JP Journal of Algebra, Number Theory and Applications
Volume …, Number …, 2010, Pages …
This paper is available online at http://www.pphmj.com

ROOT-CLASS RESIDUALITY OF SOME FREE CONSTRUCTIONS

D. TIEUDJO

Max-Planck-Institute for Mathematics
Vivagasse 7, 53111 Bonn, Germany
e-mail: tieudjo@mpim-bonn.mpg.de

University of Ngaoundere
P. O. Box 454, Ngaoundere, Cameroon
e-mail: tieudjo@yahoo.com

Abstract

This is a survey of some recent results obtained on root-class residuality. First, we review and extend some properties of root-class residuality of generalized free products and HNN-extensions. Then conditions such that, by adjoining roots to a root-class residual group, the resulting group is again root-class residual, are derived. These results are extended to generalized free product of infinitely many groups amalgamating a common subgroup and also to multiple HNN-extensions. Further, they are applied to study root-class residuality of some one-relator groups.

1. Introduction

Let $\mathcal{K}$ denote an abstract non-empty class of groups. Then $\mathcal{K}$ is called a *root-class* if the following conditions are satisfied:

1. $\mathcal{K}$ is closed under taking subgroups, i.e., if $A \in \mathcal{K}$ and $B \leq A$, then $B \in \mathcal{K}$.

2010 Mathematics Subject Classification: Primary 20E26, 20E06; Secondary 20F19, 20F05.

Keywords and phrases: root-class, root-class residuality, root-class separability, generalized free product, HNN-extensions.

Received November 12, 2009

2. $\mathcal{K}$ is closed under taking direct products, i.e., if $A \in \mathcal{K}$ and $B \in \mathcal{K}$, then $A \times B \in \mathcal{K}$.

3. If $1 \leq C \leq B \leq A$ is a subnormal sequence and A/B, $B/C \in \mathcal{K}$, then there exists a normal subgroup D in group A such that $D \leq C$ and $A/D \in \mathcal{K}$. See [6], for more details about root properties.

We recall that a group G is *root-class residual* (or $\mathcal{K}$*-residual*, for a root-class $\mathcal{K}$) if, for every non-identity element $g \in G$, there exists a homomorphism φ from G to some group G' of root-class $\mathcal{K}$ such that $g\varphi \neq 1$. Equivalently, G is $\mathcal{K}$-*residual* if, for every non-identity element $g \in G$, there exists a normal subgroup N of G such that $G/N \in \mathcal{K}$ and $g \notin N$.

Famous examples of root-classes are the class of all finite groups, the class of all finite *p*-groups, the class of all soluble groups, the class of all finitely generated nilpotent groups. For these examples, root-class residuality is just residual finiteness, finite *p*-groups residuality, residual solvability, finitely generated nilpotent residuality, respectively. Thus, root-class residuality is more general. Residual finiteness, finite *p*-groups residuality, residual solvability are the most investigated residual properties of groups. See for example [2, 3, 14-16].

In this paper, we present some results on root-class residuality of generalized free products and HNN-extensions. In [1], some properties of root-class residuality of amalgamated free products were obtained. Analogous results for HNN-extensions were proved in [19]. Here, we review and extend these results. We first recall with proofs, root-class residuality of free groups and free products of root-class residual groups. Then sufficient conditions for root-class residuality of generalized free product $G = (A * B;\, H = K,\, \varphi)$ of root-class residual groups A and B amalgamating subgroups H and K through the isomorphism φ, and for root-class residuality of HNN-extensions $G = \langle A,\, t;\, t^{-1}ht = \varphi(h),\, h \in H \rangle$ with root-class residual base group A are derived; for some particular cases, necessary and sufficient conditions (criteria) are given. Further, conditions for adjoining roots to root-class residual groups to be root-class residual are stated. The results are extended to generalized free product of infinitely many groups amalgamating a common subgroup and also to multiple HNN-extensions. Finally, we apply these results to study root-class residuality of some one-relator groups.

2. Root-class Residuality of Free Groups and Free Products

In this section, we present root-class residuality of free groups and free products of root-class residual groups.

Let $\mathcal{K}$ be a root-class of groups. The following properties are easily verified.

Lemma. *Let* $\mathcal{K}$ *be a root-class of groups. Then*:

1. *If a group* G *has a subnormal sequence with factors belonging to class* $\mathcal{K}$, *then* $G \in \mathcal{K}$.

2. *If* $F \trianglelefteq G$, $G/F \in \mathcal{K}$ *and* $F \in \mathcal{K}$, *then group* $G \in \mathcal{K}$.

3. *If* $A \trianglelefteq G$, $B \trianglelefteq G$, $G/A \in \mathcal{K}$ *and* $G/B \in \mathcal{K}$, *then* $G/(A \cap B) \in \mathcal{K}$.

Indeed, root-class is closed for extensions. This follows from the definition of root-class. So the first property of Lemma is satisfied. The second and third properties are easily verified by the definition of root-class.

In [6], Theorem 6.2, Gruenberg states that

Free product of root-class residual groups is root-class residual if and only if every free group is root-class residual.

However, it happens that the above given condition is necessary and sufficient for every root-class $\mathcal{K}$.

Theorem 2.1. *Every free group is* $\mathcal{K}$*-residual, for every root-class* $\mathcal{K}$.

Proof. We see that every root-class $\mathcal{K}$ contains a non-trivial cyclic group (Property 1 of the definition of root-class). If $\mathcal{K}$ contains an infinite cyclic group, then, by Lemma, $\mathcal{K}$ contains any group possessing subnormal sequence with infinite cyclic factors; thus all finitely generated nilpotent torsion-free groups belong to class $\mathcal{K}$. Also, if $\mathcal{K}$ contains a finite non-trivial cyclic group, then $\mathcal{K}$ contains a group of prime order p and consequently, by Lemma, $\mathcal{K}$ contains all groups possessing subnormal sequence with factors of order p; hence all finite p-groups belong to $\mathcal{K}$. So any root-class contains all finitely generated nilpotent torsion-free groups or all finite p-groups for some prime p. But free groups are residually finitely generated nilpotent torsion-free ([13], p. 347) and also residually *p*-finite ([7], p. 121). Therefore, free groups are $\mathcal{K}$-residual, for every root-class $\mathcal{K}$ and this ends the proof of Theorem 2.1.

Now, from the proof of Theorem 2.1 and the Gruenberg's result formulated above, Theorem 2.2 directly follows:

Theorem 2.2. *Free product of root-class residual groups is root-class residual.*

3. Root-class Residuality of Generalized Free Products

This section is focused on the study of root-class residuality of generalized free products.

We first give some useful properties of the construction of free product of groups with amalgamated subgroups.

Let *A* and *B* be two groups, each of which is given by the presentation:

$$A = \langle a_1, a_2, ..., a_m; W \rangle,$$

$$B = \langle b_1, b_2, ..., b_n; V \rangle.$$

Let also *H* and *K* be subgroups of group *A* and *B*, respectively, and let φ be an isomorphism of group *H* onto group *K*. Then by *free product of groups A and B, amalgamating subgroups H and K through the isomorphism* φ, we mean the group denoted $G = (A * B;\ H = K,\ \varphi)$, which is given by the presentation

$$G = \langle a_1, a_2, ..., a_m, b_1, b_2, ..., b_n; W, V, h = h\varphi\ (h \in H) \rangle.$$

Thus, the set of generators of group *G* is the disjoint union of the sets of generators of groups *A* and *B*; and the set of the defining relations of group *G* consists of the defining relations of groups *A* and *B* and every possible relation of the form $h = h\varphi$, where *h* is an element of *H* in the generators $a_1, a_2, ..., a_m$, and $h\varphi$ is an element of *K* in the generators $b_1, b_2, ..., b_n$, which is the corresponding image by the mapping φ of *h*.

To point out the fact that groups *A* and *B* are identified with the indicated subgroups of group *G*, we denote this group by $G = (A * B;\ H)$ and call it the free product of groups A and B amalgamating subgroup H (considering that isomorphism φ is given).

A *reduced form* of an element $g \in G$ is the representation of this element as product

$$g = x_1 x_2 \cdots x_s,$$

where components $x_1, x_2, \ldots, x_s$ belong, in turn, to subgroups A and B, and if $s > 1$, then any of these components does not belong to subgroup H.

In general, an element g of group $G = (A * B;\ H)$ can have more than one reduced form. In this case, components of the same index lie in the same subgroup A or B and the number of components in these forms is the same. We call this number *the length of element g* and denote $l(g)$.

Thus if element $g = x_1 x_2 \cdots x_s$ of group $G = (A * B;\ H)$ is reduced and $s > 1$, then $g \neq 1$. If $s = 1$, then $g \in A$ or $g \in B$.

From Theorem 2.2 and H. Neumann's theorem ([12], p. 212), the following result is easily established:

Theorem 3.1. *Let $\mathcal{K}$ be a root-class. Then the generalized free product $G = (A * B;\ H)$ of groups A and B amalgamating subgroup H is $\mathcal{K}$-residual if groups A and B are $\mathcal{K}$-residual and there exists a homomorphism σ from G to a group G' of root-class $\mathcal{K}$, such that σ is injective on H.*

Proof. Let $\mathcal{K}$ be a root-class. Let $G = (A * B;\ H)$ be the generalized free product of groups A and B amalgamating subgroup H and let groups A and B be $\mathcal{K}$-residual. Suppose there exists a homomorphism σ of G to a group of class $\mathcal{K}$, which is injective on H. Let N be the kernel of the homomorphism σ. Then $G/N \in \mathcal{K}$ and $N \cap H = 1$. Now, by H. Neumann's Theorem ([12], p. 212) N is the free product of a free group F and some subgroups of group G of the form

$$g^{-1}Ag \cap N, \quad g^{-1}Bg \cap N, \tag{1}$$

where $g \in G$. The subgroups of the form (1) are $\mathcal{K}$-residuals since are groups A and B. By Theorem 2.1, free group F is also $\mathcal{K}$-residual. Thus N is a free product of root-class residual groups. Therefore, by Theorem 2.2, N is root-class residual. Moreover, since $G/N \in \mathcal{K}$, by Property 2 of Lemma, it follows that group G is root-class residual. Theorem 3.1 is proven.

Remark that Theorem 2.2 can be considered as a particular case of Theorem 3.1. We also see that, if the amalgamated subgroup H is finite, then the formulated above sufficient condition of root-class residuality of group G will be as well necessary.

Another restriction permitting to obtain simple criteria of root-class residuality of generalized free product of groups A and B amalgamating subgroup H is the equality of the free factors A and B.

More precisely, let G be the generalized free product of groups A and B amalgamating subgroups H and K through the isomorphism φ. If $A = B$, $H = K$ and φ is the identity map, we denote group G by $G = A \underset{H}{\star} A$. This construction is sometimes called the *generalized free square of group A over subgroup H* (see [9]). Then for the generalized free square of group A over subgroup H we prove the following criterion:

Theorem 3.2. *Let $\mathcal{K}$ be a root-class. The group $G = A \underset{H}{\star} A$ is $\mathcal{K}$-residual if and only if group A is $\mathcal{K}$-residual and the subgroup H of A is $\mathcal{K}$-separable.*

We recall that subgroup H of a group A is *root-class separable* (or *$\mathcal{K}$-separable*, for a root-class $\mathcal{K}$) if, for any element a of A and $a \notin H$, there exists a homomorphism φ from A to a group of root-class $\mathcal{K}$ such that $a\varphi \notin H\varphi$. This means that, for each $a \in A \backslash H$, there exists a normal subgroup N of A such that $A/N \in \mathcal{K}$ and $a \notin NH$.

Let us now prove Theorem 3.2.

Proof. Let $\mathcal{K}$ be a root-class. Let $G = A \underset{H}{*} A$. For any normal subgroup N of group A one can define the generalized free square

$$G_N = A/N \underset{HN/N}{*} A/N$$

of group A/N over subgroup HN/N and the homomorphism $\varepsilon_N : G \to G_N$, extending the canonical homomorphism $A \to A/N$. It is evident that group G_N is an extension of free group with group A/N. So, if A/N belongs to root-class $\mathcal{K}$, then by Lemma and Theorem 2.1, G_N is $\mathcal{K}$-residual. Thus, to prove that G is $\mathcal{K}$-residual, it is enough to show that G is residually a group of the form G_N such that $A/N \in \mathcal{K}$.

Suppose group A is $\mathcal{K}$-residual and subgroup H of A is $\mathcal{K}$-separable. Let $g \in G$ such that $g \neq 1$. Also, let $g = a_1 \cdots a_s$ be the reduced form of element g. Then two cases arise:

1. $s > 1$. In this case, $a_i \in A \backslash H$ for all $i = 1, ..., s$. From $\mathcal{K}$-separability of H, it follows that, for every $i = 1, ..., s$, there exits a normal subgroup N_i of group A such that $A/N_i \in \mathcal{K}$ and $a_i \notin HN_i$. Let $N = N_1 \cap \cdots \cap N_s$. By Lemma, $A/N \in \mathcal{K}$ and, it is clear that, for all $i = 1, ..., s$, $a_i \notin HN$, i.e., $a_iN \notin HN/N$. So, for all $i = 1, ..., s$, $a_i\varepsilon_N \notin H\varepsilon_N$. Therefore the form

$$g\varepsilon_N = a_1\varepsilon_N \cdots a_s\varepsilon_N$$

is reduced and has length $s > 1$.

Consequently $g\varepsilon_N \neq 1$.

2. $s = 1$, i.e., $g \in A$. As group A is $\mathcal{K}$-residual, there exists a normal subgroup N of A such that $A/N \in \mathcal{K}$ and $g \notin N$, i.e., $gN \neq N$. Hence $g\varepsilon_N \neq 1$.

Thus, in any case, for an element $g \neq 1$ in group A, there exists a normal subgroup N such that $A/N \in \mathcal{K}$ and the homomorphism $\varepsilon_N : G \to G_N$ transforms g to a non-identity element. Hence group G is residually a group G_N, where $A/N \in \mathcal{K}$. Therefore G is $\mathcal{K}$-residual.

Conversely, suppose group G is $\mathcal{K}$-residual. Evidently this subgroup A has the same property. Let us prove that H be a $\mathcal{K}$-separable subgroup of group A. Let γ be an automorphism of group G canonically permuting the free factor. Let $a \in A \backslash H$. Then $a\gamma \neq a$. Since G is $\mathcal{K}$-residual, there exists a normal subgroup N of G such that $G/N \in \mathcal{K}$ and $aN \neq a\gamma N$. Let $M = N \cap N\gamma$. Then

$$M\gamma = N\gamma \cap N\gamma^2 = N\gamma \cap N = M.$$

Consequently, in the quotient-group G/M, it is possible to consider the automorphism $\bar{\gamma}$, induced by γ. Since $aN \neq a\gamma N$ and $M \leq N$, $aM \neq a\gamma M$. On the other hand, $a\gamma M = (aM)\bar{\gamma}$. Thus $aM \neq (aM)\bar{\gamma}$. Since γ acts identically on H, $\bar{\gamma}$ also acts identically on HM/M. So and since $aM \neq (aM)\bar{\gamma}$, it follows that $aM \notin HM/M$, i.e., $a\varepsilon \notin H\varepsilon$, where ε is the canonical homomorphism of group G onto G/M. Consequently, $G/M \in \mathcal{K}$ and the $\mathcal{K}$-separability of subgroup H of group A is demonstrated.

In [11], the above result is obtained for the particular case of the class of all finite p-groups.

We also remark that the necessary condition for Theorem 3.2 takes place even at more gentle restriction on class $\mathcal{K}$, namely when $\mathcal{K}$ satisfies only properties 1 and 2 of the definition of root-class.

Further, the generalized free product of infinitely many groups amalgamating subgroup is introduced in [17]. Some results on residual properties of this construction are shown in [5]. We extend Theorems 3.1 and 3.2 above to generalized free products of every family $(G_\lambda)_{\lambda\in\Lambda}$ of groups G_λ amalgamating a common subgroup H (Theorems 3.3 and 3.4).

Let $(G_\lambda)_{\lambda\in\Lambda}$ be a family of groups, where the set Λ can be infinite. Let $H_\lambda \leq G_\lambda$, for every $\lambda \in \Lambda$. Suppose also that, for every $\lambda, \mu \in \Lambda$, there exists an isomorphism $\varphi_{\lambda\mu} : H_\lambda \to H_\mu$ such that, for all $\lambda, \mu, \nu \in \Lambda$, the following conditions are satisfied: $\varphi_{\lambda\lambda} = id_{H_\lambda}$, $\varphi_{\lambda\mu}^{-1} = \varphi_{\mu\lambda}$, $\varphi_{\lambda\mu}\varphi_{\mu\nu} = \varphi_{\lambda\nu}$. Let now

$$G = (\mathop{\star}_{\lambda\in\Lambda} G_\lambda;\quad h\varphi_{\lambda\mu} = h \quad (h \in H_\lambda,\ \ \lambda, \mu \in \Lambda))$$

be the group generated by groups G_λ $(\lambda \in \Lambda)$ and defined by all the relators of these groups and moreover by all possible relations of the form $h\varphi_{\lambda\mu} = h$, where $h \in H_\lambda$, $\lambda, \mu \in \Lambda$. Then it is evident that every G_λ can be canonically embedded in group G and if we consider $G_\lambda \leq G$, then for all different $\lambda, \mu \in \Lambda$,

$$G_\lambda \cap G_\mu = H_\lambda = H_\mu.$$

Let us denote by H the subgroup of group G that equals to the common subgroups H_λ. Then G is the *generalized free product of the family* $(G_\lambda)_{\lambda\in\Lambda}$ *of groups* G_λ $(\lambda\in\Lambda)$ *amalgamating subgroup H*. We will consider, as well, that $G_\lambda \leq G$, for all $\lambda \in \Lambda$. See for example [5] or [17] for details about the generalized free product of a family of groups.

Theorem 3.3. *Let* $\mathcal{K}$ *be a root class. The generalized free product G of the family* $(G_\lambda)_{\lambda\in\Lambda}$ *of group* G_λ *amalgamating subgroup H is* $\mathcal{K}$*-residual if every group* G_λ *is* $\mathcal{K}$*-residual and there exists a homomorphism* σ *from G to a group* G' *of class* $\mathcal{K}$ *such that* σ *is injective on H*.

Proof. The proof is the same as that of Theorem 3.1.

In fact, let group G_λ be $\mathcal{K}$-residual, for all $\lambda \in \Lambda$. Suppose there exists a homomorphism σ of G to a group of class $\mathcal{K}$, which is one-to-one on H and let $N = ker\,\sigma$. Then $G/N \in \mathcal{K}$ and $N \cap H = 1$. But N is the free product of a free group F and some subgroups of group G of the form

$$g^{-1}G_\lambda g \cap N,$$

$($where $g \in G$ and $\lambda \in \Lambda)$ which are root-class residuals. Since F is also root-class residual by Theorem 2.1, N is a free product of root-class residual groups. Thus, by Theorem 2.2, N is root-class residual. Moreover, since $G/N \in \mathcal{K}$, by property 2 of Lemma, it follows that group G is root-class residual and the theorem is proven.

Suppose now that, for all $\lambda \in \Lambda$, $G_\lambda = A$. Then, in this case, the generalized free product of the family $(G_\lambda)_{\lambda\in\Lambda}$ of groups G_λ amalgamating subgroup H is called the *generalized free power of group A over subgroup H*. It is denoted by P and written as $P = A \underset{H}{\star} \cdots \underset{H}{\star} A$. For such group P, we have the following criterion:

Theorem 3.4. *Let* $\mathcal{K}$ *be a root-class. The group* $P = A \underset{H}{\star} \cdots \underset{H}{\star} A$ *is* $\mathcal{K}$*-residual if and only if group A is* $\mathcal{K}$*-residual and the subgroup H of A is* $\mathcal{K}$*-separable*.

The proof is similar to that of Theorem 3.2.

4. Root-class Residuality of HNN-extensions

In this section, we study root-class residuality of HNN-extensions. Let us recall the construction of HNN-extensions.

Let A be a group, H and K two subgroups of group A and let $\varphi : H \to K$ be an isomorphism. Then the *HNN-extension with base group A, stable letter t and associated subgroups H and K* denoted by

$$G = \langle A, t;\ t^{-1}ht = \varphi(h),\ h \in H \rangle$$

is the group generated by all the generators of the group A and one more element t and defined by all the relators of group A and all possible relations of form $t^{-1}ht = \varphi(h),\ \ h \in H.$

For this construction, every element $g \in G$ can be written as

$$g = x_0 t^{\varepsilon_1} \cdots t^{\varepsilon_r} x_r, \qquad (2)$$

where for any $i = 0, 1, \ldots, r$ element x_i belongs to the subgroup A, $\varepsilon_i = \pm 1$ and if $r > 1$, there is no consecutive subwords of type $t^{-1}x_i t$ or $tx_j t^{-1}$ with $x_i \in H$ or $x_j \in K$ in script (2).

Such form of element g is called *reduced* and r – its *length*.

By Britton’s Lemma ([12], p. 181), if $g = x_0 t^{\varepsilon_1} \cdots t^{\varepsilon_r} x_r$ is reduced and $r \geq 1$, then $g \neq 1$ in group G.

The HNN-extension with base group A, stable letter t and associated subgroups H and K can also be denoted

$$G = \langle A, t;\ t^{-1}Ht = K,\ \varphi \rangle.$$

We prove:

Theorem 4.1. *The HNN-extension* $G = \langle A, t;\ t^{-1}Ht = K,\ \varphi \rangle$ *is* $\mathcal{K}$*-residual for a given root-class* $\mathcal{K}$ *if the base group A is* $\mathcal{K}$*-residual and there exists a homomorphism* σ *of G onto some group of root-class* $\mathcal{K}$ *such that* σ *is one-to-one on H.*

We establish Theorem 4.1 from Theorem 2.2 and H. Neumann's Theorem ([12], p. 212):

Proof. Let $\mathcal{K}$ be a root-class. Let $G = \langle A, t;\, t^{-1}Ht = K,\, \varphi \rangle$ be the HNN-extension with base group A, stable letter t and associated subgroups H and K via φ. Assume that the group A is $\mathcal{K}$-residual. Suppose there exists a homomorphism σ of G onto some group of class $\mathcal{K}$, such that σ is one-to-one on H. Denote by N the kernel of the homomorphism σ. Then $G/N \in \mathcal{K}$ and $N \cap H = 1$. By Neumann's Theorem ([12], p. 212) or by [8], N is the free product of a free group F and some subgroups of group G of the form

$$g^{-1}Ag \cap N, \tag{3}$$

where $g \in G$. Since group A is $\mathcal{K}$-residual, the subgroups of form (3) are also $\mathcal{K}$-residuals. Therefore N is $\mathcal{K}$-residual as a free product of $\mathcal{K}$-residual groups (Theorem 2.2), since free group F is $\mathcal{K}$-residual (Theorem 2.1). Moreover, since $G/N \in \mathcal{K}$, then by property 2 of Lemma, it follows that G is $\mathcal{K}$-residual and Theorem 4.1 is proven.

It is evident that if $H = K = 1$ or if H is finite, then the above sufficient condition of root-class residuality of group G will be necessary as well.

Another restriction permitting to obtain criteria for root-class residuality of HNN-extension with base group A, stable letter t and associated subgroups H and K is the equality of the associated subgroups. We prove:

Theorem 4.2. *Let $\mathcal{K}$ be a given root-class. Let $G = \langle A, t;\, t^{-1}Ht = K,\, \varphi \rangle$ be the HNN-extension with base group A, stable letter t and associated subgroups H and K via φ such that $H = K$ and φ is the identity map on H. Then G is $\mathcal{K}$-residual if and only if group A is $\mathcal{K}$-residual and subgroup H is $\mathcal{K}$-separable in A.*

Proof. So let $\mathcal{K}$ be a root-class. Let $G = \langle A, t;\, t^{-1}Ht = K,\, \varphi \rangle$ be the HNN-extension with base group A, stable letter t and associated subgroups H and K such that $H = K$ and φ is the identity map on H. Then for any normal subgroup N of group A, one can define the HNN-extension

$$G_N = \langle A/N,\, t;\, t^{-1}\, HN/Nt = HN/N,\, \varphi_N \rangle,$$

where φ_N is the identity map on subgroup HN/N of group G_N, and the homomorphism $\rho_N : G \to G_N$, extending the canonical homomorphism $A \to A/N$ and $t \mapsto t$. Consider the homomorphism $\sigma : G_N \to A$ which is the identity map on A and which maps $t \mapsto 1$. Then $ker\, \sigma = \langle t \rangle^{G_N}$ is free by [12], (Theorem 6.6, p. 212). So $G_N / \langle t \rangle^{G_N} \cong A/N$ and G_N is an extension of a free group by group A/N. Therefore, if A/N belongs to root-class $\mathcal{K}$, then G_N is $\mathcal{K}$-residual. Thus, to prove $\mathcal{K}$-residuality of G, it is enough to show that G is residually a group of kind G_N, where $A/N \in \mathcal{K}$.

Suppose the group A is $\mathcal{K}$-residual and the subgroup H is $\mathcal{K}$-separable in A. Let $1 \neq g \in G$. Assume that element g has a reduced form $g = a_0 t^{\varepsilon_1} \cdots t^{\varepsilon_s} a_s$. Two cases arise:

1. $s \geq 1$. In this case, for every $i = 0, ..., s$, $a_i \in A$, $\varepsilon_i = \pm 1$ and there are no consecutive sequences of type t^{-1}, a_i, t or t, a_j, t^{-1} with $a_i, a_j \in H$. From $\mathcal{K}$-separability of H, it follows that, for every $i = 0, ..., s$, there exists a normal subgroup N_i of A such that $A/N_i \in \mathcal{K}$ and $a_i \notin HN_i$. Thus, there will be no consecutive sequences of type $t^{-1}, a_i N_i, t$ or $t, a_j N_i, t^{-1}$ with $a_i, a_j \in H$. So let $N = N_0 \cap \cdots \cap N_s$. By Lemma, $A/N \in \mathcal{K}$ and, it is clear that, for every $i = 0, ..., s$, $a_i \notin HN$ and there is no consecutive subwords of type $t^{-1}, a_i N, t$ or $t, a_j N, t^{-1}$ with $a_i, a_j \in H$. Therefore the form

$$g\rho_N = a_0 \rho_N t^{\varepsilon_1} \cdots t^{\varepsilon_s} a_s \rho_N$$

is reduced and has length $s \geq 1$. Consequently, $g\rho_N \neq 1$.

2. $s = 0$, i.e., $g \in A$. Since A is $\mathcal{K}$-residual, there exists a normal subgroup N of A such that $A/N \in \mathcal{K}$ and $g \notin N$, i.e., $gN \neq N$. So $g\rho_N \neq 1$.

Hence, for any element $g \neq 1$, there exists a normal subgroup N in A, such that $A/N \in \mathcal{K}$ and the homomorphism $\rho_N : G \to G_N$ maps element g to a non-identity element. Consequently, G is residually a group G_N, where $A/N \in \mathcal{K}$. Therefore G is $\mathcal{K}$-residual.

Conversely, suppose G is $\mathcal{K}$-residual. Evidently, its subgroup A is $\mathcal{K}$-residual. It remains to show that H is $\mathcal{K}$-separable in group A. If H is not $\mathcal{K}$-separable in A, we choose element $a \in A \backslash H$ such that $a \in NH$, for all normal subgroups N of A, where $A/N \in \mathcal{K}$. Let $g = t^{-1}ata^{-1}$. Then g has length greater than 1. By Britton's Lemma, $g \neq 1$. Let M be a normal subgroup of G with $G/M \in \mathcal{K}$ and $g \notin M$, since G is $\mathcal{K}$-residual. So let $R = M \cap A$. R is a normal subgroup of A and furthermore $A/R \in \mathcal{K}$. Consequently the canonical homomorphism $A \to A/R$ extends to an epimorphism $\pi : G \to G_R$, where $G_R = \langle A/R, t; t^{-1} HR/R\, t = HR/R, \varphi_R \rangle$. Hence $a \in RH$ by the choice of a. Thus, there exists $h \in H$ such that $\pi(a) = \overline{h}$. Then $\pi(g) = \pi(t^{-1}ata^{-1}) = t^{-1}\overline{h}t\overline{h}^{-1} = 1$. Hence, $g \in Ker(\pi) = \langle R \rangle^G \leq M$ and this is a contradiction.

Remark 1. We remark that this result generalizes for example Lemma 3.1 in [10], where analogous result is proven for the particular case of the class of all finite p-groups. We also see that, if $A = H = K$, then A is a normal subgroup of G and $G/A \cong \langle t \rangle$. Therefore G is an extension of a group of class $\mathcal{K}$ by a free group; and thus is $\mathcal{K}$-residual. We remark also that, the necessary condition for Theorem 4.2 also holds when K satisfies only Properties 1 and 2 of the definition of root-class.

Remark 2. We further remark that Theorem 4.2 can be strengthened. Indeed, if we consider that the base group A is finitely generated and $H = K$ via an isomorphism φ, where φ is induced by an automorphism of A, then the criterion of the Theorem 4.2 also holds.

Although HNN-extensions are basically defined with multiple stable letters and multiple associated subgroups, mostly HNN-extensions with only one stable letter have been studied. However Shirvani in [17] examined residual finiteness of HNN-extensions with multiple stable letters and associated subgroups (multiple HNN-extensions). We also study root-class residuality of multiple HNN-extensions. We will generalize Theorems 4.1 and 4.2 above to multiple HNN-extensions.

Let A be a group and I be an index set. Let H_i and K_i, $i \in I$ be families of subgroups of group A with $(\varphi_i)_{i \in I}$ a family of maps such that $\varphi_i : H_i \to K_i$ is an isomorphism. Then the HNN-extension with base group A, stable letters t_i, $i \in I$, and associated subgroups H_i and K_i, $i \in I$, denoted by

$$G = \langle A, t_i \ (i \in I);\ t_i^{-1} h_i t_i = \varphi_i(h_i),\ h_i \in H_i \rangle$$

is the group generated by all the generators of A and elements t_i, $(i \in I)$ and defined by all the relators of A and all possible relations of form $t_i^{-1} h_i t_i = \varphi_i(h_i)$, $h_i \in H_i$ for all $i \in I$.

The group G defined above will be called the *multiple HNN-extension* of base group A, stable letters t_i, $i \in I$, and associated subgroups H_i and K_i, $i \in I$.

In fact, let $G_0 = A$ and

$$G_1 = \langle A, t_1;\ t_1^{-1} H_1 t_1 = K_1, \varphi_1 \rangle;$$

we see that the double HNN-extension

$$G_2 = \langle A, t_1, t_2;\ t_1^{-1} H_1 t_1 = K_1,\ t_2^{-1} H_2 t_2 = K_2, \varphi_1, \varphi_2 \rangle$$

is the HNN-extension with base group G_1, stable letter t_2, and associated subgroups H_2 and K_2 via φ_2; i.e.,

$$G_2 = \langle G_1, t_2;\ t_2^{-1} H_2 t_2 = K_2, \varphi_2 \rangle.$$

Thus, for j of an index set I, G_j is the HNN-extension with base group G_{j-1}, stable letter t_j and associated subgroups H_j and K_j via φ_j, i.e.,

$$G_j = \langle A, t_1, \ldots, t_j;\ t_1^{-1} H_1 t_1 = K_1, \ldots, t_j^{-1} H_j t_j = K_j, \varphi_1, \ldots, \varphi_j \rangle$$

$$= \langle G_{j-1}, t_j;\ t_j^{-1} H_j t_j = K_j, \varphi_j \rangle.$$

For this construction, we have the following results.

Theorem 4.3. *Let $\mathcal{K}$ be a root-class. For any index set I, the multiple HNN-extension*

$$G = \langle A, t_i \ (i \in I);\ t_i^{-1} h_i t_i = \varphi_i(h_i),\ h_i \in H_i \rangle$$

with base group A, stable letters t_i, and associated subgroups H_i and K_i via φ_i

$(i \in I)$, is $\mathcal{K}$-residual if A is $\mathcal{K}$-residual and there exists a sequence $(\sigma_i)_{i \in I}$ of homomorphisms of group G_i onto some group X_i of root-class $\mathcal{K}$, such that σ_i is one-to-one on subgroup H_i for all $i \in I$.

The proof is similar to the proof of Theorem 4.1.

For other criteria of root-class residuality of multiple HNN-extensions with base group A, stable letters t_i and associated subgroups H_i and K_i $(i \in I)$, we may assume the equality of the associated subgroups H_i and K_i for all $i \in I$.

So, suppose $H_i = K_i$ and φ_i is the identity map on H_i for all $i \in I$. Then for such group we have the following criterion which generalizes Theorem 4.2 and the proof is just a repetition of it.

Theorem 4.4. *The multiple HNN-extension*

$$G = \langle A, t_i \ (i \in I); t_i^{-1} h_i t_i = \varphi_i(h_i), h_i \in H_i \rangle$$

with base group A, stable letters t_i, $i \in I$, and associated subgroups H_i and K_i via φ_i such that $H_i = K_i$ and φ_i is the identity map on H_i for all $i \in I$, is $\mathcal{K}$-residual if and only if A is $\mathcal{K}$-residual and subgroup H_i is $\mathcal{K}$-separable in G_i for all $i \in I$.

5. Adjoining Roots to Root-class Residual Groups

Let A be a group and $a \in A$. Let n be a non-negative integer. Then the group $G = \langle A, x; a = x^n \rangle$ denoted by $A \underset{a=x^n}{\star} \langle x \rangle$ is obtained by adjoining roots to group A.

Let A be a group of a root-class $\mathcal{K}$. By adjoining roots to group A, we need not to obtain a group of root-class $\mathcal{K}$. For this purpose, we have the following criteria.

Theorem 5.1. *Let A be a group with element a of infinite order. Let A be $\mathcal{K}$-residual for a root-class $\mathcal{K}$ and for some given integer $n > 1$ class $\mathcal{K}$ contains the cycle of order n. Then group $G = \langle A, x; a = x^n \rangle = A \underset{a=x^n}{\star} \langle x \rangle$ is $\mathcal{K}$-residual if and only if the infinite cycle $\langle a \rangle$, generated by element a, is $\mathcal{K}$-separable in A.*

Proof. Suppose that subgroup $\langle a \rangle$ is not $\mathcal{K}$-separable in group A. Then there exists an element $g \in A \backslash \langle a \rangle$ such that $g\varphi \in \langle a \rangle \varphi$, for any homomorphism φ of group G onto a group of class $\mathcal{K}$. Since $a = x^n$, $g\varphi \in \langle x \rangle \varphi$ and thus $[g, x]\varphi = 1$. But element $[g, x] = gxg^{-1}x^{-1}$ is reduced since $n > 1$ and its length is greater than 1. Therefore $[g, x] \neq 1$ and hence, group G is not $\mathcal{K}$-residual.

Conversely, let subgroup $\langle a \rangle$ be $\mathcal{K}$-separable in group A. By Theorem 3.4, the normal closure A^G of subgroup A in group G is $\mathcal{K}$-residual, since it is the generalized free power of group A over subgroup $\langle a \rangle$ with index $I = \{1, ..., n\}$, i.e.,

$$A^G = A \underset{\langle a \rangle}{\star} \cdots \underset{\langle a \rangle}{\star} A \quad (n \text{ times}).$$

Since $G/A^G = \langle x,\ x^n = 1 \rangle \in \mathcal{K}$. Lemma in Section 2 implies now that G is $\mathcal{K}$-residual.

We can now apply this result to study root-class residuality of any group given by the presentation $G_{mn} = \langle a, b; [a^m, b^n] = 1 \rangle$, $(m, n \geq 1)$. Observe that

$$G_{mn} = \langle a \rangle \underset{a^m = x}{\star} H \underset{y = b^n}{\star} \langle b \rangle.$$

We have the following result.

Theorem 5.2. *Let* $\mathcal{K}$ *be a root-class. Let* $G_{mn} = \langle a, b; [a^m, b^n] = 1 \rangle$, *where* $m, n \geq 1$. *Group* G_{mn} *is* $\mathcal{K}$*-residual if class* $\mathcal{K}$ *contains cyclic subgroups of order m and n.*

Proof. Let $\mathcal{K}$ be a root-class. Let $m, n > 1$. Assume that the cyclic subgroups of order m and n belong to $\mathcal{K}$. Let $H = \langle x, y; [x, y] = 1 \rangle$ be the free abelian group of rank 2. Then clearly, H is $\mathcal{K}$-residual and its subgroups $\langle x \rangle$ and $\langle y \rangle$ are $\mathcal{K}$-separable.

Let $A = H \underset{y = b^n}{\star} \langle b \rangle = \langle x, b; [x, b^n] = 1 \rangle$. By Theorem 5.1, A is $\mathcal{K}$-residual.

We claim that $\langle x \rangle$ is $\mathcal{K}$-separable in A. Indeed, one can easily verify that $H = C_A(\langle x \rangle)$, the centralizer of subgroup $\langle x \rangle$ in group A. Therefore, if $g \in A \backslash H$, then $[x, g] \neq 1;$ so there exists a homomorphism φ of group A onto a group of class $\mathcal{K}$ such that $[x, g]\varphi \neq 1,$ i.e., in particular, $g\varphi \notin \langle x \rangle \varphi.$

Let now $g \in H \backslash \langle x \rangle,$ i.e., $g = x^k y^l,$ where $l \neq 0.$ Then $g = x^k b^{nl}.$ Let $\sigma : A \to \langle b \rangle$ such that $x \mapsto 1$ and $b \mapsto b.$ Then $g\sigma = b^{nl} \neq 1$ and $\langle x \rangle \sigma = 1.$ Let σ_0 be a homomorphism of group $\langle b \rangle$ onto a group of class $\mathcal{K}.$ Then $g\sigma\sigma_0 \neq 1.$ Hence, subgroup $\langle x \rangle$ is $\mathcal{K}$ -separable in A.

Then applying again Theorem 5.1, we show that group $G_{mn} = \langle a \rangle \underset{a^m = x}{\star} A$ is $\mathcal{K}$ -residual.

Now, if $m = 1$ or $n = 1,$ then G_{mn} is isomorphic to one of the groups A or H above and thus, is $\mathcal{K}$ -residual.

Remark 3. We remark in summary that the converse of Theorem 5.2 is not true. For example, let $\mathcal{K}$ be the class of all torsion-free groups; then $G_{mn} \in \mathcal{K},$ when cyclic subgroups of finite orders do not belong to $\mathcal{K}.$ But there exists a partial converse which holds for some additional condition on class $\mathcal{K},$ namely if $\mathcal{K}$ is closed under quotient groups.

In fact, suppose in addition that $\mathcal{K}$ contains any quotient group of its group, i.e., $\mathcal{K}$ is closed under taking homomorphic images. Let G_{mn} be $\mathcal{K}$ -residual. Assume for example, that the cyclic subgroup of order m does not belong to $\mathcal{K}.$ Then there exists a prime divisor p of integer m, such that the cyclic subgroup of order p does not belong to $\mathcal{K}.$ Further, it is evident that, every element x of a group X of a root-class $\mathcal{K}$ has a finite order, relatively prime with p. Indeed, let $|\, f \,|$ be the order of an element f. If $|\, x \,| = \infty,$ then $\langle x \rangle \in \mathcal{K},$ and since $\mathcal{K}$ is closed under quotient groups, the cyclic subgroup of order p would belong to K. Hence, $|\, x \,| < \infty$ and $\gcd(|\, x \,|,\ p) = 1,$ since the cyclic subgroup of order p does not belong to $\mathcal{K}.$ So let $c = [a^{m/p}, b^n].$ Obviously $c \neq 1.$ Then there exists a homomorphism φ of group G_{mn} onto a group X of class $\mathcal{K}$ such that $c\varphi \neq 1.$ Let $k = |\,(a^{m/p}\varphi)\,|.$ Then $k < \infty$ and $\gcd(k,\ p) = 1.$ Hence $((a\varphi)^{m/p})^k = 1$ and this implies that

$$[((a\varphi)^{m/p})^k,\ b^n\varphi] = 1. \qquad (\star)$$

On the order hand,

$$[((a\varphi)^{m/p})^p,\ b^n\varphi] = 1. \qquad (\star\star)$$

Now, from $(\star)$ and $(\star\star)$ and since integers k and p are relatively primes, it follows that

$$c\varphi = [(a\varphi)^{m/p},\ b^n\varphi] = 1$$

and this is a contradiction.

Corollary. *Any group* $G_{mn} = \langle a, b; [a^m, b^n] = 1\rangle$, *where* $m, n \geq 1$ *is residually a finite p-group if and only if integers m and n are p-numbers, for some prime p.*

Acknowledgement

The author would like to thank the Max-Planck-Institute for Mathematics for the hospitality. The travel grant by the International Mathematical Union (IMU) CDE Exchange Programme is also gratefully acknowledged.

References

[1] D. N. Azarov and D. Tieudjo, On root class residuality of amalgamated free product, Naouch. Trudy IvGU 5 (2002), 6-10 (in Russian).

[2] G. Baumslag and D. Solitar, Some two-generator one-relator non-Hopfian groups, Bull. Amer. Math. Soc. 68 (1962), 199-201.

[3] B. Baumslag and M. Tretkoff, Residually finite HNN extensions, Comm. Algebra 6(2) (1978), 179-194.

[4] A. M. Brunner, On a class of one-relator groups, Canad. J. Math. 32(2) (1980), 414-420.

[5] D. Doniz, Residual properties of free products of infinitely many nilpotent groups amalgamating cycles, J. Algebra 179(3) (1996), 930-935.

[6] K. W. Gruenberg, Residual properties of infinite soluble groups, Proc. London Math. Soc. 3(7) (1957), 29-62.

[7] M. I. Kargapolov and I. I. Merzliakov, Elements of group theory, Naouka, 1972 (in Russian).

[8] A. Karrass and D. Solitar, Subgroups of HNN groups and groups with one defining relation, Canad. J. Math. 23 (1971), 627-643.

[9] G. Kim and J. McCarron, On amalgamated free products of residually *p*-finite groups, J. Algebra 162(1) (1993), 1-11.

[10] G. Kim and J. McCarron, Some residually *p*-finite one relator groups, J. Algebra 169(3) (1994), 817-826.

[11] G. Kim and C. Y. Tang, On generalized free products of residually finite *p*-groups, J. Algebra 201(1) (1998), 317-327.

[12] R. C. Lyndon and P. E. Schupp, Combinatorial Group Theory, Springer-Verlag, Berlin, New York, 1977.

[13] W. Magnus, A. Karrass and D. Solitar, Combinatorial Group Theory, Naouka, 1974 (in Russian).

[14] S. Meskin, Nonresidually finite one-relator groups, Trans. Amer. Math. Soc. 164 (1972), 105-114.

[15] D. I. Moldavanskii, *p*-finite residuality of HNN-extensions, Viesnik Ivanov. Gos. Univ. 3 (2000), 129-140 (in Russian).

[16] E. Raptis and D. Varsos, Residual properties of HNN-extensions with base group an abelian group, J. Pure Appl. Algebra 59(3) (1989), 285-290.

[17] M. Shirvani, On residually finite HNN-extensions, Arch. Math. (Basel) 44(2) (1985), 110-115.

[18] M. Shirvani, A converse to a residual finiteness theorem of G. Baumslag, Proc. Amer. Math. Soc. 104(3) (1988), 703-706.

[19] D. Tieudjo, On root class residuality of HNN-extensions, IMHOTEP J. Afr. Math. Pures Appl. 6(1) (2005), 18-23.