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On the Residual Properties of Generalized Direct Products of Groups

E. A. Tumanova*

(Submitted by M. M. Arslanov)

Ivanovo State University, Ivanovo, 153025 Russia

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Abstract—Let $\mathcal{K}$ be a class of groups closed under taking subgroups, direct products of a finite number of factors, and quotient groups. Let also G be a generalized direct product of two groups with amalgamated subgroups. We obtain necessary and sufficient conditions for G to be residually a $\mathcal{K}$-group. These results generalize and strengthen the known conditions for the residual finiteness of the group G.

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1. INTRODUCTION

In the course of describing and studying the properties of basic group-theoretical constructions, other, more complicated structures are often introduced. So, for example, in order to describe the subgroups of a free product of two groups with an amalgamated subgroup, Karras and Solitar [1] introduce the notion of a tree product of groups and reveal the conditions for its existence. In [2], the construction of the generalized direct product associated with a graph of groups is defined, which is used in studying the residual properties of free constructions of groups. In this paper, we consider a special case of the last construction—when the graph consists of two vertices and the edge connecting them. The direct product associated with such a graph is called *the generalized direct product of two groups with amalgamated subgroups* or *the central product of two groups* [3]. This construction is also a special case of a generalized direct product of a family of groups introduced by B. Neumann and H. Neumann in [4]. Recall how it is defined.

Let A and B be some groups, H be a central subgroup of A, K be a central subgroup of B, $\varphi\colon H \to K$ be an isomorphism of subgroups. Then the generalized direct product of the groups A and B with the subgroups H and K amalgamated in accordance with the isomorphism φ is the group $G = \langle A \times B;\ H = K,\ \varphi\rangle$ whose generators are the generators of A, B and whose defining relations are the relations of A, B as well as all possible relations of the form $[a, b] = 1$ ($a \in A$, $b \in B$) and $h = h\varphi$ ($h \in H$). All denotations just introduced are assumed to be fixed until the end of the paper.

We note that, due to the centrality of the amalgamating subgroups in the factors, the identity mappings of the generators of A and B to G determine injective homomorphisms, and therefore A and B can be considered as subgroups of their generalized direct product G (see, for example, [2]). A description of some other properties of the group G is given in [5–8].

This paper studies residual properties of generalized direct products of two groups. Let $\mathcal{K}$ be a class of groups. Recall that a group X is said to be *residually a $\mathcal{K}$-group* if, for each non-unit element $x \in X$, there exists a homomorphism of X onto a group of $\mathcal{K}$ (*a $\mathcal{K}$-group*) mapping x to a non-unit element. Further, we will assume that every class of groups under consideration contains at least one non-identity group.

*E-mail: helenfog@bk.ru

One of the most effective methods for studying residual properties of group-theoretic constructions is the so-called "filtration technique". Initially, it was proposed by Baumslag [9] with the aim of studying the residual finiteness of generalized free products of two groups. A similar method for researching the residual finiteness of HNN-extensions was indicated in [10]. The adaptation of the filtration technique, which made possible its using to study the residual finiteness of generalized direct products of two groups, was performed by Moldavanskii and Agafonova [5].

In [11, 12], the author proposed an approach that allowed using the filtration technique to research the approximability of generalized free products and HNN-extensions not by a specific class, but by a set of classes of groups satisfying certain conditions. This approach led to a number of results on the approximability of these constructions by various classes of groups (see, for example, [11–14]). The aim of this paper is to extend the methodology for studying approximability by a set of classes of groups to generalized direct products of two groups.

For the construction of a generalized direct product, the closeness under taking subgroups, extensions, and quotient groups is fixed as a requirement imposed on the approximating class. Moreover, in proving the main results, the requirement of closeness of the class under taking extensions is weakened to the closeness under taking direct products of a finite number of factors. Further, for the sake of brevity, the last condition will be called simply the closeness under taking direct products. The examples of classes closed under taking subgroups, extensions, and quotient groups are the classes of all finite groups, finite p-groups (where p is a prime number), finite π-groups (where π is a non-empty set of primes), solvable groups, polycyclic groups, all periodic groups. If, instead of the closeness under taking extensions, we require the closeness of the class under taking direct products, then the above list can be supplemented by the classes of all nilpotent groups, solvable groups of derived length at most n, nilpotent groups having nilpotency class at most n (where $n \geqslant 1$). Note that the closeness under taking subgroups, quotient groups, extensions and direct products is preserved when intersecting the classes of groups.

The main results of the paper are contained in Theorems 1 and 2 below. To state the first of them, we recall a number of concepts going back to [9].

A family $\{Y_i\}_{i\in\mathcal{I}}$ of normal subgroups of a group X is said to be *a filtration* if $\bigcap_{i\in\mathcal{I}} Y_i = 1$. Subgroups R and S of the groups A and B respectively are called (H,K,φ)-*compatible* if $(H\cap R)\varphi = K\cap S$. In addition, for a class $\mathcal{K}$ and a group X, we denote by $\mathcal{K}^*(X)$ the family of subgroups of X such that $Y \in \mathcal{K}^*(X)$ if and only if Y is normal in X and $X/Y \in \mathcal{K}$.

Theorem 1. *Let $\mathcal{K}$ be an arbitrary class of groups, and let $\{(R_\lambda, S_\lambda)\}_{\lambda\in\Lambda}$ be the family of all pairs of (H,K,φ)-compatible subgroups of $\mathcal{K}^*(A)$ and $\mathcal{K}^*(B)$ respectively.*

1. *If $\mathcal{K}$ is closed under taking subgroups and G is residually a $\mathcal{K}$-group, then each of the families $\{R_\lambda\}_{\lambda\in\Lambda}$, $\{S_\lambda\}_{\lambda\in\Lambda}$ is a filtration.*

2. *Let $\mathcal{K}$ be closed under taking direct products and quotient groups. If $\{R_\lambda\}_{\lambda\in\Lambda}$, $\{S_\lambda\}_{\lambda\in\Lambda}$ are filtrations and at least one of the quotient groups A/H, B/K is residually a $\mathcal{K}$-group, then G is residually a $\mathcal{K}$-group.*

This theorem generalizes and strengthens the conditions for the residual finiteness of a generalized direct product with amalgamated subgroups, which were obtained in [5].

If the class $\mathcal{K}$ is closed under taking subgroups, direct products, and quotient groups, then the difference between the necessary and sufficient conditions delivered by Theorem 1 consists only in the requirement for one of the groups A/H and B/K to be residually a $\mathcal{K}$-group. The paper ends with the examples showing that this requirement is not necessary for G to be residually a $\mathcal{K}$-group (Example 1), but at the same time it cannot be discarded (Example 2). This situation is different from the case of a free product of two groups with central amalgamated subgroups (see [11, Proposition 13] and Proposition 3 below).

It is a non-trivial problem to check whether the families $\{R_\lambda\}_{\lambda\in\Lambda}$ and $\{S_\lambda\}_{\lambda\in\Lambda}$ indicated in Theorem 1 are filtrations. Therefore, certain restrictions imposed on the factors and the amalgamated subgroups are found, which make it possible to simplify the solution to this problem. The main of them is the requirement for one of the factors to be $\mathcal{K}$-regular in the corresponding amalgamated subgroup [11]. Recall what it consists of.

Let $\mathcal{K}$ be an arbitrary class of groups and X be a group. We say that X is *$\mathcal{K}$-regular* in its normal subgroup Y if, for any subgroup $M \in \mathcal{K}^*(Y)$, which is normal in X, there exists a subgroup $N \in \mathcal{K}^*(X)$ such that $N \cap Y = M$.

Theorem 2. *Let $\mathcal{K}$ be a class of groups closed under taking subgroups, direct products, and quotient groups. If A and B/K are residually $\mathcal{K}$-groups and B is $\mathcal{K}$-regular in K, then G is residually a $\mathcal{K}$-group.*

Theorem 2 has a number of corollaries, which, in turn, simplify the verification of the $\mathcal{K}$-regularity condition.

Corollary 1. *Let $\mathcal{K}$ be a class of groups closed under taking subgroups, extensions, and quotient groups. If A is residually a $\mathcal{K}$-group and $K \in \mathcal{K}^*(B)$, then G is residually a $\mathcal{K}$-group.*

If a class $\mathcal{K}$ consists only of periodic groups, then we denote by $\pi(\mathcal{K})$ the set of all prime divisors of the orders of the elements of all $\mathcal{K}$-groups. A subgroup Y of a group X is called *$\pi(\mathcal{K})'$-isolated* in X if, for any element $x \in X$ and for any prime number $q \notin \pi(\mathcal{K})$, it follows from the inclusion $x^q \in Y$ that $x \in Y$.

Corollary 2. *Let $\mathcal{K}$ be a class of groups closed under taking subgroups, extensions, and quotient groups, A be residually a $\mathcal{K}$-group, B be a finitely generated nilpotent group.*

1. *If $\mathcal{K}$ contains at least one non-periodic group, then G is residually a $\mathcal{K}$-group.*

2. *If $\mathcal{K}$ consists only of periodic groups and K is $\pi(\mathcal{K})'$-isolated in B, then G is residually a $\mathcal{K}$-group.*

We note that, as Example 1 below shows, the requirement for K to be $\pi(\mathcal{K})'$-isolated in B is not necessary for G to be residually a $\mathcal{K}$-group.

Corollary 3. *Let $\mathcal{K}$ be a class of groups closed under taking subgroups, extensions, and quotient groups, A and B be residually $\mathcal{K}$-groups, H and K be subgroups of finite rank. If at least one of the groups A and B is residually a torsion-free $\mathcal{K}$-group, then G is residually a $\mathcal{K}$-group.*

2. PROOF OF THEOREMS

Before proceeding directly to the proof of the formulated statements, we note some easily verifiable properties of the group G.

Since H and K lie in the centers of A and B respectively, then the set $L = \{h(h\varphi)^{-1} | h \in H\}$ is a central subgroup of the direct product $D = A \times B$ and G is isomorphic to the quotient group D/L. An arbitrary element of G either is contained in one of the factors, or is represented as a product of two elements contained in the factors, but not included in the amalgamated subgroups. Indeed, let $g \in G$ and $g = ab$, where $a \in A$, $b \in B$. If $a \in H$, then $a = a\varphi \in K$ and $g \in B$. If $b \in K$, then $b = b\varphi^{-1} \in H$ and $g \in A$. The case remains when $a \in A \setminus H$ and $b \in B \setminus K$.

Just as for the construction of a generalized free product of two groups, the combination of the normality of subgroups in the factors and their (H, K, φ)-compatibility allows us to consider the generalized direct product of the corresponding quotient groups along with the generalized direct product G. Namely, if $R \leqslant A$ and $S \leqslant B$ are normal (H, K, φ)-compatible subgroups, then the map $\varphi_{R,S}\colon HR/R \to KS/S$ acting by the rule $(hR)\varphi_{R,S} = (h\varphi)S$, where $h \in H$, is correctly defined and is an isomorphism of the subgroup HR/R of the quotient group A/R onto the subgroup KS/S of the quotient group B/S. The presence of the isomorphism $\varphi_{R,S}$ allows us to construct the generalized direct product $G_{R,S} = \langle A/R \times B/S;\ HR/R = KS/S,\ \varphi_{R,S}\rangle$ of the groups A/R and B/S with the subgroups HR/R and KS/S amalgamated in accordance with the isomorphism $\varphi_{R,S}$. It is easy to see that the word mapping, which continues the identity mapping of the generators of G to the group $G_{R,S}$, takes all the defining relations of G into the equalities valid in $G_{R,S}$. Therefore, it determines a surjective homomorphism $\rho_{R,S}\colon G \to G_{R,S}$, whose action on A and B coincides with the action of the natural homomorphisms $A \to A/R$ and $B \to B/S$.

Proof of Theorem 1.

1. Take an arbitrary non-unit element $a \in A$. Since G is residually a $\mathcal{K}$-group, then there exists a homomorphism σ of G onto a $\mathcal{K}$-group X such that $a\sigma \neq 1$. Let $N = \ker\sigma$, $R = A \cap N$, and $S = B \cap N$.

Obviously, R and S are normal in A and B respectively. Because $\mathcal{K}$ is closed under taking subgroups, it follows from the relations $G/N \cong X \in \mathcal{K}$, $A/R \cong AN/N$, and $B/S \cong BN/N$ that $A/R \in \mathcal{K}$ and $B/S \in \mathcal{K}$. Let us show that R and S are (H, K, φ)-compatible, i.e. $(H \cap R)\varphi = K \cap S$.

Since $H \cap R = H \cap (A \cap N) = H \cap N$ and $K \cap S = K \cap (B \cap N) = K \cap N$, it is sufficient to prove that $(H \cap N)\varphi = K \cap N$. If x is an arbitrary element of $(H \cap N)\varphi$, then $x = h\varphi$ for a suitable $h \in H \cap N$. Because $h = h\varphi$, it follows that $x \in K \cap N$, and so $(H \cap N)\varphi \subseteq K \cap N$. Conversely, if $x \in K \cap N$, then there exists $h \in H$ such that $x = h\varphi$. As above, using the relation $h = h\varphi$, we get $h \in N$, and so $K \cap N \subseteq (H \cap N)\varphi$.

Thus, R and S are (H, K, φ)-compatible, and hence the pair (R, S) belongs to $\{(R_\lambda, S_\lambda)\}_{\lambda\in\Lambda}$. Since $a \notin N$ by the choice of σ, we have $a \notin R$. Therefore, $\bigcap_{\lambda\in\Lambda} R_\lambda = 1$, i.e. $\{R_\lambda\}_{\lambda\in\Lambda}$ is a filtration.

It is similarly proved that $\{S_\lambda\}_{\lambda\in\Lambda}$ is a filtration.

2. For an arbitrary non-unit element $g \in G$, we construct a homomorphism of G onto a group of $\mathcal{K}$ such that the image of g is different from 1. Consider three cases.

Case 1. $g \in A$.

Because $\{R_\lambda\}_{\lambda\in\Lambda}$ is a filtration, there exists $\lambda \in \Lambda$ such that $g \notin R_\lambda$. Let S_λ be a subgroup of $\mathcal{K}^*(B)$ which is (H, K, φ)-compatible with R_λ. Consider the homomorphism $\rho_{R_\lambda,S_\lambda}\colon G \to G_{R_\lambda,S_\lambda}$.

Since $R_\lambda \in \mathcal{K}^*(A)$, $S_\lambda \in \mathcal{K}^*(B)$, and $\mathcal{K}$ is closed under taking direct products, then $A/R_\lambda \times B/S_\lambda \in \mathcal{K}$. The group G_{R_λ,S_λ} is isomorphic to a quotient group of $A/R_\lambda \times B/S_\lambda$ and so belongs to $\mathcal{K}$ because this class is closed under taking quotient groups. Since $g \in A \setminus R_\lambda$ and $\rho_{R_\lambda,S_\lambda}$ acts on A as the natural homomorphism $A \to A/R_\lambda$, then $g\rho_{R_\lambda,S_\lambda} \neq 1$. Thus, $\rho_{R_\lambda,S_\lambda}$ is the desired homomorphism.

Case 2. $g \in B$.

This case is considered similarly.

Case 3. $g = ab$, where $a \in A \setminus H$ and $b \in B \setminus K$.

For definiteness, we assume that A/H is residually a $\mathcal{K}$-group. It is easy to see that H and B are normal (H, K, φ)-compatible subgroups of A and B respectively. This allows us to consider the group $G_{H,B} \cong A/H$ and the homomorphism $\rho_{H,B}\colon G \to G_{H,B}$. Since $a \notin H$, then $a\rho_{H,B} = aH$ is a non-unit element of the residually $\mathcal{K}$-group $G_{H,B}$. Therefore, there exists a homomorphism σ of $G_{H,B}$ onto a $\mathcal{K}$-group such that $(a\rho_{H,B})\sigma \neq 1$. Because $g(\rho_{H,B}\sigma) = (a\rho_{H,B})\sigma$, we have that $\rho_{H,B}\sigma$ is the desired homomorphism. □

Proof of Theorem 2. Let $\{(R_\lambda, S_\lambda)\}_{\lambda\in\Lambda}$ be the family of all pairs of (H, K, φ)-compatible subgroups of $\mathcal{K}^*(A)$ and $\mathcal{K}^*(B)$ respectively. First, we show that $\{R_\lambda\}_{\lambda\in\Lambda}$ is a filtration.

Let a be an arbitrary non-unit element of A. Since A is residually a $\mathcal{K}$-group, there exists a subgroup $R \in \mathcal{K}^*(A)$ which does not contain a. Obviously, $H \cap R$ is a normal subgroup of H and $H/H \cap R \cong HR/R \leqslant A/R$. Since $R \in \mathcal{K}^*(A)$ and $\mathcal{K}$ is closed under taking subgroups, it follows that $H \cap R \in \mathcal{K}^*(H)$. Let $V = (H \cap R)\varphi$. Then $V \in \mathcal{K}^*(K)$ because φ is an isomorphism, and V lies in the center of B. By the $\mathcal{K}$-regularity of B in K, there exists a subgroup $S \in \mathcal{K}^*(B)$ such that $V = K \cap S$. Then R and S are (H, K, φ)-compatible subgroups of $\mathcal{K}^*(A)$ and $\mathcal{K}^*(B)$ respectively, and so $R \in \{R_\lambda\}_{\lambda\in\Lambda}$. Thus, an arbitrary non-unit element of A does not belong to a subgroup of $\{R_\lambda\}_{\lambda\in\Lambda}$, and hence this family is a filtration.

Next, we verify that $\{S_\lambda\}_{\lambda\in\Lambda}$ is also a filtration. To do this, we take an arbitrary non-unit element $b \in B$ and find a subgroup $S \in \{S_\lambda\}_{\lambda\in\Lambda}$ such that $b \notin S$. Consider two cases.

Case 1. $b \in K$.

Since $b\varphi^{-1}$ is a non-unit element of A, we can, as in case 1, find a pair $(R, S) \in \{(R_\lambda, S_\lambda)\}_{\lambda\in\Lambda}$ such that $b\varphi^{-1} \notin R$. Then $b\varphi^{-1} \notin H \cap R$ and $b \notin K \cap S$ because $(H \cap R)\varphi = K \cap S$. Since $b \in K$, it follows that $b \notin S$, and so S is the desired subgroup.

Case 2. $b \notin K$.

Since $b \notin K$, then bK is a non-unit element of the residually $\mathcal{K}$-group B/K and there exists a homomorphism σ of B/K onto a $\mathcal{K}$-group such that $(bK)\sigma \neq 1$. Let $S = \ker(\varepsilon\sigma)$, where $\varepsilon\colon B \to B/K$ is the natural homomorphism. Then $S \in \mathcal{K}^*(B)$ and $b \notin S$. Since $K \subseteq S$, we have $K \cap S = K$, and so $(A, S) \in \{(R_\lambda, S_\lambda)\}_{\lambda\in\Lambda}$. Therefore, S is the desired subgroup.

Thus, $\{R_\lambda\}_{\lambda\in\Lambda}$ and $\{S_\lambda\}_{\lambda\in\Lambda}$ are filtrations, and G is residually a $\mathcal{K}$-group by Theorem 1. □

3. PROOF OF COROLLARIES

Proof of Corollary 1. It follows from the condition $K \in \mathcal{K}^*(B)$ that B is $\mathcal{K}$-regular in K. Indeed, consider the series $1 \leqslant M \leqslant K \leqslant B$, where M is an arbitrary subgroup of $\mathcal{K}^*(K)$. Since K lies in the center of B, then M is normal in B. The quotient group B/M is an extension of the $\mathcal{K}$-group K/M by the $\mathcal{K}$-group B/K and hence belongs to the class $\mathcal{K}$ due to its closeness under taking extensions. Therefore, $M \in \mathcal{K}^*(B)$ and $M = M \cap K$ because $M \leqslant K$. Thus, B is $\mathcal{K}$-regular in K, $B/K \in \mathcal{K}$, and G is residually a $\mathcal{K}$-group by Theorem 2. □

To prove Corollary 2, the following auxiliary statements are used.

Proposition 1 [15, Proposition 1]. *Let $\mathcal{K}$ be a class of groups closed under taking subgroups and direct products, X be an arbitrary group. Then the intersection of a finite number of subgroups of $\mathcal{K}^*(X)$ is again a subgroup of this family.*

Proposition 2. *Let $\mathcal{K}$ be a class of groups closed under taking subgroups, extensions, and quotient groups. If $\mathcal{K}$ contains at least one non-periodic group, then it includes all finitely generated nilpotent groups.*

Proof. Let X be a non-periodic $\mathcal{K}$-group and x be an element of X of infinite order. Then the infinite cyclic group generated by x belongs to the class $\mathcal{K}$ due to its closeness under taking subgroups. Since $\mathcal{K}$ is closed under taking quotient groups, it also contains all finite cyclic groups. Therefore, all polycyclic groups belong to $\mathcal{K}$ because this class is closed under taking extensions. It is well known that every finitely generated nilpotent group is polycyclic (see, for example, [16, Theorem 17.2.2]). Thus, all finitely generated nilpotent groups belong to $\mathcal{K}$. □

Let $\mathcal{K}$ be a class of groups. A subgroup Y of a group X is called *$\mathcal{K}$-separable* in this group if, for each $x \in X \setminus Y$, there exists a homomorphism σ of X onto a $\mathcal{K}$-group such that $x\sigma \notin Y\sigma$ [17].

Proposition 3. *Let $\mathcal{K}$ be a class of groups closed under taking quotient groups, X be a group, Y be a normal $\mathcal{K}$-separable subgroup of X. Then X/Y is residually a $\mathcal{K}$-group.*

Proof. Let xY be an arbitrary non-unit element of X/Y. Then $x \notin Y$ and, by the $\mathcal{K}$-separability of Y in X, there exists a subgroup $N \in \mathcal{K}^*(X)$ such that $x \notin YN$. The last relation implies that $xY \notin YN/Y$. Since $N \in \mathcal{K}^*(X)$ and $\mathcal{K}$ is closed under taking quotient groups, then $(X/Y)/(YN/Y) \cong X/YN \cong (X/N)/(YN/N) \in \mathcal{K}$. Therefore, the natural homomorphism $X/Y \to (X/Y)/(YN/Y)$ maps xY to a non-unit element of a $\mathcal{K}$-group, and so X/Y is residually a $\mathcal{K}$-group. □

Proposition 4. *Let $\mathcal{K}$ be a class of groups consisting only of periodic groups and closed under taking subgroups and extensions. Then every $\pi(\mathcal{K})'$-isolated subgroup of a finitely generated nilpotent group is $\mathcal{K}$-separable.*

Proof. Let X be an arbitrary finitely generated nilpotent group, Y be its $\pi(\mathcal{K})'$-isolated subgroup. Let also $\mathcal{F}_{\pi(\mathcal{K})}$ and $\mathcal{FN}_{\pi(\mathcal{K})}$ denote the classes of all finite $\pi(\mathcal{K})$-groups and finite nilpotent $\pi(\mathcal{K})$-groups respectively.

By assumption, $\mathcal{K}$ contains at least one non-identity group. Therefore, the set $\pi(\mathcal{K})$ is non-empty, and Y is $\mathcal{F}_{\pi(\mathcal{K})}$-separable in X by [18, Theorem 3]. Since any homomorphic image of a nilpotent group is again a nilpotent group, then Y turns out to be $\mathcal{FN}_{\pi(\mathcal{K})}$-separable in X. It is easy to see that any $\mathcal{FN}_{\pi(\mathcal{K})}$-group possesses a finite subnormal series, the orders of factors of which are prime numbers. All these numbers divide the order of the group and hence belong to $\pi(\mathcal{K})$. It follows from the definition of $\pi(\mathcal{K})$ and the closeness of $\mathcal{K}$ under taking subgroups that, for every $p \in \pi(\mathcal{K})$, the cyclic group of order p belongs to $\mathcal{K}$. Since $\mathcal{K}$ is closed under taking extensions, this implies that $\mathcal{FN}_{\pi(\mathcal{K})} \subseteq \mathcal{K}$. Thus, Y is $\mathcal{K}$-separable in X. □

Proposition 5. *Let $\mathcal{K}$ be a class of groups consisting only of periodic groups and closed under taking subgroups, extensions, and quotient groups. If X is a finitely generated nilpotent group, Y is a $\pi(\mathcal{K})'$-isolated normal subgroup of X, then X is $\mathcal{K}$-regular in Y.*

Proof. Take an arbitrary subgroup M of $\mathcal{K}^*(Y)$, normal in X, and find a subgroup $N \in \mathcal{K}^*(X)$ such that $N \cap Y = M$. To do this, we first show that M is $\pi(\mathcal{K})'$-isolated in X.

Let x be an element of X such that $x^q \in M$, where q is a prime number not belonging to $\pi(\mathcal{K})$. Then $x^q \in Y$ and $x \in Y$ because Y is $\pi(\mathcal{K})'$-isolated in X. Consider the element xM of the quotient group Y/M. By the choice of M, Y/M is a periodic $\pi(\mathcal{K})$-group, and so the order s of xM is finite and is a $\pi(\mathcal{K})$-number. We have that $x^s \in M$, $x^q \in M$, and the numbers s, q are coprime. Therefore, $x \in M$, and so M is $\pi(\mathcal{K})'$-isolated in X.

By Proposition 4, M is $\mathcal{K}$-separable in X. Since $\mathcal{K}$ is closed under taking quotient groups, this implies that X/M is residually a $\mathcal{K}$-group by Proposition 3. Therefore, we can find, for each non-unit element $yM \in Y/M$, a subgroup $N_y/M \in \mathcal{K}^*(X/M)$ such that $yM \notin N_y/M$. Since Y/M is a finitely generated periodic nilpotent group, it is finite. Hence, by Proposition 1, the intersection N/M of all the subgroups N_y/M ($yM \in Y/M$) belongs to $\mathcal{K}^*(X/M)$. It is also clear that $N/M \cap Y/M = 1$. Therefore, $N \in \mathcal{K}^*(X)$ and $N \cap Y = M$.

Thus, X is $\mathcal{K}$-regular in Y. □

Proof of Corollary 2

1. By Proposition 2, $B \in \mathcal{K}$. Since $\mathcal{K}$ is closed under taking quotient groups, it follows that $B/K \in \mathcal{K}$. Therefore, G is residually a $\mathcal{K}$-group by Corollary 1.

2. We note that B is $\mathcal{K}$-regular in K by Proposition 5, and B/K is residually a $\mathcal{K}$-group by Propositions 3 and 4. Thus, G is residually a $\mathcal{K}$-group by Theorem 2. □

Now we turn to the proof of Corollary 3 and start with the auxiliary statements.

Proposition 6. *Let $\mathcal{K}$ be a class of groups closed under taking subgroups, extensions, and quotient groups, A and B be residually $\mathcal{K}$-groups. If there exists a subgroup $N \in \mathcal{K}^*(B)$ such that $N \cap K = 1$, then G is residually a $\mathcal{K}$-group.*

Proof. Since B is residually a $\mathcal{K}$-group, then, for each non-unit element of B, there exists a subgroup of $\mathcal{K}^*(B)$ which does not contain this element. By Proposition 1, the intersection of any two subgroups of $\mathcal{K}^*(B)$ is also a subgroup of this family. Therefore, we can assume that N is chosen in such a way that it does not contain a predefined non-unit element of B.

Obviously, 1 and N are normal (H, K, φ)-compatible subgroups of A and B respectively. This allows us to construct the generalized direct product $G_{1,N} = \langle A \times B/N;\ H = KN/N,\ \varphi_{1,N}\rangle$.

Since $N \in \mathcal{K}^*(B)$, then $B/N \in \mathcal{K}$ and $(B/N)/(KN/N) \in \mathcal{K}$ because $\mathcal{K}$ is closed under taking quotient groups. It follows that $KN/N \in \mathcal{K}^*(B/N)$ and $G_{1,N}$ is residually a $\mathcal{K}$-group by Corollary 1. Take an arbitrary non-unit element $g \in G$ and prove that $g\rho_{1,N} \neq 1$ (with a suitable choice of N). In view of the above, this will mean that G is residually a $\mathcal{K}$-group.

If $g \in A$, then $g\rho_{1,N} = g$, and so $g\rho_{1,N} \neq 1$. If $g \in B$, then, by virtue of the remark made above, the subgroup N can be considered as not containing g. So $g\rho_{1,N}$ is a non-unit element of the quotient group B/N. Let $g = ab$, where $a \in A \setminus H$ and $b \in B \setminus K$. Then $g\rho_{1,N} = a\rho_{1,N}b\rho_{1,N}$, where $a\rho_{1,N} \notin H$ because $a \in A \setminus H$. If $b\rho_{1,N} \notin KN/N$, then $g\rho_{1,N} \neq 1$. If $b\rho_{1,N} \in KN/N$, then $b\rho_{1,N} \in H$ and $g\rho_{1,N} \in A \setminus H$. Thus, in each case, $g\rho_{1,N} \neq 1$. □

A group is said to have *a finite Hirsch–Zaitsev rank* (see, for example, [19]) if it possesses a finite subnormal series, each factor of which is either a periodic or an infinite cyclic group.

Proposition 7 [11, Proposition 18]. *Let $\mathcal{K}$ be a class of torsion-free groups closed under taking subgroups and direct products, X be residually a $\mathcal{K}$-group, Y be a subgroup of X having a finite Hirsch–Zaitsev rank. Then there exists a subgroup $Z \in \mathcal{K}^*(X)$ such that $Z \cap Y = 1$.*

Proof of Corollary 3. For definiteness, let B is residually a torsion-free $\mathcal{K}$-group. Since K is a subgroup of finite rank, it is an extension of a free abelian group of finite rank by a periodic group and so has a finite Hirsch–Zaitsev rank. Therefore, by Proposition 7, there exists a normal subgroup N of B such that $N \cap K = 1$ and B/N is a torsion-free $\mathcal{K}$-group. It remains to apply Proposition 6, which states that G is residually a $\mathcal{K}$-group. □

4. EXAMPLES

Example 1. Let $\mathcal{K}$ be a class of groups closed under taking direct products and satisfying the following conditions:

1) $\mathbb{Z}$ is residually a $\mathcal{K}$-group;

2) $\mathbb{Z}_p \notin \mathcal{K}$ for some prime number p.

Let also A and B be the free abelian groups of rank 2 with the bases $\{a_1; a_2\}$ and $\{b_1; b_2\}$ respectively, $H = \mathrm{sgp}\{a_1^p, a_2\}$, $K = \mathrm{sgp}\{b_1, b_2^p\}$, and $\varphi\colon H \to K$ be the isomorphism acting by the rule: $a_1^p\varphi = b_1$, $a_2\varphi = b_2^p$. Then G has the presentation

$$G = \big\langle a_1, a_2, b_1, b_2;\ [a_1, a_2] = [b_1, b_2] = [a_1, b_1] = [a_1, b_2] = [a_2, b_1] = [a_2, b_2] = 1,$$
$$a_1^p = b_1, a_2 = b_2^p\big\rangle = \big\langle a_1, b_2;\ [a_1, b_2] = 1\big\rangle.$$

The groups A, B, and G are residually $\mathcal{K}$-groups because each of them is the direct product of two infinite cyclic residually $\mathcal{K}$-groups. The quotient groups A/H and B/K are isomorphic to $\mathbb{Z}_p$ and so are not residually $\mathcal{K}$-groups. Thus, the requirement that at least one of the groups A/H and B/K is residually a $\mathcal{K}$-group is not necessary for G to be residually a $\mathcal{K}$-group.

We note also that, if $\mathcal{K}$ consists of only periodic groups, then H and K are not $\pi(\mathcal{K})'$-isolated in A and B respectively, whereas A, B, and G are finitely generated nilpotent residually $\mathcal{K}$-groups. This confirms the remark made after the statement of Corollary 2.

Example 2. Let $\mathcal{K}$ be a class of groups which consists of only periodic groups and satisfies the following conditions:

1) $\mathbb{Z}$ is residually a $\mathcal{K}$-group;

2) $\mathbb{Z}_p \notin \mathcal{K}$ for some prime number p.

Let also A, B, H, and K be infinite cyclic groups generated by a, b, a^p, and b^p respectively, $\varphi\colon H \to K$ be the isomorphism mapping a^p to b^p. Then G has the presentation

$$G = \big\langle a, b;\ [a, b] = 1, a^p = b^p\big\rangle.$$

As above, the quotient groups A/H and B/K are isomorphic to $\mathbb{Z}_p$ and hence are not residually $\mathcal{K}$-groups. Let $\psi\colon A \to B$ be the isomorphism mapping a to b. It is directly verified that the set $\big\{(R, R\psi) | R \in \mathcal{K}^*(A)\big\}$ is contained in the family of all pairs of normal (H, K, φ)-compatible subgroups of $\mathcal{K}^*(A)$ and $\mathcal{K}^*(B)$ respectively. Since A and B are residually $\mathcal{K}$-groups, then $\mathcal{K}^*(A)$ and $\mathcal{K}^*(A)\psi$ are filtrations. It follows that the necessary condition for G to be residually a $\mathcal{K}$-group is fulfilled.

Consider the non-unit element $g = a^{-1}b \in G$ and take an arbitrary homomorphism σ of G onto a $\mathcal{K}$-group. Since $\mathcal{K}$ consists of only periodic groups, the order q of $g\sigma$ is finite and is a $\pi(\mathcal{K})$-number. We have $(g\sigma)^q = 1$ and $(g\sigma)^p = \big((a^{-1}b)^p\big)\sigma = (a^{-p}b^p)\sigma = 1$. Since $\mathbb{Z}_p \notin \mathcal{K}$, then $p \notin \pi(\mathcal{K})$ and hence $(p, q) = 1$. Therefore, $g\sigma = 1$, and so G is not residually a $\mathcal{K}$-group.

Thus, if both the groups A/H and B/K are not residually $\mathcal{K}$-groups, then G may not be residually a $\mathcal{K}$-group. Therefore, the condition on the groups A/H and B/K from the statement of Theorem 1 cannot be discarded.

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