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Le paramètre θ peut être considéré comme une fonction paramétrique associée à la loi P par une fonctionnelle T définie sur l'ensemble **P** des lois de probabilités et à valeurs dans Θ

$$\theta = T(P)$$

L'estimateur naturel de ϑ est alors $T(P_n)$ où P_n est la loi empirique associée à P .

Toute statistique T_n pouvant s'écrire comme fonction de P_n , $T_n = T(P_n)$ est appelée fonctionnelle statistique.

La notion de fonctionnelle statistique a été introduite par Von Mises (1947).

Soit $P, Q \in$ **P**, on appelle contaminée de P par Q au taux t, $0 \le t \le 1$, la probabilité notée $P_t(Q)$

$$P_t(Q) = (1-t)P + tQ$$

Q est la probabilité contaminante.

Définition 6.2
T est dite dérivable au sens de Gâteaux si la quantité $T'_Q(P)$ définie par

$$T'_Q(P) = \lim_{t\to 0^+} \frac{T((1-t)P + tQ) - T(P)}{t}$$

existe.
$T'_Q(P)$ est appelée dérivée de Gâteaux de T en P dans la direction Q.

Définition 6.3
On appelle fonction d'influence de T en P la dérivée au sens de Gâteaux de T en P dans la direction δ_x

$$IF(x;P,T) = \lim_{t\to 0^+} \frac{T((1-t)P + t\delta_x) - T(P)}{t}$$

δ_x est la répartition de masse 1 en x.

Il existe plusieurs approches de la robustesse, dans cet article nous avons retenu la définition de Hampel (1971) et qui repose sur la continuité de la fonctionnelle T au voisinage de P

$$\sup_x |IF(x;P,T)| < \infty.$$

Un M-estimateur (Huber (1981)) T_n est solution du problème

$$\min_\theta \sum_{i=1}^n \rho(X_i, \theta).$$

Si ρ est dérivable par rapport à θ et de dérivée ψ, T_n est solution de

$$\sum_{i=1}^n \psi(X_i, \theta) = 0.$$

D'où $\frac{1}{n}\sum_{i=1}^n \psi(X_i, T_n) = 0$, ceci implique que $E_{P_n}(\psi(X, T_n)) = 0$.

On peut donc associer à la fonctionnelle statistique T_n la fonctionnelle T définie sur **P** comme solution de l'équation $E_P(\psi(X, T(P))) = 0$.

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ENDOMORPHISMS OF THE GROUP $G_{mn} = < a,b;[a^m,b^n]=1 >$ m,n > 1

by

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Abstract :
Characterisation up to inner automorphisms of the endomorphisms of the group $G_{mn} = < a,b;[a^m,b^n]=1 >$, m, n > 1.

Keywords :
Free products of groups with amalgamations (amalgamated free products). Tietze transformations. Reduced forms, length of an element. Cyclically irreducible elements.

§0 Introduction.

The purpose of this note is to characterise, up to inner automorphisms, the endomorphisms of the group $G_{mn} = < a,b;[a^m,b^n]=1 >$, where m and n are integers greater than 1.

§1 Properties of the group G_{mn}.

Let $H = < c,d;\ cd = dc >$ be the free abelian group of rank 2; let $A = (< a > * H;\ a^m = c)$ be the free product of $< a >$, the free group generated by the element a, and the free abelian group H, amalgamated by $< c >$, the subgroup generated by the element $a^m = c$; let $B = (H * < b >;\ d = b^n)$ be the free product of the free abelian group H and $< b >$, the free group generated by the element b, amalgamated by $< d >$, the subgroup generated by the element $b^n = d$. Then by Tietze transformations,

$$A = < a,c,d;\ a^m = c,\ cd = dc >, B = < c,d,b;\ b^n = d,\ cd = dc > \text{ and}$$

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$G_{mn} = (A*B;H)$, that is the free product of groups A and B, amalgamated by the subgroup H.

We recall that $P = (X*Y;U)$ is the free product of groups X and Y amalgamated by the subgroup U means that P is generated by X and Y, where $X \cap Y = U$. Further more any product of elements taken in tours in X and Y such that the number of the components is greater than one and consecutive components are not both in one of the subgroup X or Y, is different from 1, the unity of the group P. More formally, P = sgp(X,Y), where $X \cap Y = U$. And any element p of P has the form :

$$p = p_1 p_2 \cdots p_n \qquad \text{(I)}$$

where any component p_i $(i = 1,2,...,n)$ belongs to X or Y, consecutive components are not both in one of the subgroup X or Y and for $n > 1$, $p \neq 1$.

The amalgamated free product P is uniquely defined, up to isomorphisms, by the generated factors X and Y and the amalgamated subgroup U.

The form (I) with these conditions is called the **reduced form** of the element p and then p is **irreducible**.

$f_1 f_2 \cdots f_r$ and $g_1 g_2 \cdots g_s$ are reduced forms of the same element p iff r = s and there exist elements $u_1, u_2, ..., u_{r-1}$ of the amalgamated subgroup U such that :

$$\begin{cases} f_1 = g_1 u_1 \\ f_i = (u_{i-1})^{-1} g_i u_i, \ (1 < i < r) \\ f_r = (u_{r-1})^{-1} g_r . \end{cases} \qquad \text{(II)}$$

So, the reduced forms are not unique for a given element. But the number of the components is the same for all these forms. This number is called the **length** of the element p or the **length of the reduced form** of p and will be denoted **l(p)**. An element p of P is said to be **cyclically irreducible** if for its reduced form $p = p_1 p_2 \cdots p_n$, either n = 1, or n > 1 and the components p_1 and p_n are not both in one of the subgroup X or Y. The form is then said to be **cyclically reduced**. If p is not cyclically irreducible, then it is conjugated to a cyclically irreducible element, that is :

$$p = x_1 x_2 \cdots x_r (y_1 y_2 \cdots y_s) x_r^{-1} x_{r-1}^{-1} \cdots x_1^{-1}, \qquad \text{(III)}$$

where $r \geq 1$, $s \geq 1$, $x_1 x_2 \cdots x_r$ is reduced, $y_1 y_2 \cdots y_s$, cyclically reduced and components x_r and y_s are not both in one of the subgroup X or Y, and for s > 1, $y_s x_r^{-1} \notin U$.

We refer to [1] or [2] for details.

Let's prove some properties of amalgamated free product.

Let $P = (X*Y;U)$ be the free product of groups X and Y amalgamated by the subgroup U.

Proposition 1.1 If an element a belongs to X and is not conjugated to elements of U, then for any element b of P such that ab = ba, it follows that b belongs to X.

If a is cyclically irreducible in P, l(a) > 1 and ab = ba, then either b belongs to U, or b is cyclically irreducible and l(b) > 1.

Proof. Let $a \in X$ and is not conjugated in U. Let $b = x_1 x_2 \cdots x_r$ be irreducible. If ab = ba, then $x_r^{-1} x_{r-1}^{-1} \cdots x_1^{-1} a x_1 x_2 \cdots x_r = a$: for $x_1 \in Y$, the left hand side of the equality is irreducible with 2r + 1 components. This is impossible as the right hand side has only one component. So $x_1 \in X$ and since $x_1^{-1} a x_1 \notin U$, if r > 1, the left hand side of the same equality has 2r - 1 components, which is again impossible. Consequently, r = 1 and $b = x_1 \in X$.

Let now $a = y_1 y_2 ... y_s$ cyclically irreducible with s > 1.

If r > 1 and b is not cyclically irreducible, then both x_1 and x_r belong to one of the subgroup X or Y; this subgroup contains either y_1 or y_s.

Let it be y_1. Then the sides of the equality $x_1 x_2 \cdots x_r y_1 y_2 \cdots y_s = y_1 y_2 \cdots y_s x_1 x_2 \cdots x_r$ have different length; s + r - 1 and s + r. This is impossible and the proposition is proven. □

Corollary 1.1 For any element g of A\H (respectively B\H), $g^{-1}Hg \cap H = <c>$ (respectively $<d>$). Consequently, for elements g in A or B and h in H, $g^{-1}hg \in H$ implies gh = hg.

Moreover, if $g \in G_{mn}$, $1 \neq h \in H$ and gh = hg, then $g \in A$ (or B).

Proof. In fact, if $g \in$ A\H, then g centralises $<c>$. Hence $<c> \subseteq (g^{-1} <c> g) \subseteq (g^{-1}Hg)$ since $<c> \subseteq H$. In the other side, let $x \in H$ and $g^{-1}xg \in H$, where $g = g_1 g_2 \cdots g_r$ is irreducible in $A = (<a>*H;<c>)$. Two cases arise as r = 1 or r > 1.

i) if r = 1 : then $g \in <a>$ and $x \in <c>$ for otherwise $g^{-1}xg$ will be irreducible and $l(g^{-1}xg) > 1$; that is $g^{-1}xg$ could not belongs to H. So $x \in <c>$ and $g^{-1}xg = x$ since g centralises $<c>$.

ii) if r > 1 : then $g^{-1}xg = g_r^{-1} g_{r-1}^{-1} \cdots g_1^{-1} x g_1 g_2 \cdots g_r$. If $g_1 \in <a>$ or $g_1 \in H$, then x must belong to $<c>$ for the same reason as above. Hence $g^{-1}Hg \cap H \subseteq <c>$ and the equality is established.

Similarly we showed that $<d> = g^{-1}Hg \cap H$, for $g \in B \setminus H$.

The assumption $g^{-1}Hg \in H$ implies gh = hg immediately follows from the properties showed above. For the last step of the corollary, let $g = g_1 g_2 \cdots g_r$ is irreducible with $r \geq 2$. As gh = hg we have : $g_1 g_2 \cdots g_r h g_r^{-1} g_{r-1}^{-1} \cdots g_1^{-1} = h$. If $g_r \in A$, then $g_r h g_r^{-1} \in H$ and by proposition 1.1, $h \in <c>$; that is : $h = c^k$, for $k \in Z$. Then $g_1 g_2 \cdots g_{r-1} c^k g_{r-1}^{-1} g_{r-2}^{-1} \cdots g_1^{-1} = c^k$ and $g_{r-1}^{-1} c^k g_{r-1} \in H$ and by the same proposition $c^k \in <d>$, which is possible only if k = 0; this contradicts the fact that $r \geq 2$. So $r \leq 1$; that is : $g \in A$ (or B) and the corollary is proven. □

Proposition 1.2 let U be a central subgroup of groups X and Y and g an element of P\X such that $g^k \in X$, for an integer k > 1. Then g is conjugated in P to an element z of X\U or Y\U and $z^k \in U$; that is g is transformed.

Proof. We first remind that U is a central subgroup of X and Y if $U \subseteq Z(X)$ and $U \subseteq Z(Y)$, where Z(#) is the centralizer of the subgroup #. So, Z(P) = U since $Z(P) = Z(X) \cap Z(Y) \cap U$.

Let g be irreducible. If l(g) = 1, then $g \in Y$. So $g^k \in X \cap Y = U$ and we take z = g.

Let l(g) > 1. If g is cyclically irreducible, then $g^k \notin X$. So g is conjugated to a cyclically irreducible element; that is : $g = x_1 x_2 \cdots x_r (y_1 y_2 \cdots y_s) x_r^{-1} x_{r-1}^{-1} \cdots x_1^{-1}$ with the conditions of (II).

If s > 1, then $l(g^k) = 2r + ks - 1 > 1$. Hence s = 1 and $g = x_1 x_2 \cdots x_r (y_1) x_r^{-1} x_{r-1}^{-1} \cdots x_1^{-1}$, and $g^k \in X$ only if $y_1^k \in U$. This proves the proposition. □

Corollary 1.2 Let P be one of the groups A, B or G_{mn}. If g is cyclically irreducible element of P and l(g) > 1, then for any element h of P and any integer k > 1, $g^k = h^k$ implies g = h.

Proof. If P is A or B, apply proposition 1.2. If P is G_{mn}, apply the consequence of the first part of corollary 1.1. □

Proposition 1.3 i) Let $v \in G_{mn} \setminus A$ such that $v^k \in A$, for an integer k > 1. Then either $v \in B$ and is conjugated in B to an element of the subgroup $<b>$, or $v = xyx^{-1}$, where $x \in A \setminus H$, $y \in B \setminus H$ and is conjugated in B to an element of the subgroup $<b>$.

ii) Let $v \in G_{mn} \setminus B$ such that $v^k \in B$, for an integer k > 1. Then either $v \in A$ and is conjugated in A to an element of the subgroup $<a>$, that is $v = t^{-1} a^l t$, where $t \in A$, $l \in Z$ and m divides kl, or $v = xyx^{-1}$, where $x \in B \setminus H$, $y \in A \setminus H$ and is conjugated in A to an element of the subgroup $<a>$, that is : $y = s^{-1} a^l s$, where $s \in A$, $l \in Z$ and m divides kl.

Remark. We note that ii) is obtained from i) by permuting A and B and by seeing that if $v = t^{-1} a^l t$, where $t \in A$, $l \in Z$, then $v \in A$ and $v^k = t^{-1} a^{kl} t \in B$ iff $a^{kl} \in Z(A) = <c>$; that is $m | kl$. Thus it suffices here to prove only i).

Proof. As $v \in G_{mn} \setminus A$ and $v^k \in A$, it follows that v is transformed; that is : $v = x_1 x_2 \cdots x_r y x_r^{-1} x_{r-1}^{-1} \cdots x_1^{-1}$ by proposition 1.2. We distinguish 2 cases as r = 0 or r > 0.

1) If r = 0 then $v = y \in B \setminus H$ and $v^k = y^k \in A$. But $v^k \in B$; so $v^k \in A \cap B = H$ and by proposition 1.2, $y = t^{-1} b^l t$, where $t \in B$, $l \in Z$ and n doesn't divide l.

2) If r > 0, again 2 cases arise as $y \in B$ or $y \in A$.

a) If $y \in B$, as $v^k = x_1 x_2 \cdots x_r y^k x_r^{-1} x_{r-1}^{-1} \cdots x_1^{-1} \in A$, then $y^k \in H$; by proposition 1.2, $y = t^{-1} b^l t$ and $y^k = b^{kl} = d^q$ for kl = nq, $t \in B$, $l, q \in Z$. Since $x_r \notin B$, then $x_r d^k x_r^{-1} \in H$ and since $v^k \in A$, then r = 1 and $v = xyx^{-1}$, where $x \in A \setminus H$, $y \in B \setminus H$ and $y = t^{-1} b^k t$, $t \in B$.

b) If $y \in A$, as $v^k = x_1 x_2 \cdots x_r y^k x_r^{-1} x_{r-1}^{-1} \cdots x_1^{-1} \in A$, then $x_r \in B$ and $y^k \in H$. By corollary 1.1, $y^k = c^l$, $l \in Z$. So in B, $l(x_r y^k x_r^{-1}) = 3$ and $x_r y^k x_r^{-1} \notin H$; So this case is impossible. □

§2. Endomorphisms of the group G_{mn}.

Let φ be a map defined on the set of generators of G_{mn} by :

$$a\varphi = u,\ b\varphi = v, \text{ where } u, v \in G_{mn}.$$

Then φ can be extended to an endomorphism of G_{mn} iff $[u^m, v^n] = 1$. (See [2] or [3]).

Let now φ be an endomorphism of G_{mn} as defined above. For the sake of simplicity, we can assume u cyclically irreducible. For otherwise, u is conjugated to a cyclically irreducible element; and by multiplying φ by a suitable inner automorphism, we again obtain an endomorphism of G_{mn}, where the image of the generator a, is cyclically irreducible.

Proposition 2.1 Let u be an element of the group A or B such that u^m is not conjugated in A or in B respectively to elements of H. Let v be an element of G_{mn}, such that $[u^m, v^n] = 1$. Then $v \in A$ or B respectively.

Proof. Let $u \in A$. Since u^m is not conjugated in A to elements of H and $[u^m, v^n] = 1$, then $v^n \in A$ by proposition 1.1.

If $v \in A$, then there is nothing to prove.

Let $v \notin A$; then by proposition 1.3 we have 2 cases.

1) $v = t^{-1}b^l t$ and then $v^n = t^{-1}b^{nl}t = d^l$, where $t \in B$, $l \in Z$. But in A, d^l is not conjugated in H to elements of $<c>$: so, by proposition 1.1, $[u^m, d^l] = 1$ implies $u^m \in H$; this contradicts the hypothesis.

2) $v = xyx^{-1}$ and $y = s^{-1}b^l s$, where $x \in A$, $s \in B$, $l \in Z$. Then $y^n = s^{-1}b^{nl}s = d^l$, that is $v^n = xd^l x^{-1}$. But in A, $[u^m, v^n] = 1$ implies $[x^{-1}u^m x, d^l] = 1$ which means that $x^{-1}u^m x \in H$ by proposition 1.1; this again contradicts the hypothesis. Hence $v \in A$.

Similarly, it can be shown that $v \in B$ if $u \in B$ and u^m is not conjugated in B to elements of H to end our proof. □

Theorem 2.1 An endomorphism φ of G_{mn}, up to inner automorphisms, has the form $a\varphi = u$, $b\varphi = v$, where either one of the following properties holds :

(1) u and v are cyclically irreducible elements of lengths greater than one and $uv = vu$.

(2) u and v belong to the same subgroup A (or B) and $u^m v^n = v^n u^m$.

(3) $u = x^{-1}a^k x$, $v = y^{-1}b^l y$, where $x \in A$, $y \in B$ and $k, l \in Z$.

(4) $u = y^{-1}b^k y$, $v = x^{-1}a^l x$, where $x \in A$, $y \in B$ and $k, l \in Z$, n divides km and m divides nl.

Proof. For the sake of simplicity, let u be cyclically irreducible. Then two cases arise :

I) If $l(u) > 1$, then u^m is cyclically irreducible and $l(u^m) > 1$: so by proposition 1.1, either $v^n \in H$, or v^n is cyclically irreducible and $l(v^n) > 1$.

If $v^n \in H$, then $u^m \in A$ (or B) by corollary 1.1; this is impossible since $l(u^m) > 1$. So v^n is cyclically irreducible and $l(v^n) > 1$. From $[u^m, v^n] = 1$ we have $u^m = v^{-n}u^m v^n = (v^{-n}uv^n)^m$; and by corollary 1.2, $u = v^{-1}uv$, that is $uv = vu$.

Similarly $v = u^{-1}vu$, that is $uv = vu$. The elements u and v then satisfy property (1) of the theorem.

II) $l(u) = 1$; let us suppose that u and v do not belong to the same subgroup conjugated to A (or B). We again distinguish 2 cases as $u \in A$ or $u \in B$.

A) Let $u \in A$.

If u^m is not conjugated in A to elements of H, then u and v must belong to A by proposition 2.1 and this contradicts the supposition. So u^m is conjugated to the some element of H in A and we can consider $u^m \in H$. Then either $u \in H$ or $u = x^{-1}a^k x$, ($x \in A$, $k \in Z$) by proposition 1.2. Since v^n commutes with an element of H, then $v^n \in A$ (or B).

a) Let $v^n \in A$. Since $u \in A$ and u and v do not belong to the same subgroup, then $v \notin A$. So by corollary 1.1, either $v = y^{-1}b^l y$, where $y \in B$ and $l \in Z$, or $v = t^{-1}yt$, where $t \in A\backslash H$, $y \in B\backslash H$ and $y = s^{-1}b^l s$, $s \in B$, $l \in Z$.

In the first case where $v \in B$, $u \notin H$ as u and v do not belong to the same subgroup A or B. So then $u = x^{-1}a^k x$, where $x \in A$, $k \in Z$ and u and v satisfy property (3) of the theorem.

In the second case, $v = t^{-1}s^{-1}b^l st$ and $v^n = t^{-1}d^l t \in A\backslash H$. Since $v^{-n}u^m v^n = u^m$ in A and $v^n \in A\backslash H$, then by proposition 1.1, $u^m \in H$ and must be conjugated to some element of $<c>$. So, if $u \in H$, then $u^m \in <c>$ implies $u \in <c>$; that is $u = c^k \in Z(A)$, $k \in Z$ and $tut^{-1} = u$ and $tvt^{-1} = s^{-1}b^l s$ belong to the same subgroup B: this leads to contradiction. So $u \notin H$; that is $u = x^{-1}a^k x$ and the elements $tut^{-1} = (xt^{-1})^{-1}a^k(xt^{-1})$ and $tvt^{-1} = s^{-1}b^l s$ satisfy property (3) of the theorem.

b) Let $v^n \in B$ and let's again distinguish 2 cases.

1) If $v \in B$, then $u \notin H$. So $u = x^{-1}a^k x \in A$ ($x \in A$, $k \in Z$) and $u^m \in <c> \subseteq H$; so $[u^m, v^n] = 1$ implies $v^n \in H$ by proposition 1.1.

By proposition 1.2, either $v \in B$, which is impossible, or $v = y^{-1}b^l y$, ($y \in B$, $l \in Z$) and property (3) of the theorem is satisfied.

2) $v \notin B$.

i) If $v = t^{-1}a^{l}t \in A$, ($t \in A$, $l \in Z$ and m divides nl); then the cases $u \in H \subseteq A$ and $u = x^{-1}a^{k}x$ ($x \in A$) are impossible for u and v will belong to the same subgroup.

ii) If $v = txt^{-1}$, where $t \in B\backslash H$, $x \in A\backslash H$ and $x = sa^{l}s^{-1}$, $s \in A$, $l \in Z$ and m divides nl, then $v^n \in B\backslash H$:

*) if $u \in H \subseteq A$, then u and $t^{-1}vt = x$ will belong to the same subgroup; contradiction again:

**) if $u = x^{-1}a^{k}x$, ($x \in A$, $k \in Z$) then $u^m \in \langle c \rangle \subseteq H$; by proposition 1.1, $[u^m, v^n] = 1$ and u^m is not conjugated to elements of $\langle d \rangle$ both imply $v^n \in H$; this contradicts the fact that $v^n \in B\backslash H$.

B) Let now $u \in B$.

This case is the same as A) for we simply interchange subgroups A and B and take in consideration the remark of proposition 1.4. Elements u and v will then satisfy property (4) of the theorem.

To end the proof, we note that any correspondence φ such that aφ = u and bφ = v, where u and v satisfy one of the properties (1), (2), (3) or (4) of the theorem is an endomorphism of G_{mn}. □

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J-DIVISEURS TOPOLOGIQUES DE ZERO D'UNE ALGEBRE DE JORDAN LOCALEMENT MULTIPLICATIVEMENT CONVEXE

par

N. El Yacoubi et H. Kemmoun

ABSTRACT.

In this paper, we give some characterizations of J-topological divisors of zero (J-t.d.z.). We show, in particular, that in a locally multiplicatively convex Jordan algebra (l.m.c.J.a.), $\left(A,(p_i)_{i\in I}\right)$; $\bigcap_{i\in I} Z_i$ is a closed set in A, where Z_i is the set of J-t.d.z. in (A,p_i), and we prove that in a l.m.c.J.a. A with continuous inverse ([9]), each non J-invertible element which is a limit of a sequence of J-invertible elements is a J-t.d.z. in some Jordan semi-normed algebra.

INTRODUCTION.

C.Viola ([9]) et A.M.Kaïdi ([6]) se sont exclusivement interessés aux J-diviseurs topologiques de zéro (J-d.t.z.) dans les algèbres de Jordan normées. Récemment, A.Tajmouâti, dans ([8]), a étudié en collaboration avec M.Akkar, les J-d.t.z. d'une algèbre de Jordan localement multiplicativement convexe (J-a.l.m.c.) métrisable $\left(A,(p_n)_n\right)$, où $(p_n)_n$ est la famille de semi-normes sous-multiplicatives définissant la topologie de A, en utilisant la suite $(\chi_n)_{n\in IN}$ associée à l'opérateur quadratique U_x et, qui est définie par :

$$\chi_n(x) = \inf_{y\in A}\left[d(0,U_x(y))\frac{1+p_n(y)}{p_n(y)}\right]$$

et où $U_x(y) = 2x.(x.y) - x^2.y$, pour $x, y \in A$.

Nous utilisons ici une technique différente pour étudier les J-d.t.z. d'une J-a.l.m.c. quelconque $\left(A,(p_i)_{i\in I}\right)$ unitaire et séparée. Nous montrons alors que $\bigcap_{i\in I} Z_i$ est fermé, (Z_i désignant l'ensemble des J-d.t.z. de (A,p_i)), même si la J-a.l.m.c. $\left(A,(p_i)_{i\in I}\right)$ n'est ni métrisable, ni complète. Puis, nous mettons en évidence le lien qui existe entre les J-d.t.z. de la J-a.l.m.c. $\left(A,(p_i)_{i\in I}\right)$ et ceux des algèbres de Jordan semi-normées (A,p_i).

Enfin, dans le cas d'une J-a.l.m.c. à inverse continu, nous montrons que tout élément de A non J-inversible, limite d'une suite d'éléments J-inversibles, est un J-d.t.z. dans (A,p_i), à partir d'un certain rang i.

I.PRELIMINAIRES.

CONTENTS
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