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ON THE AUTOMORPHISMS OF SOME ONE-RELATOR GROUPS

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The description of the automorphism group of group $\langle a, b; [a^m, b^n] = 1\rangle$ $(m, n > 1)$ in terms of generators and defining relations is given. This result is applied to prove that any normal automorphism of every such group is inner.

Key Words: Automorphism; Free product with amalgamation; Normal automorphism; One-relator group.

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INTRODUCTION

The automorphism group of certain one-relator groups was studied by several authors. Collins (1978) obtained the presentation by generators and defining relations of the automorphism group of the Baumslag–Solitar groups $G(l, m) = \langle a, b; a^{-1}b^l a = b^m\rangle$ when $|l| = 1$ or $|m| = 1$ or $|l| > 1$, $|m| > 1$ and l and m are coprime; in particular, in these cases the group $\mathrm{Aut}\, G(l, m)$ turns out to be finitely related. Later Collins and Levin (1983) found the presentation of the group $\mathrm{Aut}\, G(l, m)$ when $m = ls$, $|l| > 1$, and $|s| > 1$ and showed thereby that in this case the group $\mathrm{Aut}\, G(l, m)$ is not finitely generated. In the same article, the more extensive class of groups $G = \langle a_1, a_2, \ldots, a_n, t; t^{-1}w^l t = w^m\rangle$ where w is a word in $a_1, a_2, \ldots, a_n$ was considered. When $n \geq 2$, w is neither a proper power nor primitive in the free group $\langle a_1, a_2, \ldots, a_n\rangle$ and $m = ls$ with $|s| > 1$, authors gave the presentation of group $\mathrm{Aut}\, G$ and this group turns out to be infinitely generated too. Some HNN-extensions of Baumslag–Solitar groups $G(l, m; k) = \langle a, t; t^{-1}a^{-k}ta^l t^{-1}a^k t = a^m\rangle$ were considered by Brunner (1980). In the case when $|l| \neq |m|$, he described all endomorphisms of such groups and noted that if $|l| = 1$ or $|m| = 1$ then the group $\mathrm{Aut}\, G(m, n; k)$ is not finitely generated. Using Brunner's results, Kavutskii and Moldavanskii (1988) under assumption $|l| \neq |m|$ obtained the presentation of $\mathrm{Aut}\, G(m, n; k)$ and proved that this group is finitely generated if and only if none of the integers l and m is divisor of another. Furthermore, if the group $\mathrm{Aut}\, G(m, n; k)$ is finitely generated, then it is finitely related. It should be mentioned here that it is still unknown whether the automorphism group of any one-relator group is finitely presented if it is finitely generated.

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Results listing above are relative to one-relator groups which in either case are connected with Baumslag–Solitar groups. In the present article, we consider another class of one-relator groups consisting of groups G_{mn} with presentation

$$G_{mn} = \langle a, b; [a^m, b^n] = 1 \rangle$$

where m and n are arbitrary integers satisfying inequalities $m > 1$ and $n > 1$. We obtain the presentation of group $\operatorname{Aut} G_{mn}$ by generators and defining relations and thereby prove that it is finitely related. We prove also that any normal automorphism of every group G_{mn} is inner.

As can be immediately verified the following mappings of generators of group G_{mn} define the automorphisms of G_{mn} (which will be denoted by the same symbols):

$$\lambda : a \mapsto a^{-1}, \qquad b \mapsto b;$$
$$\mu : a \mapsto a, \qquad b \mapsto b^{-1};$$
$$\nu : a \mapsto a^{-1}, \qquad b \mapsto b^{-1}.$$

It is evident that $\lambda^2 = \mu^2 = 1$, $\lambda\mu = \mu\lambda$, and $\lambda\mu = \nu$ and therefore these automorphisms together with the identity mapping constitute a subgroup K of group $\operatorname{Aut} G_{mn}$, where K is the Klein four-group. If $m = n$, the mapping

$$\eta : a \mapsto b, \qquad b \mapsto a$$

defines one more automorphism of G_{mn}. The relations $\eta^2 = 1$, $\eta^{-1}\lambda\eta = \mu$, and $\eta^{-1}\mu\eta = \lambda$ (which can also be immediately checked) show that the subgroup L of $\operatorname{Aut} G_{mn}$ generated by subgroup K and element η is the split extension of K by the 2-cycle $\langle \eta \rangle$. It will be shown here that if $m \neq n$ then $\operatorname{Aut} G_{mn} = K \cdot \operatorname{Inn} G_{mn}$ and if $m = n$, then $\operatorname{Aut} G_{mn} = L \cdot \operatorname{Inn} G_{mn}$. More explicitly, we shall prove the following theorem.

Theorem 1. *Let λ, μ, and η be the automorphisms of group G_{mn} defined above and α and β be the inner automorphisms of G_{mn} generated by elements a and b, respectively.*

If $m \neq n$, then group $\operatorname{Aut} G_{mn}$ is generated by the automorphisms λ, μ, α, and β, and defined by the relations

1. $\lambda^2 = \mu^2 = 1$;
2. $\lambda\mu = \mu\lambda$;
3. $\lambda^{-1}\alpha\lambda = \alpha^{-1}$;
4. $\lambda^{-1}\beta\lambda = \beta$;
5. $\mu^{-1}\alpha\mu = \alpha$;
6. $\mu^{-1}\beta\mu = \beta^{-1}$;
7. $\alpha^m\beta^n = \beta^n\alpha^m$.

If $m = n$, then group $\operatorname{Aut} G_{mn}$ is generated by the automorphisms λ, μ, η, α, and β and defined by the relations 1 – 7 and the additional relations

8. $\eta^2 = 1$;
9. $\eta^{-1}\lambda\eta = \mu$;
10. $\eta^{-1}\alpha\eta = \beta$.

Theorem 1 can be applied to characterize the normal automorphisms of groups G_{mn}. Let us recall that an automorphism of a group G is said to be *normal* if it maps onto itself every normal subgroup of G. It is evident that any inner automorphism is normal. In general, the converse is not true. It was proved in Lubotski (1980) and Lue (1980) that any normal automorphism of a noncyclic free group must be inner. Generalizing this result, Neschadim (1996) exhibited that the same assertion is true for any group which is a nontrivial free product. Also he gave the example of one-relator group possessing a normal automorphism which is not inner. Nevertheless, for groups G_{mn} we have the following theorem.

Theorem 2. *Any normal automorphism of group G_{mn} is inner.*

We note that the residual finiteness of group G_{mn} (i.e., recall, for any nonidentity element $g \in G_{mn}$ there exists a homomorphism φ of group G_{mn} onto some finite group X such that $g\varphi \neq 1$) is well known; it follows, for example, from the result of Baumslag (1983). Since G_{mn} is finitely generated, then by Mal'cev (1940) Theorem, it is Hopfian; i.e., every of its surjective endomorphism is an automorphism. Some other properties of these groups were considered in Tieudjo and Moldavanskii (1998) where, in particular, their construction as amalgamated free product and the description of their endomorphisms were given. These results can be used for somewhat shortening of the proof of our Theorem 1 but for completeness we shall give here an independent proof.

1. PRELIMINARIES

As we have just mentioned, the group G_{mn} can be constructed as amalgamated free product and we begin from some properties of this group-theoretic construction.

Let $G = (A * B; H)$ be a free product of groups A and B with amalgamated subgroup H. Then any element $g \in G$ can be written in the form $g = x_1 x_2 \dots x_s$, where elements $x_1, x_2, \dots, x_s$ belong in turns to one of groups A and B and if $s > 1$ then no one of them belongs to subgroup H. Such representation is called a *reduced form* of element g and the number s of factors of it (uniquely determined by g) is called a *length* of g and denoted by $l(g)$. An element g is said to be *cyclically reduced* if either $l(g) = 1$ or the factors x_1 and x_s of its reduced form $g = x_1 x_2 \dots x_s$ do not belong to the same subgroup A or B (the definition is correct since all reduced forms of element g have or do not have this property simultaneously). It is easy to see that any element of G is conjugate with a cyclically reduced element. Moreover, an immediate induction gives the following proposition.

Proposition 1.1. *If element g of group $G = (A * B; H)$ is not cyclically reduced and $l(g) > 1$, then g can be written in the form*

$$g = u \cdot v \cdot u^{-1},$$

where elements u and v have reduced forms $u = x_1 x_2 \dots x_r$ and $v = y_1 y_2 \dots y_s$ with $r \geq 1$ and $s \geq 1$, element v is cyclically reduced, elements x_r and y_1 do not belong to the same subgroup A or B and if $s > 1$, then element $y_s x_r^{-1}$ does not belong to subgroup H.

By means of Proposition 1.1 it is easy to prove the following proposition.

Proposition 1.2. *If the element g of the group $G = (A * B; H)$ does not belong to subgroup A and if $g^k \in A$ for some integer $k \neq 0$, then $g = x^{-1}yx$ for some $x, y \in G$ where the element y belongs to one of subgroups A or B and $y^k \in H$.*

Also we need the following simple proposition.

Proposition 1.3. *Let $G = (A * B; H)$ and suppose that the amalgamated subgroup H is contained in the centre of both groups A and B. If element $g \in G$ does not belong to subgroup A, then $g^{-1}Ag \cap A = H$.*

Indeed, the inclusion $H \subseteq g^{-1}Ag \cap A$ is evident. To prove the inverse inclusion let ρ be the natural homomorphism of group G onto quotient group G/H which is the ordinary free product of quotients A/H and B/H. Then since $g\rho \notin A\rho$, we have

$$(g^{-1}Ag \cap A)\rho \subseteq (g\rho)^{-1}(A\rho)(g\rho) \cap (A\rho) = 1$$

and therefore $g^{-1}Ag \cap A \subseteq H$.

Further, we need the construction of group G_{mn} in terms of amalgamated free product. For this purpose let $H = \langle c, d; [c, d] = 1\rangle$ be the free Abelian group of rank 2, $A = (\langle a\rangle * H; a^m = c)$ be the amalgamated free product of infinite cycle $\langle a\rangle$ and H and $B = (H * \langle b\rangle; d = b^n)$ be the amalgamated free product of H and infinite cycle $\langle b\rangle$. Then it is easy to show by means of Tietze transformations that group G_{mn} is isomorphic to the free product $(A * B; H)$ of groups A and B with amalgamated subgroup H. These notations are assumed in what follows.

Since in constructions of groups A and B the amalgamated subgroups are central in the free factors, Proposition 1.3 gives the following proposition.

Proposition 1.4. *If an element g of group A does not belong to subgroup H, then $g^{-1}Hg \cap H = \langle c\rangle$, and if an element g of group B does not belong to subgroup H, then $g^{-1}Hg \cap H = \langle d\rangle$.*

Proposition 1.5. *Any element g of group G_{mn} such that $g^{-1}Hg \cap H \neq 1$ is contained in subgroup A or in subgroup B.*

For the proof it is enough to show that if $g = x_1x_2 \ldots x_s$ is the reduced form of g with $s > 1$, then $g^{-1}Hg \cap H = 1$. Let us suppose that $x_1 \in A$; the case $x_1 \in B$ is considered similarly. For any element $h \in H$, the inclusion $g^{-1}hg \in H$ implies the inclusions $x_1^{-1}hx_1 \in H$ and $x_2^{-1}(x_1^{-1}hx_1)x_2 \in H$. Thus, $x_1^{-1}hx_1 \in x_1^{-1}Hx_1 \cap H$ and since $x_1 \in A\backslash H$ it follows from Proposition 1.4 that $x_1^{-1}hx_1 = c^k$ for some integer k. Similarly, inclusion $x_2^{-1}c^kx_2 \in x_2^{-1}Hx_2 \cap H$ gives $x_2^{-1}c^kx_2 = d^l$ for some integer l, and since element d^l lies in the centre of group B, we have the equality $c^k = d^l$. As elements c and d form the basis of free Abelian group H, hence $k = l = 0$ and $h = 1$.

Proposition 1.6. *Any Abelian subgroup of group G_{mn} which contains a cyclically reduced element of length greater than 1 is cyclic.*

Proof. Let U be Abelian subgroup of group G_{mn} and let U contain a cyclically reduced element u of length greater than 1. It is not difficult to see that any element of G_{mn} commuting with u is either element of H or cyclically reduced of length greater than 1. Since Proposition 1.5 implies $U \cap H = 1$ we conclude that all nonidentity elements of U are cyclically reduced of length greater than 1.

Let u be the nonidentity element of U of the smallest length and $u = u_1 u_1 \dots u_r$ be a reduced form of it. We claim that subgroup U is generated by u. Namely, for any nonidentity element $v \in U$ we shall prove by induction on $l(v)$ that v is equal to some power of u.

Let $v = v_1 v_2 \dots v_s$ be a reduced form of element v. Replacing, if necessary, element v by element v^{-1}, we can assume that elements u_1 and v_1 belong to the same subgroup A or B. Then since elements v_s and u_1 do not belong to the same subgroup A or B and the right side of equation

$$u_r^{-1} \dots u_2^{-1} u_1^{-1} v_1 v_2 \dots v_s u_1 u_2 \dots u_r = v_1 v_2 \dots v_s$$

is cyclically reduced the product $h = u_r^{-1} \dots u_2^{-1} u_1^{-1} v_1 v_2 \dots v_r$ must be element of H. So, if $s = r$ we have $h = u^{-1} v \in U$ and since $U \cap H = 1$ we obtain the equality $v = u$ giving the basis of induction.

If $s > r$, then $v = uv'$, where $v' = hv_{r+1} \dots v_s$. Since $l(v') < s$, then by induction $v' = u^k$ for some integer k. Hence $v = u^{k+1}$ and the proof is complete.

2. PROOF OF THEOREM 1

We first prove the following necessary results.

Proposition 2.1. *For any automorphism φ of group G_{mn} there exists an inner automorphism ψ of G_{mn} such that either $a(\varphi\psi) \in A$ and $b(\varphi\psi) \in B$ or $a(\varphi\psi) \in B$ and $b(\varphi\psi) \in A$.*

Proof. Let φ be an automorphism of group G_{mn} and $u = a\varphi$, $v = b\varphi$. At first, we note that elements u and v cannot be cyclically reduced of length greater than 1.

If, on the contrary, element u is cyclically reduced and $l(u) > 1$, then element u^m is also cyclically reduced of length greater than 1, and since $[u^m, v^n] = 1$, by Proposition 1.6, elements u^m and v^n generate the (infinite) cyclic subgroup. Therefore, $u^{mr} = v^{ns}$ for some nonzero integers r and s. But this equation implies that $a^{mr} = b^{ns}$ which is not satisfied in group G_{mn}.

On the other hand, element u is conjugate with a cyclically reduced element and after multiplying φ by suitable inner automorphism, we can assume that u is cyclically reduced. Consequently, by the remark above, $u \in A$ or $u \in B$.

Suppose firstly that $u \in A$. We claim that if $v \notin B$ then $v = xyx^{-1}$ where $x \in A$ and $y \in B$ and therefore $x^{-1}ux \in A$ and $x^{-1}vx \in B$. So, multiplying φ by one more inner automorphism we obtain the desired result.

Since u and v generate the group G_{mn}, then $v \notin A$. Hence if $v \notin B$, then $l(v) > 1$ and since v is not cyclically reduced it has by Proposition 1.1 the form

$$v = x_1 x_2 \dots x_r \cdot y_1 y_2 \dots y_s \cdot (x_1 x_2 \dots x_r)^{-1},$$

where $r \geq 1$, $s \geq 1$, element $x_1 x_2 \ldots x_r$ is reduced, element $y_1 y_2 \ldots y_s$ is cyclically reduced, elements x_r and y_1 do not belong to the same subgroup A or B and if $s > 1$ then element $y_s x_r^{-1}$ does not belong to subgroup H.

We assert now that the assumption $x_1 \in B$ leads to the contradiction. To prove this, let us note firstly that if $x_1 \in B$, then $l(v^n) > 1$ and the first syllable of reduced form of v^n is x_1. This is evident if $s > 1$ or if $s = 1$ and $y_1^n \notin H$. If $s = 1$, then $y_1 \in B$ since if $y_1 \in A$, then elements u and v are contained in the normal closure in G_{mn} of subgroup A and therefore cannot generate the group G_{mn}. Hence $x_r \in A$ and $r > 1$. If $y_1^n \in H$ then $y_1^n \in y_1^{-1} H y_1 \cap H$ and by Proposition 1.4 $y_1^n = d^k$ for some integer $k \neq 0$. Therefore $x_r y_1^n x_r^{-1} \in A \backslash H$, $l(v^n) = 2r - 1 > 1$ and the first syllable of reduced form of v^n is x_1.

Now, since $x_1 \in B$ the equality $v^{-n} u^m v^n = u^m$ implies inclusions $u^m \in H$ and $x_1^{-1} u^m x_1 \in H$. Since $u \in A \backslash H$ (because the quotient group of G_{mn} by the normal closure of H is not cyclic) and $x_1 \in B \backslash H$ the Proposition 1.4 implies that $u^m = 1$, a contradiction.

So, $x_1 \in A$. If v does not have the form claimed above, then $r > 1$ and elements $u_1 = x_1^{-1} u x_1$ and $v_1 = x_1^{-1} v x_1$ turn out to be in the previous case.

Thus, we have proved that if element $u = a\varphi$ belongs to subgroup A, then after multiplying, if necessary, automorphism φ by one more inner automorphism we have $u \in A$ and $v \in B$. Similar arguments will show that if element $u = a\varphi$ belongs to subgroup B, then after multiplying, if necessary, automorphism φ by one more inner automorphism we get $u \in B$ and $v \in A$.

Proposition 2.2. *Let elements u and v of group G_{mn} be such that $u \in A \backslash H$, $v \in B \backslash H$ and $[u^r, v^s] = 1$ for some integers $r \neq 0$ and $s \neq 0$. Then $u = x^{-1} a^k x$ and $v = y^{-1} b^l y$ where $x \in A$, $y \in B$, and nonzero integers k and l are such that kr is divided by m and ls is divided by n.*

Proof. We note, firstly, that $u^r \in H$ and $v^s \in H$. Indeed, if, say, $u^r \notin H$, then since $u^r \in A$, $v^s \in B$ and $[u^r, v^s] = 1$ we get $v^s \in H$. Hence $v^s = u^{-r} v^s u^r \in H \cap u^{-r} H u^r$ and $v^s = v^{-1} v^s v \in H \cap v^{-1} H v$. Proposition 1.4 implies now that $v^s = 1$ which is impossible.

Since $u \in A \backslash H$ and $u^r \in H$, then Proposition 1.2, applied to the group $A = (\langle a \rangle * H; a^m = c)$, gives $u = x^{-1} z x$ where $x \in A$, element z is contained in subgroup $\langle a \rangle$ or in subgroup H and element z^r belongs to subgroup $\langle a^m \rangle$. But if $z \in H$, then the inclusion $z^r \in \langle c \rangle$ is possible only if $z \in \langle c \rangle$. Thus, in any case $z = a^k$ for some integer $k \neq 0$ and m divides kr because $z^r \in \langle a^m \rangle$.

So, we have proved that u has the required form. The assertion on the element v is proved similarly.

We remind that group G_{mn} is isomorphic to the amalgamated free product $(A * B; H)$, where $A = (\langle a \rangle * H; a^m = c)$, $B = (H * \langle b \rangle; d = b^n)$, and $H = \langle c, d; [c, d] = 1 \rangle$.

Proposition 2.3. *Let F be the subgroup of group G_{mn} generated by elements $x^{-1} a^k x$ and $y^{-1} b^l y$ where $x \in A$, $y \in B$, and $k, l \in \mathbb{Z}$. Then $F = G_{mn}$ if and only if $|k| = 1 = |l|$ and $x \in \langle a \rangle \cdot \langle d \rangle$, $y \in \langle b \rangle \cdot \langle c \rangle$.*

Proof. If $|k| = 1 = |l|$ and $x = a^p d^q$, $y = b^r c^s$ for some integers p, q, r, s, then elements $c = (x^{-1}ax)^m$ and $d = (y^{-1}by)^n$ belong to subgroup F and since $a = d^q(x^{-1}ax)d^{-q}$ and $b = c^s(y^{-1}by)c^{-s}$ we have $a \in F$ and $b \in F$ and therefore $F = G_{mn}$.

Conversely, let us suppose that $F = G_{mn}$. Then the quotient group of G_{mn} by its commutator subgroup G'_{mn} is generated by elements a^k and b^l and since G_{mn}/G'_{mn} is the free Abelian group with basis a, b we must have $|k| = 1 = |l|$.

Let A_1 denote the subgroup of G_{mn} generated by subgroup H and element $x^{-1}ax$ and let B_1 denote the subgroup of G_{mn} generated by subgroup H and element $y^{-1}by$. Since $H \leq A_1 \leq A$ and $H \leq B_1 \leq B$, it follows by the theorem of H. Neumann (see e.g., Neumann, 1954, p. 512) that the subgroup F_1 generated by A_1 and B_1 is the free product of groups A_1 and B_1 with amalgamated subgroup H and $A \cap F_1 = A_1$, $B \cap F_1 = B_1$. Therefore, since $F \leq F_1$ the equality $F = G_{mn}$ implies $A_1 = A$ and $B_1 = B$.

Now we shall prove that if $A_1 = A$, then $x \in \langle a \rangle \cdot \langle d \rangle$. Let $x = x_1 x_2 \dots x_r$ be the reduced form of element x (in the decomposition of group A in amalgamated product $A = (\langle a \rangle * H; a^m = c)$).

If $x \notin \langle a \rangle \cdot \langle d \rangle$, then $r \geq 2$ and if $r = 2$, then $x_1 \in H$ and $x_2 \in \langle a \rangle$. If $r > 2$ and $x_1 \in \langle a \rangle$, then letting $x' = x_2 x_3 \dots x_r$ we see that subgroup A_1 is generated by subgroup H and element $(x')^{-1}a(x')$. Now if $x_r \in H$, then $r > 3$ and letting $x'' = x_2 x_3 \dots x_{r-1}$ we see that subgroup A_1 is generated by subgroup H and element $(x'')^{-1}a(x'')$. Thus, we have shown that if $x \notin \langle a \rangle \cdot \langle d \rangle$, then we can assume without loss of generality that $r \geq 2$ and $x_1 \in H$, $x_r \in \langle a \rangle$.

Let $\overline{A}$ be the quotient of group A by central subgroup $\langle a^m \rangle$ and $\overline{g}$ denote the image of element $g \in A$ under the natural homomorphism of A to $\overline{A}$. Then $\overline{A}$ is the ordinary free product of cyclic group $\langle \overline{a} \rangle$ of order m and of infinite cycle $\langle \overline{d} \rangle$. The image $\overline{A}_1$ of subgroup A_1 is generated by element $\overline{d}$ and by image $\overline{x^{-1}ax}$ of element $x^{-1}ax$. Our assumptions about x imply that

$$\overline{x}_r^{-1} \dots \overline{x}_2^{-1} \overline{x}_1^{-1} \overline{a}\, \overline{x}_1 \overline{x}_2 \dots \overline{x}_r$$

is the reduced form of element $\overline{x^{-1}ax}$. This in turn implies that any alternating product of nonidentity powers of elements $\overline{x^{-1}ax}$ and $\overline{d}$ is reduced as written. Thus, $\overline{A}_1 \neq \overline{A}$ (since $\overline{a} \notin \overline{A}_1$) and hence $A_1 \neq A$. Consequently, the equality $A_1 = A$ really implies the inclusion $x \in \langle a \rangle \cdot \langle d \rangle$ and the same arguments will show that the equality $B_1 = B$ implies the inclusion $y \in \langle b \rangle \cdot \langle c \rangle$.

Now we can complete the proof of Theorem 1. Let φ be an automorphism of group G_{mn}. Proposition 2.1 implies that for some inner automorphism ψ of G_{mn} we shall get either $a(\varphi\psi) \in A$ and $b(\varphi\psi) \in B$ or $a(\varphi\psi) \in B$ and $b(\varphi\psi) \in A$.

Firstly, let us consider the case when $a(\varphi\psi) \in A$ and $b(\varphi\psi) \in B$. Since elements $a(\varphi\psi)$ and $b(\varphi\psi)$ generate the group G_{mn} and hence no one of them belong to subgroup H, it follows from Proposition 2.2 that $a(\varphi\psi) = x^{-1}a^k x$ and $b(\varphi\psi) = y^{-1}b^l y$ for some $x \in A$, $y \in B$, and nonzero integers k and l. Now, Proposition 2.3 implies that $a(\varphi\psi) = d^{-p}a^{\varepsilon}d^p$ and $b(\varphi\psi) = c^{-q}b^{\delta}c^q$ for some integers p and q and $\varepsilon, \delta = \pm 1$. Then the product of $\varphi\psi$ by the inner automorphism generated by element $c^{-q}d^{-q}$ belongs to subgroup K and therefore $\varphi \in K \cdot \operatorname{Inn} G_{mn}$.

Now, let $a(\varphi\psi) \in B$ and $b(\varphi\psi) \in A$. Then by Proposition 2.2 $a(\varphi\psi) = y^{-1}b^l y$ and $b(\varphi\psi) = x^{-1}a^k x$ for some $x \in A$, $y \in B$, and nonzero integers k and l where kn is divided by m and lm is divided by n. Since Proposition 2.3 again gives $|k| = 1 = |l|$, conditions of divisibility imply the equality $m = n$. Thus, if $m \neq n$, then $\operatorname{Aut} G_{mn} = K \cdot \operatorname{Inn} G_{mn}$.

If $m = n$, then the group G_{mn} has the automorphism η and since $A\eta = B$ and $B\eta = A$ we obtain $a(\varphi\psi\eta) \in A$ and $b(\varphi\psi\eta) \in B$. Therefore, automorphism $\varphi\psi\eta$ belongs to subgroup $K \cdot \operatorname{Inn} G_{mn}$. This means that $\varphi \in L \cdot \operatorname{Inn} G_{mn}$. Thus, in the case $m = n$ we obtain $\operatorname{Aut} G_{mn} = L \cdot \operatorname{Inn} G_{mn}$.

The validity of relations 1–10 in the statement of Theorem 1 can be checked immediately (and this in part was singled out above) and it remains to show that these relations do define the group $\operatorname{Aut} G_{mn}$. Making use of relations 3–6 in the case $m \neq n$ and of relations 3–6 and 10 in the case $m = n$, any relation in the pointed out generators of $\operatorname{Aut} G_{mn}$ can be transformed to the form $uv = 1$ where u is a product of elements λ and μ (or λ, μ and η) and v is a product of elements α and β. Since the unit is the only element of subgroups K and L inducing the identity automorphism of quotient group G_{mn}/G'_{mn}, we can conclude that

$$K \cap \operatorname{Inn} G_{mn} = 1 \qquad \text{and} \qquad L \cap \operatorname{Inn} G_{mn} = 1$$

and therefore the relation $uv = 1$ implies $u = 1$ and $v = 1$. Since relations 1 and 2 define the group K and relations 1, 2, 8, and 9 define the group L, the relation $u = 1$ is derivable from the relations singled out in Theorem. Since the presentation above of group G_{mn} as amalgamated free product with regard to Corollary 4.5 in Magnus et al. (1966) makes evident the triviality of its centre, the group $\operatorname{Inn} G_{mn}$ is isomorphic to G_{mn} and therefore the relation $v = 1$ must be derivable from relation 7. Thus, any relation in the indicated generators of group $\operatorname{Aut} G_{mn}$ is derivable from the relations 1–10 and the proof is complete.

3. PROOF OF THEOREM 2

We begin with a rather obvious remark. If φ is a normal automorphism of a group G and if N is a normal subgroup of group G, then the mapping $\overline{\varphi}$ of the factor group G/N onto itself, defined by

$$(gN)\overline{\varphi} = (g\varphi)N \qquad (g \in G),$$

is an automorphism of group G/N and this automorphism is normal too. The automorphism $\overline{\varphi}$ is said to be *induced* by automorphism φ.

Now, let φ be a normal automorphism of group G_{mn}. Then by Theorem 1 $\varphi = \xi\psi$ where $\psi \in \operatorname{Inn} G_{mn}$ and $\xi \in K$ if $m \neq n$ or $\xi \in L$ if $m = n$. Since automorphism φ is normal if and only if the automorphism ξ is normal, it remains to show that any nonidentity element of subgroups K and L is not normal automorphism.

Let M and N denote the normal closure in group G_{mn} of elements a^m and b^n respectively. Then the quotient group G_{mn}/M is the free product of cycle $\langle a \rangle$ of order m and infinite cycle $\langle b \rangle$ and the quotient group G_{mn}/N is the free product of infinite cycle $\langle a \rangle$ and cycle $\langle b \rangle$ of order n.

Since the orders of elements aM and bM of the group G_{mn}/M are different, then any automorphism of form $\kappa\eta$ where $\kappa \in K$ does not induce any automorphism of this quotient and therefore is not normal by the remark above.

In the same quotient group G_{mn}/M the elements bM and $(bM)^{-1}$ are not conjugate, since two elements of a free factor of an ordinary free product are conjugate if and only if they are conjugate in the factor. Therefore, automorphisms $\overline{\mu}$ and $\overline{\nu}$ of group G_{mn}/M, induced by the automorphisms μ and ν respectively, are not inner and consequently, by the mentioned above result in Neschadim (1996), $\overline{\mu}$ and $\overline{\nu}$ are not normal. Hence, from the remark above, it follows that automorphisms μ and ν of group G_{mn} are not normal. Analogously, automorphism λ induces a noninner automorphism in the quotient G_{mn}/N and therefore is not normal. Theorem 2 is demonstrated.

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