \in V$, and, by the definition of σ, the inclusion $g\sigma \in \langle x \rangle \setminus \{1\}$ holds. Therefore, $g\sigma$ is mapped to a non-unit element under the homomorphism of X onto the group

$$\mathrm{BS}(1, -1) = \big\langle x, a_{-1};\ a_{-1}^{-1} x a_{-1} = x^{-1} \big\rangle.$$

The latter is residually a finite 2-group by Theorem 1. Thus, the constructed homomorphism $G \to \mathrm{BS}(1, -1)$ can be extended to the required one.

3. Since $\operatorname{Im}\Delta \nsubseteq \{1, -1\}$, then G is not residually finite by Theorem 3 and is not residually nilpotent by Proposition 2.5. □

Proof of Theorem 6. $1 \Rightarrow 3$. Since G is residually a torsion-free nilpotent group, then, by Proposition 2.5, it is residually a finite p-group for any prime number p. Therefore, by Theorem 1, G cannot be isomorphic to $\mathrm{BS}(1, n)$, where $n \neq 1$. Obviously, $\mathrm{BS}(1, 1)$ satisfies Statement 3. So, further, we can assume that G is non-solvable and the labeled graph $\mathcal{L}(\Gamma)$ defining it is reduced. Then, by Theorem 3, $\operatorname{Im}\Delta = \{1\}$ and $|\lambda(\varepsilon e)| = 1$ for all $e \in E$, $\varepsilon = \pm 1$. This means that Γ has one vertex v and G is an extension of the vertex group G_v by the free group generated by the elements t_e ($e \in E$). Since such an extension is splittable, G contains a free subgroup F such that $G = G_v F$ and $G_v \cap F = 1$. It follows from the equality $\operatorname{Im}\Delta = \{1\}$ and Proposition 4.4 that G_v lies in the center of G. Therefore, $G = G_v \times F$, as required.

$3 \Rightarrow 2$. The direct product of two free groups is residually free by [12, Lemma 1.1].

$2 \Rightarrow 1$. It is well known that, for an arbitrary free group, the intersection of the members of its lower central series is trivial [19] and the factors of this series are free abelian groups [13]. Therefore, every free group is residually a torsion-free nilpotent group. □

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