ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2008, Vol. 29, No. 2, pp. 56–59. © Pleiades Publishing, Ltd., 2008.

A Generalization of Residual Finiteness

D. Tieudjo[1*] and D. I. Moldavanskii[2]**

(Submitted by M.M. Arslanov)

[1]Department of Mathematics and Computer Sciences, University of Ngaoundere, P. O. Box 455, Ngaoundere, Cameroon

[2]Dept. of Algebra and Mathematical Logic, Ivanovo State University, ul. Ermaka 39, Ivanovo 153025, Russia

Abstract—The concept of residual finiteness with respect to automorphic equivalence, a property generalizing residual finiteness and conjugacy separability is introduced. A sufficient condition for a group G to be residually finite with respect to automorphic equivalence is proven (Theorem). It is then used to give some examples of automorphic equivalent residually finite groups.

2000 Mathematics Subject Classification: 20E26, 20E06, 20E28, 20F05

DOI: 10.1134/S1995080208020029

Key words and phrases: *Residual properties, automorphic equivalence, residual finiteness with respect to automorphic equivalence, free products with amalgamation, HNN extensions.*

1. INTRODUCTION

A group G is *residually finite* if, for any element $g \neq 1$ in G, there exists a normal subgroup N of finite index in G such that $gN \neq 1$ in the quotient group $\overline{G} = G/N$.

A well-known generalization of residual finiteness is conjugacy separability.

A group G is *conjugacy separable* if, for any two nonconjugate elements f and g of G, there exists a normal subgroup N of finite index in G such that elements fN and gN are nonconjugate in the quotient group $\overline{G} = G/N$. It is evident that, if a group G is conjugacy separable, then it is residually finite.

Some other residual properties generalizing residual finiteness are considered in [2, 4].

In this paper, we introduce the concept of residual finiteness with respect to automorphic equivalence. We see that this concept generalizes residual finiteness and conjugacy separability. We prove a sufficient condition for a group to be residually finite with respect to some automorphic equivalence, namely:

Theorem *Let a subgroup Φ of* Aut G*, the group of the automorphisms of a given group G, contain a group* Inn G *of the inner automorphisms of this group, and let* Inn G *have a finite index in group Φ. If group G is finitely generated and conjugacy separable, then G is Φ-equivalent residually finite.*

We then use this theorem to give some examples of automorphic equivalent residually finite groups.

*E-mail address: tieudjo\@yahoo.com
**E-mail address: moldav\@ivanovo.ac.ru

2. DEFINITION

Let G be an arbitrary group and let φ be an automorphism of G.

We remind the reader that a normal subgroup N of group G is said to be *φ-invariant* if $N\varphi = N$. If N is a φ-invariant normal subgroup of group G, then the mapping $\overline{\varphi}$ of the quotient group G/N onto itself defined by

$$(gN)\overline{\varphi} = (g\varphi)N \quad (g \in G)$$

is an automorphism of group G/N. This automorphism is called the automorphism *induced* by φ.

Let Φ be a subgroup of the group Aut G of all automorphisms of G.

A normal subgroup N of group G is said to be *Φ-invariant* if it is φ-invariant for any automorphism $\varphi \in \Phi$.

If N now is a Φ-invariant normal subgroup of the group G, then we denote by $\overline{\Phi}$ the subgroup of the group Aut(G/N) of all automorphisms of the group G/N induced by the automorphisms of Φ.

Let G be a group and $\Phi \leqslant$ Aut G. Elements a and b of group G are said to be *Φ-equivalent* if there exists an automorphism $\varphi \in \Phi$ such that $a = b\varphi$.

Let us now formulate our main concept.

Let G be an arbitrary group. Let Φ be a subgroup of the group Aut G of all the automorphisms of G.

Group G is said to be *residually finite with respect to Φ-equivalence* (or *Φ-equivalent residually finite*) if, for any non-Φ-equivalent elements a and b of group G, there exists a normal Φ-invariant subgroup N of finite index in G such that elements aN and bN of the quotient group G/N are not $\overline{\Phi}$-equivalent.

It is clear that particular cases of this notion are residual finiteness (when subgroup Φ consists of only the identical automorphism) and conjugacy separability (when Φ coincides with Inn G, the group of all inner automorphisms of the group G).

If $\Phi =$ Aut G, then Φ-equivalent residual finiteness is just Aut G-equivalent residual finiteness, i.e., residual finiteness with respect to any automorphic equivalence.

We now prove the sufficient condition of Φ-equivalent residual finiteness (Theorem).

3. PROOF OF THEOREM

Let $\psi_1, \psi_2, \ldots, \psi_r$ be a fixed representative system of cosets of the subgroup Inn$G$ in group Φ.

Let a and b be non-Φ-equivalent elements of group G. Assume for $i = 1, 2, \ldots, r$, that $b_i = b\psi_i$. Since subgroup Φ contains group Inn G, element a can not be conjugate in group G to elements $b_1, b_2, \ldots, b_r$. Further, since group G is conjugacy separable, there exists a normal subgroup M of finite index of G such that in the quotient group G/M element aM is not conjugate to elements $b_1M, b_2M, \ldots, b_rM$.

It is well known that an arbitrary subgroup of finite index of a finitely generated group G contains some characteristic subgroup that has a finite index in G. Let N be the characteristic (and consequently the Φ-invariant) subgroup of finite index of our group G, contained in subgroup M. Then, in the quotient group G/N, element aN is not conjugate to elements $b_1N, b_2N, \ldots, b_rN$.

We now assert that elements aN and bN of the quotient group G/N are not $\overline{\Phi}$-equivalent. Assume by contradiction that, for some automorphism $\varphi \in \Phi$, the equality $aN = (bN)\overline{\varphi}$ takes place; i.e., $aN = (b\varphi)N$. Let us write the automorphism φ as $\varphi = \psi_i\gamma$ for some $i \in \{1, 2, \ldots, r\}$ and some inner automorphism γ of group G. Then, in group G, element $b\varphi$ is conjugate to some element b_i, and consequently, in the quotient group G/N, element $aN = (b\varphi)N$ is conjugate to some element b_iN. This contradicts the selection of the subgroup N. So, the theorem is proved.

4. EXAMPLES

Now, using the above Theorem, we have

Example 1. If $k = \pm p^e$, where p is a prime integer and $e \geq 1$, then the group G_k with presentation

$$G_k = \langle a, b; a^{-1}ba = b^k \rangle$$

is Aut G_k-equivalent residually finite.

Indeed, group G_k is conjugacy separable [5]. From [1], group Aut G_k can be described as follows:

Proposition 1. *Let $G_k = \langle a, b; a^{-1}ba = b^k \rangle$, where $|k| \neq 1$. Let $k = \delta p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, where $\delta = \pm 1$, $p_1, p_2, \ldots, p_r$ are distinct primes and $e_i \geq 1$ $(i = 1, \ldots, r)$. Then the group Aut G_k has the presentation*

$$\langle \varphi, \psi_1, \psi_2, \ldots, \psi_r, \tau; \psi_i^{-1}\varphi\psi = \varphi^{p_i}, \psi_i\psi_j = \psi_j\psi_i, \tau^2 = 1, \tau\psi_i = \psi_i\tau, \tau^{-1}\varphi\tau = \varphi^{-1} (i, j = 1, \ldots, r) \rangle.$$

In this presentation the automorphisms are defined by

(a) $a\varphi = ab$, $b\varphi = b$;

(b) $a\psi_i = a$, $b\psi_i = b^{p_i}$ $(i = 1, \ldots, r)$;

(c) $a\tau = a$, $b\tau = b^{-1}$.

Now, the following proposition can be derived.

Proposition 2. *Subgroup* Inn G_k *has a finite index in group* Aut G_k *if and only if $k = \pm p^e$, where p is a prime integer and $e \geq 1$.*

Proof. Assume first that subgroup Inn G_k has a finite index in group Aut G_k. Then any automorphism of G_k should have a finite order modulo Inn G_k. In particular, for some integer $n > 0$, the automorphism ψ_1^n should be inner; i.e., for some element $g \in G_k$, the equalities $g^{-1}ag = a$ and $g^{-1}bg = b^{p_1^n}$ should be satisfied. Since condition $|k| \neq 1$ implies that $Z_{G_k}(a) = \langle a \rangle$ ($Z_{G_k}(a)$ is the centralizer in group G_k of element a), we have $g = a^m$ for some integer m. So, the equality $g^{-1}bg = b^{p_1^n}$ has the form $a^{-m}ba^m = b^{p_1^n}$. But G_k is the HNN-extension with the base group of the infinite cycle $\langle b \rangle$ and stable letter a. Since the left-hand side of the last equality is not reduced, $|k| \neq 1$ and $n > 0$, we should have $m > 0$. Consequently, using the defining relations of group G_k, this equality has the form $b^{k^m} = b^{p_1^n}$, which gives $k^m = p_1^n$. Hence, p_1 is the unique prime divisor of k, which is required.

Conversely, let $k = \delta p^e$, where $\delta = \pm 1$, p is a prime number, and $e > 0$. By Proposition 1, for this case, group Inn G_k is generated by the automorphisms φ, τ, and ψ (where φ and τ are defined above and ψ_1 is defined by: $a\psi = a$, $b\psi = b^p$) and is defined by the relations

$$\psi_i^{-1}\varphi\psi = \varphi^p, \quad \tau^2 = 1, \quad \tau\psi = \psi\tau, \quad \tau^{-1}\varphi\tau = \varphi^{-1}.$$

Since $a\varphi^n = ab^n$ for any integer n and the defining relations of group G_k yield $bab^{-1} = ab^{k-1}$, it follows that automorphism φ^{k-1} is inner. But $k - 1 \neq 0$; thus, automorphism φ has a finite order modulo Inn G_k.

Similarly, the equality $b\psi^n = b^{p^n}$, which is satisfied for any integer $n \geq 0$, shows that if $\delta = 1$, then automorphism ψ^e is inner and if $\delta = -1$, then automorphism ψ^{2e} is inner.

So let f : Aut $G_k \longrightarrow$ Aut G_k/Inn G_k be the canonical homomorphism of group Aut G_k onto the quotient group Aut G_k/Inn G_k and let X and Y be the image by f of the subgroups generated by φ and ψ, respectively. We see that

$$1 \leq X \leq Y \leq \text{Aut } G_k/\text{Inn } G_k$$

is a subnormal sequence with finite cyclic factors. Thus, the factor group Aut G_k/Inn G_k is finite and Proposition 2 is proved.

So, G_k is conjugacy separable and subgroup Inn G_k has a finite index in group Aut G_k if $k = \pm p^e$, where p is a prime number and $e \geq 1$. Thus, applying the theorem, we fond that G_k is Aut G_k-equivalent residually finite.

Example 2. Any group with the presentation

$$G_{mn} = \langle a, b; [a^m, b^n] = 1 \rangle, \quad \text{where} \quad m > 1 \text{ and } n > 1,$$

is Aut G_{mn}-equivalent residually finite.

Group G_{mn} is conjugacy separable [3] and Inn G_{mn} has a finite index in Aut G_{mn} [6]. So, by the theorem, group G_{mn} is Aut G_{mn}-equivalent residually finite.

We mention here that groups of the form G_k are Baumslag–Solitar groups, which are HNN extensions [1, 5], whereas groups of the form G_{mn} are free products with amalgamations [6].

5. ACKNOWLEDGEMENTS

The first author gratefully acknowledges financial support from the Institut des Hautes Etudes Scientifiques (IHES) for his visit to IHES.

REFERENCES

1. D. Collins, *The Automorphism Towers of Some One-Relator Groups*, Proc. London Math. Soc. **36**(3), 480 (1973).
2. K. W. Gruenberg, *Residual Properties of Infinite Soluble Groups*, *Proc. London. Math. Soc.* **7**(3), 29 (1957).
3. G. Kim, J. McCarron and C. Y. Tang, *Adjoining Roots to Conjugacy Separable Groups*, J. Algebra **176**, 327 (1995).
4. A. I. Mal'cev, *On Homomorphisms Onto Finite Groups*, Am. Math. Soc. Transl. **119**(2), 67 (1983).
5. D. I. Moldavanskii, N. V. Kravchenko, E. N. Frolova, *Conjugacy Separability of Certain One-Relator Groups*, In "Algorithmic Problems in Theory of Groups and Semigroups" (Tula State Pedagogical University, 1986), p. 81.
6. D. Tieudjo, D. I. Moldavanskii, *On the Automorphisms of Some One Relator Groups*, Comm. Algebra **34** 3975 (2006).