Строго позитивные логики доказуемости

Л.Д. Беклемишев

Математический институт им. В.А. Стеклова РАН

Алгебра и теория алгоритмов

Всероссийская конференция, посвященная 100-летию Ивановского государственного университета

Иваново, 21–24 марта 2018 г.

Strictly positive modal formulas

The language of modal logic extends that of propositional calculus by a family of unary connectives $\{\Diamond_i : i \in I\}$.

Strictly positive modal formulas are defined by the grammar:

$$A ::= p \mid \top \mid (A \wedge A) \mid \Diamond_i A, \quad i \in I.$$

We are interested in the implications $A \to B$ where A and B are strictly positive.

Strictly positive logics

- *Strictly positive fragment* of a modal logic L is the set of all implications $A \to B$ such that A and B are strictly positive and $L \vdash A \to B$.
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Basic strictly positive logic

We derive *sequents* of the form $A \vdash B$ with A, B s.p.

K$^+$: the s.p. fragment of **K**

1. $A \vdash A$; $A \vdash \top$; from $A \vdash B$ and $B \vdash C$ infer $A \vdash C$;
2. $A \wedge B \vdash A, B$; from $A \vdash B$ and $A \vdash C$ infer $A \vdash B \wedge C$;
3. from $A \vdash B$ infer $\Diamond A \vdash \Diamond B$.

Fact. **K**$^+$ is closed under substitution and positive replacement:

- if $A(p) \vdash B(p)$ then $A(C) \vdash B(C)$;
- if $A \vdash B$ then $C(A) \vdash C(B)$.

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Normal positive logics

A *normal s.p. logic* is a set of sequents closed under the rules of $\mathbf{K}^+$ and the substitution rule.

Other standard logics:

(4) $\Diamond\Diamond A \vdash \Diamond A$;

(T) $A \vdash \Diamond A$;

(5) $\Diamond A \wedge \Diamond B \vdash \Diamond(A \wedge \Diamond B)$.

Semilattices with monotone operators

We consider lower semilattices with top equipped with a family of unary operators $\mathfrak{A} = (A; \wedge, 1, \{\Diamond_i : i \in I\})$ where each $\Diamond_i$ is a monotone operator.

An operator $R : \mathfrak{A} \to \mathfrak{A}$ is:

- *monotone* if $x \leq y$ implies $R(x) \leq R(y)$;
- *semi-idempotent* if $R(R(x)) \leq R(x)$;
- *closure* if R is m., s.i. and $x \leq R(x)$.

We call such structures *SLO*.

Algebraic semantics

We identify s.p. formulas and SLO terms. Then each sequent $A \vdash B$ represents an inequality (i.e. the identity $A \wedge B = A$):

- $A \vdash B$ *holds* in $\mathfrak{A}$ if $\mathfrak{A} \vDash \forall \vec{x}\,(A(\vec{x}) \leq B(\vec{x}))$.

Facts:

- $A \vdash B$ is provable in $\mathbf{K}^+$ iff $A \vdash B$ holds in all SLO $\mathfrak{A}$.
- Varieties of SLO = normal strictly positive logics.

Gödel's 2nd Incompleteness Theorem

A theory T is Gödelian if

- Natural numbers and operations $+$ and $\cdot$ are definable in T;
- T proves basic properties of these operations (contains EA);
- There is an algorithm (and a Σ_1-formula) recognizing the axioms of T.

$$\mathrm{Con}(T) = \text{`}T \text{ is consistent'}$$

K. Gödel (1931): If a Gödelian theory T is consistent, then $\mathrm{Con}(T)$ is true but unprovable in T.

Semilattice of Gödelian theories

Def. $\mathfrak{G}_{\mathsf{EA}}$ is the set of all Gödelian extensions of EA mod $=_{\mathsf{EA}}$.

$S \leq_{\mathsf{EA}} T \iff \mathsf{EA} \vdash \forall x\, (\Box_T(x) \to \Box_S(x))$;

$S =_{\mathsf{EA}} T \iff (S \leq_{\mathsf{EA}} T$ and $T \leq_{\mathsf{EA}} S)$.

Then $(\mathfrak{G}_{\mathsf{EA}}, \wedge_{\mathsf{EA}}, 1_{\mathsf{EA}})$ is a lower semilattice with $1_{\mathsf{EA}} = \mathsf{EA}$ and
$S \wedge_{\mathsf{EA}} T := S \cup T$
(defined by the disjunction of the Σ_1-definitions of S and T)

Reflection principles

Let T be a Gödelian theory.

- *Reflection principles* $R_n(T)$ for T are arithmetical sentences expressing *"every* Σ_n*-sentence provable in* T *is true"*.

$R_n(T)$ can be seen as a relativization of the consistency assertion $Con(T) = R_0(T)$.

- Every formula R_n induces a monotone semi-idempotent operator $R_n : T \longmapsto R_n(T)$ on $\mathfrak{G}_{EA}$.
- We consider the SLO $(\mathfrak{G}_{\mathrm{EA}}; \wedge_{\mathrm{EA}}, 1_{\mathrm{EA}}, \{R_n : n \in \omega\})$.

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Reflection calculus RC

RC axioms (over K^+ for all $\Diamond_n$):

1. $\Diamond_n \Diamond_n A \vdash \Diamond_n A$;
2. $\Diamond_n A \vdash \Diamond_m A$ for $n > m$;
3. $\Diamond_n A \wedge \Diamond_m B \vdash \Diamond_n (A \wedge \Diamond_m B)$ for $n > m$.

Example. $\Diamond_3 \top \wedge \Diamond_2 \Diamond_3 p \vdash \Diamond_3 (\top \wedge \Diamond_2 \Diamond_3 p) \vdash \Diamond_3 \Diamond_2 \Diamond_3 p$.

Main results on RC

Theorems (E. Dashkov, 2012).

1. $A \vdash_{RC} B$ iff $A \vdash B$ holds in $(\mathfrak{G}_{\mathrm{PA}}; \wedge_{\mathrm{PA}}, 1_{\mathrm{PA}}, \{R_n : n \in \omega\})$;
2. RC is polytime decidable;
3. RC enjoys the finite model property (многообразие конечно аппроксимируемо).

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RC^0 *as an ordinal notation system*

Let RC^0 denote the variable-free fragment of RC.
Let W denote the set of all RC^0-formulas. For $A, B \in W$ define:

- $A \sim B$ if $A \vdash B$ and $B \vdash A$ in RC^0;
- $A <_n B$ if $B \vdash \Diamond_n A$.

Theorem.

1. Every $A \in W$ is equivalent to a *word* (formula without $\wedge$);
2. $(W/{\sim}, <_0)$ is isomorphic to $(\varepsilon_0, <)$.

Rem. $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$ is the characteristic ordinal of Peano arithmetic.

Conservativity modalities

We consider operators associating with a theory S the theory generated by its consequences of logical complexity Π_{n+1}:

$$\Pi_{n+1}(S) := \{\pi \in \Pi_{n+1} : S \vdash \pi\}.$$

Notice that each Π_{n+1} is a closure operator.

We consider the SLO $(\mathfrak{G}_{\mathsf{EA}}; \wedge_{\mathsf{EA}}, 1_{\mathsf{EA}}, \{R_n, \Pi_{n+1} : n \in \omega\})$, the RC^{∇} *algebra of* EA.

Open problem: Characterize the logic/identities of this structure. Is it (polytime) decidable?

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Why conservativity?

Comparison of theories:

- $U \vdash R_n(T)$ means U *is much stronger than* T.
- $U \vdash \Pi_{n+1}(T)$ means T *is* Π_{n+1}*-conservative over* U.
- $\Pi_{n+1}(U) = \Pi_{n+1}(T)$ means T and U are equivalent up to quantifier complexity Π_{n+1}.

The logic combining both R_n and Π_{n+1} is able to express both the distance and the proximity of theories.

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Results

- A strictly positive logic RC^{∇} that is conjecturally complete;
- Expressibility of transfinitely iterated reflection up to ε_0;
- Arithmetical completeness and decidability of the variable-free fragment of RC^{∇};
- A (constructive) characterization of the Lindenbaum–Tarski algebra of the variable-free fragment;
- A relation of this algebra to proof-theoretic ordinals of arithmetical theories (*conservativity spectra*).

The system RC^{∇}

RC^{∇} is a strictly positive logic with modalities $\{\Diamond_n, \nabla_n : n \in \omega\}$ ($\Diamond_n$ for R_n, ∇_n for Π_{n+1}).

Axioms and rules:

1. RC for $\Diamond_n$;
2. RC for ∇_n;
3. $A \vdash \nabla_n A$; thus, each ∇_n satisfies $S4^+$;
4. $\Diamond_n A \vdash \nabla_n A$;
5. $\Diamond_m \nabla_n A \vdash \Diamond_m A$ if $m \leq n$;
6. $\nabla_n \Diamond_m A \vdash \Diamond_m A$ if $m \leq n$.

Transfinite iterations

Def. $R : \mathfrak{G}_T \to \mathfrak{G}_T$ is computable if it can be defined by a computable map on the Gödel numbers of numerations (of extensions of T).

Suppose $(\Omega, \prec)$ is an elementary recursive well-ordering and R is a computable m.s.i. operator on $\mathfrak{G}_T$.

Theorem

There exist theories $R^\alpha(S)$ (where $\alpha \in \Omega$):
$R^0(S) =_T S$ and, if $\alpha \succ 0$,

$$R^\alpha(S) =_T \bigcup \{R(R^\beta(S)) : \beta \prec \alpha\}.$$

Each R^α is computable and m.s.i.. Under some natural additional conditions the family R^α is unique modulo provable equivalence.

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Expressibility of iterations

Let $\mathsf{EA}^+ = I\Delta_0(\mathsf{supexp}) = \mathsf{EA} + R_1(\mathsf{EA})$.

Theorem

For each $n < \omega$ and $0 < \alpha < \varepsilon_0$ there is an RC-formula $A(p)$ s.t.

$$\forall S \in \mathfrak{G}_{\mathsf{EA}^+}\ \Diamond_n^\alpha(S) =_{\mathsf{EA}^+} \nabla_n A(S).$$

For example, $\nabla_0 \Diamond_1 \Diamond_0 \varphi$ is arithmetically equivalent to $\{\Diamond_0^{1+n}\varphi : n < \omega\}$.

Ignatiev RC^{∇}*-algebra*

Named after K. Ignatiev who introduced a universal Kripke model for Japaridze's logic based on sequences of ordinals (1993).

- I is the set of all ω-sequences $\vec{\alpha} = (\alpha_0, \alpha_1, \dots)$ such that $\alpha_i < \varepsilon_0$ and $\alpha_{i+1} \leq \ell(\alpha_i)$, for all $i \in \omega$.
- $\ell(\beta) = 0$ if $\beta = 0$, and $\ell(\beta) = \gamma$ if $\beta = \delta + \omega^{\gamma}$, for some δ, γ.
- $\vec{\alpha} \leq_{\mathfrak{I}} \vec{\beta} \iff \forall i\, \alpha_i \geq \beta_i$.

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- $\nabla_n^{\mathfrak{I}}(\vec{\alpha}) := (\alpha_0, \alpha_1, \ldots, \alpha_n, 0, \ldots)$;
- $\Diamond_n^{\mathfrak{I}}(\vec{\alpha}) := (\beta_0, \beta_1, \ldots, \beta_n, 0, \ldots)$, where $\beta_{n+1} := 0$ and $\beta_i := \alpha_i + \omega^{\beta_{i+1}}$, for all $i \leq n$.

Fact. The SLO $\mathfrak{I} = (I, \wedge_{\mathfrak{I}}, \{\Diamond_n^{\mathfrak{I}}, \nabla_n^{\mathfrak{I}} : n \in \omega\})$ is an RC^{∇}-algebra.

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Back to arithmetic

Let $\mathfrak{G}^0_{\mathsf{EA}}$ denote the subalgebra of $(\mathfrak{G}_{\mathsf{EA}}; \wedge_{\mathsf{EA}}, 1_{\mathsf{EA}}, \{R_n, \Pi_{n+1} : n \in \omega\})$ generated by 1_{EA}.

Theorem

The following structures are isomorphic:

1. $\mathfrak{G}^0_{\mathsf{EA}}$;
2. The free 0-generated RC^{∇}-algebra;
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Conservativity spectra

Let S be a Gödelian extension of EA and $(\Omega, <)$ a (natural) elementary recursive well-ordering.

- Π^0_{n+1}*-ordinal of* S, denoted $ord_n(S)$, is the sup of all $\alpha \in \Omega$ such that $S \vdash R^{\alpha}_n(\mathsf{EA})$;
- *Conservativity spectrum of* S is the sequence $(\alpha_0, \alpha_1, \alpha_2, \dots)$ such that $\alpha_i = ord_i(S)$.

Examples of spectra:

$I\Sigma_1 : \ (\omega^\omega, \omega, 1, 0, 0, \dots)$

$\mathsf{PA} : \quad (\varepsilon_0, \varepsilon_0, \varepsilon_0, \dots)$

$\mathsf{PA} + PH : \quad (\varepsilon_0^2, \varepsilon_0 \cdot 2, \varepsilon_0, \varepsilon_0, \dots)$

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Spectra and $\mathfrak{I}$

An extension T of EA is *bounded*, if T is contained in a finite subtheory of PA.

Theorem

1. Let T be bounded and $\vec{\alpha}$ be the conservativity spectrum of T. Then $\forall n < \omega\ \alpha_{n+1} \leq \ell(\alpha_n)$ and $\alpha_n < \varepsilon_0$, that is, $\vec{\alpha} \in \mathfrak{I}$.
2. Let $\vec{\alpha} \in \mathfrak{I}$, A be a variable-free RC^{∇}-formula corresponding to $\vec{\alpha}$ via the isomorphism, and $A_{\mathsf{EA}} \in \mathfrak{G}^0_{\mathsf{EA}}$ its arithmetical interpretation. Then $\vec{\alpha}$ is the conservativity spectrum of A_{EA}.
3. A_{EA} is the weakest theory with the given conservativity spectrum $\vec{\alpha}$.

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Conclusion

- The set of Gödelian extensions of EA obtained from $\mathbf{1}_{\mathrm{EA}}$ by the operations of Σ_n-reflection and Π_{n+1}-conservativity forms a natural semilattice with monotone operators satisfying the identities of RC^∇.
- The algebra has several natural (isomorphic) presentations including the free 0-generated RC^∇-algebra. It bijectively corresponds to the set of all conservativity spectra of bounded extensions of EA.

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