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RESIDUAL FINITENESS OF DESCENDING HNN-EXTENSION OF GROUPS

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The paper examines the special case of the general construction of HNN-extensions of groups in which at least one of the associated subgroups is the base group. A criterion is determined for a group obtained in this way to be residually finite. Any group obtained as such an extension from a free nilpotent group of finite rank is residually finite.

1. Let G be a group, H and K isomorphic subgroups of G and $\varphi : H \to K$ an isomorphism. Let $G^* = (G,t;\ t^{-1}Ht = K,\ \varphi)$ be an HNN-extension of G with stable letter t and associated subgroups H and K. This means that G* is defined in the system of generators consisting of the generating group G and element t by all relations of G and the relations $t^{-1}ht = h\varphi$, where $h \in H$.

Following Baumslag [1], we will say that a subgroup N of G is (H, K, φ)-compatible if $(H \cap N)\varphi = K \cap N$. It is easy to see that if U is a normal subgroup of G*, then the subgroup N = G ∩ U of G is (†, K, φ)-compatible. This clearly implies a known fact (see, e.g., [2]): if G* is residually finite (r.f.), then the intersection of all (H, K, φ)-compatible normal subgroups of finite index in G is the trivial subgroup. It is also known that this condition is not sufficient for G* to be r.f.; counterexamples may already be found among the Baumslag–Solitar groups $G(l, m) = \langle a, b;\ a^{-1}b^{l}a = b^{m}\rangle\ (lm \neq 0)$.

The group G(ℓ, m) is an HNN-extension of an infinite cyclic group <b> with associated subgroups $\langle b^{\ell}\rangle$ and $\langle b^{m}\rangle$, where the isomorphism φ maps b^{ℓ} onto b^{m}. For any integer k > 0, the subgroup $\langle b^{k}\rangle$ of <b> is $(\langle b^{l}\rangle, \langle b^{m}\rangle, \varphi)$-compatible if and only if (ℓ, k) = (m, k), and therefore, the intersection of all such subgroups is the identity. On the other hand, G(ℓ, m) is r.f. just when either |ℓ| = 1, or |m| = 1, or |ℓ| = |m| [3, 4].

We shall say that an HNN-extension G* of G is descending if one of the associated subgroups is G. In that particular case, the above necessary condition for residual finiteness of G* is also sufficient.

THEOREM 1. Let G be a group, K a subgroup of G isomorphic to it and $\varphi : G \to K$ an isomorphism. Let $G^* = (G,\ t;\ t^{-1}Gt = K,\ \varphi)$ be a descending HNN-extension of G. Then G* is r.f. if and only if the intersection of all (G, K, φ)-compatible normal subgroups of finite index in G is the trivial subgroup.

When H = G the condition for (H, K, φ)-compatibility of a subgroup N of G becomes $N\varphi =$ K ∩ N. Therefore, if K is equal to G, we obtain the following

COROLLARY 1. Let G* be a split extension of G by an infinite cyclic group <t> and φ the automorphism of G induced by conjugation with t. Then G* is r.f. if and only if the

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intersection of all $\langle\varphi\rangle$-invariant normal subgroups of finite index in G is the trivial subgroup.

Since every subgroup of finite index in a finitely generated group G contains a characteristic subgroup of finite index in G, this implies, in turn, the following special case of a theorem of Mal'tsev [5, Theorem 1]:

COROLLARY 2. An extension of a finitely generated r.f. group by an infinite cyclic group is r.f.

More profound applications of Theorem 1 are based on the following

THEOREM 2. Let $G^* = (G,t;\ t^{-1}Gt = K,\ \varphi)$ be a descending HNN-extension of a finitely generated group G, where the subgroup K is of finite index modulo the commutator subgroup G' of G. Assume, moreover, that, for every number p in some infinite set of primes, G is residually a finite p-group. Then G^* is r.f.

Since a free group is residually a finite p-group for any prime p, this implies

COROLLARY 3. Let $G^* = (G,\ t;\ t^{-1}Gt = K,\ \varphi)$ be a descending HNN-extension of a free group G of finite rank. If KG' is a subgroup of finite index in G, then G^* is r.f.

The question as to whether any descending HNN-extension of a free group is r.f. is still open.

If G is a free nilpotent group of finite rank and K a subgroup of G is isomorphic to it, then it follows from a theorem of Mostowski [6, Theorem 42.51] that the index of K modulo G' is finite. Since in addition free nilpotent groups are approximated by finite p-groups for any prime p, we obtain the following

COROLLARY 4. An arbitrary descending HNN-extension of a finitely generated free nilpotent group is r.f.

Using Theorem 1 and a few simple manipulations, one can also show that an arbitrarily descending HNN-extension of the group $G(1, m) = \langle a, b;\ a^{-1}ba = b^m\rangle$ is r.f. As to Theorem 2, it is interesting to observe that G(1, m) is residually a finite p-group if and only if p is a divisor of $m - 1$.

Some of these results were announced in [7].

2. We proceed to prove the theorems. Let $G^* = (G,\ t;\ t^{-1}Ht = K,\ \varphi)$ be an HNN-extension of G. If N is an (H, K, φ)-compatible normal subgroup of G, then the map φ_N defined by $(aN)\varphi_N = (a\varphi)N,\ a \in H$, is an isomorphism of the subgroup HN/N of G/N onto the subgroup KN/N. Let $G_N^* = (G/N,\ t;\ t^{-1}(HN/N)t = KN/N,\ \varphi_N)$ be an HNN-extension of G/N. Clearly, the map defined on G as the natural homomorphism of G onto G/N and as the identity on t defines a homomorphism ρ_N of G^* onto the group G_N^*. If N has finite index in G, then G_N^* is r.f. [8]. Therefore, in order to prove that G^* is r.f., it will suffice to show that, for any element $g \in G$, $g \neq 1$, there exists an (H, K, φ)-compatible normal subgroup N of finite index in G, such that $g\rho_N \neq 1$.

Let us assume now that H = G, i.e., G^* is a descending HNN-extension of G. It is easy to see that any element $g \in G^*$ can be expressed as $g = t^m a t^{-n}$ for a suitable element $a \in G$ and nonnegative integers m and n (in fact, one can even require that if $mn \neq 0$ then a is not an element of K, and then the above expression for g is also unique; but we shall not need this). Hence it follows that any element of G^* is conjugate to some element $t^k a,\ a \in G$, and we can confine ourselves to such elements.

Let $g = t^k a$ be an element of G^* other than 1. If $k \neq 0$, then for any (G, K, φ)-compatible normal subgroup N of G the element $g\rho_N = t^k(aN)$ of G_N^* is clearly different from 1. But if $k = 0$, then $a \neq 1$ and by assumption there exists a (G, K, φ)-compatible normal subgroup N of finite index in G such that $a \notin N$. Then $g\rho_N = aN$ is a nontrivial element of G_N^*, so Theorem 1 is proved.

To prove Theorem 2, we first observe that a normal subgroup N of finite index in G is (G, K, φ)-compatible, i.e., it satisfies the equality $N\varphi = K \cap N$, if and only if $N\varphi \subseteq N$ and KN = G.

Indeed, if $N\varphi = K \cap N$, then obviously $N\varphi \subseteq N$, while KN = G follows from the fact that the map φ_N defined above is an isomorphism of the finite group G/N onto its subgroup KN/N.

Conversely, if the subgroup $N\varphi$-is admissible and KN = G, then the endomorphism $\bar{\varphi}$ of the quotient group G/N induced by φ is surjective, hence also injective. Therefore, if $x \in K \cap N$ and $y \in G$ is the element such that $x = y\varphi$, then $(yN)\bar{\varphi} = (y\varphi)N = xN = N$ and therefore $y \in N$. Hence $K \cap N \subseteq N\varphi$. The reverse inclusion is obvious.

Now let G satisfy the assumptions of Theorem 2. By Theorem 1 and the preceding remark, it will suffice to show that for any element $g \in G$ other than 1 there exists a fully invariant subgroup N of finite index in G that does not contain g and is such that KN = G.

Let m = [G:KG']. By assumption, there exists a prime p, not a divisor of m, such that G is residually a finite p-group. Therefore, for any element $g \in G$, $g \neq 1$, there exists a normal subgroup N of finite index in G, not containing g, modulo which the quotient group is a p-group. If necessary replacing N by the verbal subgroup of G defined by all identities of G/N, we may assume that N is fully invariant (see [6, Theorem 15.71]). It remains to show that KN = G.

Since the quotient group G/N is nilpotent, it follows that N contains some term $\gamma_c(G)$ of the lower central series of G. By [9, Lemma 4.4], the subgroup $K\gamma_c(G)$ has finite index in G, and the index is an m-number (that is, all its prime divisors are divisors of m). Since $K\gamma_c(G) \subseteq KN$, it follows that [G:KN] is also an m-number. On the other hand, the inclusion $N \subseteq KN$ implies that [G:KN] must be a power of p. By the choice of this number, [G:KN] = 1, i.e., G = KN. This completes the proof of Theorem 2.

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