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ON ROOT CLASS RESIDUALITY OF HNN-EXTENSIONS

DANIEL TIEUDJO

Abstract. A sufficient condition for root-class residuality of HNN-extensions with root-class residual base group is proven; namely if $G = \langle A, t;\ t^{-1}Ht = K, \varphi\rangle$ is the HNN-extension with base group A, stable letter t and associated subgroups H and K via the isomorphism φ, then G is root-class residual if group A is root-class residual and there exists a homomorphism σ of group G onto some group of a root-class such that σ is one-to-one on H. For the particular case when $H = K$ and φ is the identity map, it is shown that G is root-class residual if and only if A is root-class residual and subgroup H of A is root-class separable. These results are generalized to multiple HNN-extensions.

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1. Introduction

Let $\mathcal{K}$ be an abstract class of groups. Suppose $\mathcal{K}$ contains at least a non-trivial group. Then $\mathcal{K}$ is called a root-class if the following conditions are satisfied:

1. If $A \in \mathcal{K}$ and $B \leq A$, then $B \in \mathcal{K}$.
2. If $A \in \mathcal{K}$ and $B \in \mathcal{K}$, then $A \times B \in \mathcal{K}$.
3. If $1 \leq C \leq B \leq A$ is a subnormal sequence and $A/B,\ B/C \in \mathcal{K}$, then there exists a normal subgroup D in group A such that $D \leq C$ and $A/D \in \mathcal{K}$. For more details about root properties, see [3].

In this paper, we study root-class residuality of HNN-extensions.

We recall that a group G is *root-class residual* (or *$\mathcal{K}$-residual* for a root-class $\mathcal{K}$) if, for every $1 \neq g \in G$, there exists a homomorphism φ of the group G onto some group X of root-class $\mathcal{K}$ such that $g\varphi \neq 1$. Equivalently, G is $\mathcal{K}$-residual if, for every $1 \neq g \in G$, there exists a normal subgroup N of G such that $G/N \in \mathcal{K}$ and $g \notin N$. The most investigated residual properties of groups are residual finiteness (i.e. finite groups residuality), p-finite groups residuality and residual solvability (i.e. solvable groups residuality) [1,2,7,8,9]. All these three classes of groups are root-classes. Therefore results about root-class residuality have enough general character.

So let A be a group. Let H and K be two subgroups of A and let $\varphi : H \longrightarrow K$ be an isomorphism. Let

$$G = \langle A, t;\ t^{-1}ht = \varphi(h),\ h \in H\rangle$$

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be the HNN-extension with base group A, stable letter t and associated subgroups H and K via φ. We shall prove:

Theorem 1.1. *Let $\mathcal{K}$ be a given root-class. The HNN-extension $G = \langle A, t;\ t^{-1}Ht = K, \varphi\rangle$ is $\mathcal{K}$-residual if the group A is $\mathcal{K}$-residual and there exists a homomorphism σ of G onto some group of root-class $\mathcal{K}$ such that σ is one-to-one on H.*

It is evident that if $H = K = 1$ or if H is finite, then the above sufficient condition of root-class residuality of group G will be necessary as well.

Another restriction permitting to obtain criteria for root-class residuality of HNN-extension with base group A, stable letter t and associated subgroups H and K is the equality of the associated subgroups. We prove:

Theorem 1.2. *Let $\mathcal{K}$ be a given root-class. Let $G = \langle A, t;\ t^{-1}Ht = K, \varphi\rangle$ be the HNN-extension with base group A, stable letter t and associated subgroups H and K via φ such that $H = K$ and φ is the identity map on H. Then G is $\mathcal{K}$-residual if and only if group A is $\mathcal{K}$-residual and subgroup H is $\mathcal{K}$-separable in A.*

This result generalizes for example Lemma 3.1 in [5] where analogous result is proven for the particular case of the class of all finite p-groups.

Although HNN-extensions are basically defined with multiple stable letters and multiple associated subgroups, mostly HNN-extensions with only one stable letter have been studied. However M. Shirvani in [10] examined residual finiteness of HNN-extensions with multiple stable letters and associated subgroups (multiple HNN-extensions). We also study root-class residuality of multiple HNN-extensions. We will generalize Theorems 1.1 and 1.2 above to multiple HNN-extensions.

2. Proof of Theorems 1.1–1.2

We first prove some useful results.

Proposition 2.1. *Let $\mathcal{K}$ be a root-class. Then*

1. *If a group G has a subnormal sequence with factors belonging to class $\mathcal{K}$, then $G \in \mathcal{K}$.*
2. *If $F \trianglelefteq G$, $G/F \in \mathcal{K}$ and F is $\mathcal{K}$-residual, then group G is also $\mathcal{K}$-residual.*
3. *If $A \trianglelefteq G$, $B \trianglelefteq G$, $G/A \in \mathcal{K}$ and $G/B \in \mathcal{K}$, then $G/(A \cap B) \in \mathcal{K}$.*

In fact, from Property 3 of the definition of root-class, it follows that root-class is closed under any extension. So, the first property of Proposition 2.1 is satisfied. The second and third properties are also easily verified by the definition of root-class.

In [3] Theorem 6.2, Gruenberg states that:
free product of root-class residual groups is root-class residual if and only if every free group is root-class residual.

It happens, and we prove that, the given necessary and sufficient condition is satisfied for every root-class. Thus,

Proposition 2.2. *Every free group is root-class residual, for every root-class.*

Proof. For the proof, we remark that every root-class $\mathcal{K}$ contains a nontrivial cyclic group (Property 1 of the definition of root-class). If $\mathcal{K}$ contains an infinite cyclic group then, by Proposition 2.1, $\mathcal{K}$ contains any group possessing a subnormal sequence with infinite cyclic factors; thus, all finitely generated nilpotent torsion-free groups belong to class $\mathcal{K}$. If $\mathcal{K}$ contains a finite non trivial cyclic group, then $\mathcal{K}$ contains group of prime order p and consequently, by Proposition 2.1, $\mathcal{K}$ contains all groups possessing a subnormal sequence with factors of order p; hence all finite p-groups belong to $\mathcal{K}$. So any root-class contains all finitely generated nilpotent torsion-free groups or all finite p-groups, for some prime p. But free groups are residually finitely generated torsion-free and also residually p-finite. Therefore, free groups are $\mathcal{K}$-residual, for every root-class $\mathcal{K}$ and this ends the proof of Proposition 2.2.

Now, Proposition 2.3 below directly follows from Proposition 2.2 and Gruenberg's result formulated above.

Proposition 2.3. *A free product of root-class residual groups is root-class residual.*

Let's recall the construction of HNN-extensions.

Let A be a group, H and K two subgroups of group A and let $\varphi : H \longrightarrow K$ be an isomorphism. Then the *HNN-extension* with base group A, stable letter t and associated subgroups H and K denoted by

$$G = \langle A, t;\ t^{-1}ht = \varphi(h),\ h \in H\rangle$$

is the group generated by all the generators of the group A and one more element t and defined by all the relators of group A and all possible relations of form $t^{-1}ht = \varphi(h),\ h \in H$.

For this construction, every element $g \in G$ can be written as

$$g = x_0 t^{\epsilon_1} \cdots t^{\epsilon_r} x_r \tag{2.1}$$

where for any $i = 0, 1, \ldots, r$ element x_i belongs to the subgroup A, $\epsilon_i = \pm 1$ and if $r > 1$, there is no consecutive subwords of type $t^{-1}x_i t$ or $tx_j t^{-1}$ with $x_i \in H$ or $x_j \in K$ in script (2.1).

Such form of element g is called *reduced* and r – its *length*.

By Britton's Lemma [see 6, p. 181], if $g = x_0 t^{\epsilon_1} \cdots t^{\epsilon_r} x_r$ is reduced and $r \geq 1$, then $g \neq 1$ in group G.

The HNN-extension with base group A, stable letter t and associated subgroups H and K can also be denoted

$$G = \langle A, t;\ t^{-1}Ht = K,\ \varphi\rangle.$$

We now establish Theorem 2.1 from Proposition 2.3 and H. Neumann's theorem ([6], p. 212):

Let $\mathcal{K}$ be a root-class. Let $G = \langle A, t;\ t^{-1}Ht = K, \varphi\rangle$ be the HNN-extension with base group A, stable letter t and associated subgroups H and K via φ. Assume that the group A is $\mathcal{K}$-residual. Suppose there exists a homomorphism σ of G onto some group of class $\mathcal{K}$, such that σ is one-to-one on H. Let us denote by N the kernel of the homomorphism σ. Then $G/N \in \mathcal{K}$ and $N \cap H = 1$. By H. Neumann's theorem ([6], p. 212) or by [4], N is the free product of a free group F and some subgroups of group G of the form

$$g^{-1}Ag \cap N, \tag{2.2}$$

where $g \in G$. Since group A is $\mathcal{K}$-residual, the subgroups of form (2.2) are also $\mathcal{K}$-residual. Therefore N is $\mathcal{K}$-residual as a free product of $\mathcal{K}$-residual groups (Proposition 2.3), since free group F is $\mathcal{K}$-residual (Proposition 2.2). Moreover, since $G/N \in \mathcal{K}$, then by Property 2 of Proposition 2.1, it follows that G is $\mathcal{K}$-residual and Theorem 1.1 is proven.

We also see that, if $A = H = K$, then A is a normal subgroup of G and $G/A \cong \langle t\rangle$. Therefore G is an extension of a group of class $\mathcal{K}$ by a free group; and thus is $\mathcal{K}$-residual.

We now prove Theorem 1.2.

We first recall that a subgroup H of a group A is *root-class separable* (or $\mathcal{K}$*-separable* for a root-class $\mathcal{K}$) if, for any element a of A, where $a \notin H$, there exists a homomorphism φ of group A onto some group X of root-class $\mathcal{K}$ such that $a\varphi \notin H\varphi$. This means that, for any $a \in A \setminus H$, there exists a normal subgroup N of A such that $A/N \in \mathcal{K}$ and $a \notin NH$.

So let $\mathcal{K}$ be a root-class. Let $G = \langle A, t;\ t^{-1}Ht = K, \varphi\rangle$ be the HNN-extension with base group A, stable letter t and associated subgroups H and K such that $H = K$ and φ is the identity map on H. For any normal subgroup N of group A one can define the HNN-extension

$$G_N = \langle A/N,\ t;\ t^{-1}HN/Nt = HN/N,\ \varphi_N\rangle$$

where φ_N is the identity map on subgroup HN/N of group G_N, and the homomorphism $\rho_N : G \longrightarrow G_N$, extending the canonical homomorphism $A \longrightarrow A/N$ and $t \longmapsto t$. Consider the homomorphism $\sigma : G_N \longrightarrow A$ which is the identity map on A, and maps $t \longmapsto 1$. Then $ker\sigma = \langle t\rangle^{G_N}$ is free by [6], Theorem 6.6. So, $G_N/\langle t\rangle^{G_N} \cong A/N$ and G_N is an extension of a free group by group A/N. Therefore, if A/N belongs to root-class $\mathcal{K}$ then, G_N is $\mathcal{K}$-residual. Thus, to prove $\mathcal{K}$-residuality of G, it is enough to show that G is residually a group of kind G_N, where $A/N \in \mathcal{K}$.

Suppose the group A is $\mathcal{K}$-residual and the subgroup H is $\mathcal{K}$-separable in A. Let $1 \neq g \in G$. Assume that element g has a reduced form $g = a_0t^{\epsilon_1}\cdots t^{\epsilon_s}a_s$. Two cases arise:

1. $s \geq 1$. In this case, for every $i = 0, \ldots, s$, $a_i \in A$, $\epsilon_i = \pm 1$ and there is no consecutive sequences of type t^{-1}, a_i, t or t, a_j, t^{-1} with $a_i, a_j \in H$. From $\mathcal{K}$-separability of H, it follows that, for every $i = 0, \ldots, s$, there exists a normal subgroup N_i of A such that $A/N_i \in \mathcal{K}$ and $a_i \notin HN_i$. Thus, there will be no consecutive sequences of type t^{-1}, a_iN_i, t or t, a_jN_i, t^{-1} with $a_i, a_j \in H$. So let $N = N_0 \cap \cdots \cap N_s$. By Proposition 2.1, $A/N \in \mathcal{K}$ and, it is clear that, for every $i = 0, \ldots, s$, $a_i \notin HN$ and there is no consecutive subwords of type t^{-1}, a_iN, t or t, a_jN, t^{-1} with $a_i, a_j \in H$. Therefore the form

$$g\rho_N = a_0\rho_N t^{\epsilon_1}\cdots t^{\epsilon_s}a_s\rho_N$$

is reduced and has length $s \geq 1$. Consequently $g\rho_N \neq 1$.

2. $s = 0$ i.e. $g \in A$. Since A is $\mathcal{K}$-residual, there exists a normal subgroup N of A such that $A/N \in \mathcal{K}$ and $g \notin N$, i.e. $gN \neq N$. So, $g\rho_N \neq 1$.

Hence, for any element $g \neq 1$, there exists a normal subgroup N in A, such that $A/N \in \mathcal{K}$ and the homomorphism $\rho_N : G \longrightarrow G_N$ maps element g to a non identity element. Consequently, G is residually a group G_N, where $A/N \in \mathcal{K}$. Therefore G is $\mathcal{K}$-residual.

Conversely, suppose G is $\mathcal{K}$-residual. Evidently, its subgroup A has the same property. It remains to show that H is $\mathcal{K}$-separable in group A. If H is not $\mathcal{K}$-separable in A, we choose element $a \in A \setminus H$ such that $a \in NH$, for all normal subgroup N of A such that $A/N \in \mathcal{K}$. Let $g = t^{-1}ata^{-1}$. Then g has length greater than 1. By Britton's lemma, $g \neq 1$. Let M be a normal subgroup of G such that $G/M \in \mathcal{K}$ and $g \notin M$, since G is $\mathcal{K}$-residual. So let $R = M \cap A$. R is a normal subgroup of A and furthermore $A/R \in \mathcal{K}$. Consequently the canonical homomorphism $A \longrightarrow A/R$ extends to an epimorphism $\pi : G \longrightarrow G_R$, where $G_R = \langle A/R, t;\ t^{-1}HR/R\ t = HR/R,\ \varphi_R\rangle$. Hence $a \in RH$ by the choice of a. Thus, there exists $h \in H$ such that $\pi(a) = \bar{h}$. Then $\pi(g) = \pi(t^{-1}ata^{-1}) = t^{-1}\bar{h}t\bar{h}^{-1} = 1$. Hence, $g \in Ker(\pi) = \langle R\rangle^G \leq M$ and this is a contradiction.

We remark that, the necessary condition for Theorem 1.2 will also holds when $\mathcal{K}$ satisfies only Properties 1 and 2 of the definition of root-class.

3. Generalization

Let A be a group and let I be an index set. Let H_i and K_i, $i \in I$ be families of subgroups of group A with $(\varphi_i)_{i\in I}$ a family of maps such that $\varphi_i : H_i \longrightarrow K_i$ is an isomorphism. Then the HNN-extension with base group A, stable letters t_i, $i \in I$, and associated subgroups H_i and K_i, $i \in I$, denoted by

$$G = \langle A, t_i \ (i \in I);\ t_i^{-1} h_i t_i = \varphi_i(h_i), h_i \in H_i \rangle$$

is the group generated by all the generators of A and elements t_i, $(i \in I)$ and defined by all the relators of A and all possible relations of form $t_i^{-1} h_i t_i = \varphi_i(h_i)$, $h_i \in H_i$ for all $i \in I$.

The group G defined above will be called the *multiple HNN-extension* of base group A, stable letters t_i, $i \in I$, and associated subgroups H_i and K_i, $i \in I$.

In fact, let $G_0 = A$ and

$$G_1 = \langle A, t_1;\ t_1^{-1} H_1 t_1 = K_1,\ \varphi_1 \rangle;$$

we see that the double HNN-extension

$$G_2 = \langle A, t_1, t_2;\ t_1^{-1} H_1 t_1 = K_1, t_2^{-1} H_2 t_2 = K_2,\ \varphi_1, \varphi_2 \rangle$$

is the HNN-extension with base group G_1, stable letter t_2, and associated subgroups H_2 and K_2 via φ_2; i.e.

$$G_2 = \langle G_1, t_2;\ t_2^{-1} H_2 t_2 = K_2,\ \varphi_2 \rangle.$$

Thus, for j of an index set I, G_j is the HNN-extension with base group G_{j-1}, stable letter t_j and associated subgroups H_j and K_j via φ_j i.e.

$$\begin{aligned} G_j &= \langle A, t_1, \dots, t_j;\ t_1^{-1} H_1 t_1 = K_1, \dots, t_j^{-1} H_j t_j = K_j,\ \varphi_1, \dots, \varphi_j \rangle \\ &= \langle G_{j-1}, t_j;\ t_j^{-1} H_j t_j = K_j,\ \varphi_j \rangle \end{aligned}$$

For this construction, we have the following results.

Theorem 3.1. *Let $\mathcal{K}$ be a root-class. For any index set I, the multiple HNN-extension*

$$G = \langle A, t_i \ (i \in I);\ t_i^{-1} h_i t_i = \varphi_i(h_i),\ h_i \in H_i \rangle$$

with base group A, stable letters t_i, $i \in I$, and associated subgroups H_i and K_i via φ_i ($i \in I$), is $\mathcal{K}$-residual if A is $\mathcal{K}$-residual and there exists a sequence $(\sigma_i)_{i\in I}$ of homomorphisms of group G_i onto some group X_i of root-class $\mathcal{K}$, such that σ_i is one-to-one on subgroup H_i for all $i \in I$.

The proof is similar to the proof of Theorem 1.1.

For other criteria of root-class residuality of multiple HNN-extensions with base group A, stable letters t_i and associated subgroups H_i and K_i ($i \in I$), we may assume the equality of the associated subgroups H_i and K_i for all $i \in I$.

So, suppose $H_i = K_i$ and φ_i is the identity map on H_i for all $i \in I$. Then for such group we have the following criterium which generalizes Theorem 1.2 and the proof is just a repetition of its.

Theorem 3.2. *The multiple HNN-extension*

$$G = \langle A, t_i \ (i \in I);\ t_i^{-1} h_i t_i = \varphi_i(h_i),\ h_i \in H_i \rangle$$

with base group A, stable letters t_i, $i \in I$, and associated subgroups H_i and K_i via φ_i such that $H_i = K_i$ and φ_i is the identity map on H_i for all $i \in I$, is $\mathcal{K}$-residual if and only if A is $\mathcal{K}$-residual and subgroup H_i is $\mathcal{K}$-separable in G_i for all $i \in I$.

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(Daniel Tieudjo) Department of Mathematics and Computer Science, University of Ngaoundere, P.O. Box 454, Ngaoundere, Cameroon.
E-mail address: tieudjo@yahoo.com