Rigig solvable groups. Model theory

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The talk will be based on the following papers:

N.S.Romanovskiy, *Equational Noetherianess of rigid solvable groups*, Algebra and Logic, 48(2), 2009, pp. 147-160.

N.S.Romanovskiy, *Divisible rigid groups*, Algebra and Logic, 47(6), 2008, pp. 426-434.

N.S.Romanovskiy, *Irreducible algebraic sets over divisible decomposed rigid groups*, Algebra and Logic, 48(6), 2009, pp. 449-464.

A.Myasnikov, N.Romanovskiy, *Krull dimension of solvable groups*, J.Algebra, 324 (10), 2010, pp. 2814-2831.

N.S.Romanovskiy, *Coproducts of rigid groups*, Algebra and Logic, 49 (6), 2010, pp. 539-550.

A.Myasnikov, N.S.Romanovskiy, *On universal theories of rigid solvable groups*, Algebra and Logic, 50 (6), 2011, pp. 539-552.

N.S.Romanovskiy, *Universal theories for free solvable groups*, Algebra and Logic, 51 (3), 2012, pp. 259-263.

N.S.Romanovskiy, *Presentations for rigid solvable groups* , J.Group Theory, 15 (6), 2012, pp. 793-810.

A.Myasnikov, N.S.Romanovskiy, *Logical aspects of the theory of divisible rigid groups*, Doklady Mathematics, 90 (3), 2014, pp. 697-698.

N.S.Romanovskiy, *Hilbert's Nullstellensatz in algebraic geometry over rigid solvable groups*, Izvestiya: Mathematics, 79 (5), 2015, pp. 1051-1063.

A.Myasnikov, N.S.Romanovskiy, *Model-theoretic aspects of the theory of divisible rigid soluble groups*, Algebra and Logic, 56 (1), 2017, pp. 82-84.

N.S.Romanovskiy, *Divisible rigid groups. algebraic closedness and elementary theory*, Algebra and Logic, 56 (5), 2017, pp. 395-408.

A.Myasnikov,N.S.Romanovskiy, *Divisible rigid groups. II. Stability, saturation and elementary submodels*, Algebra and Logic, 57 (1), 2018, pp. -.

N.S.Romanovskiy, *Divisible rigid groups. III. Homogeneity and quantifier elimination*, Algebra and Logic, 57, 2018, pp. -.

$G \triangleright A$, A is abelian. G acts by conjugation: $a \to a^g = g^{-1}ag$, G/A acts. We set $\overline{G} = G/A$ and $\overline{g} = gA$ for $g \in G$. The group A may be viewed as a right $\mathbb{Z}\overline{G}$-module with the action of an element $u = \alpha_1\overline{g}_1 + \ldots + \alpha_n\overline{g}_n \in \mathbb{Z}[G/A]$ on $a \in A$ defined by the formula $a^u = (a^{g_1})^{\alpha_1} \cdot \ldots \cdot (a^{g_n})^{\alpha_n}$.

Let $K \ni 1$ be an associative ring without zero divisors. K is called a right Ore domain if, for arbitrary elements $a, b \in K$, there exists a nonzero pair of elements $x, y \in K$ such that $ax = by$. A left Ore domain is defined similarly. The right Ore domain K is embedded in the right division ring (skew field) of fractions $Q(K)$. In this ring, every element can be represented in the form ab^{-1}, where $a, b \in K,\ b \neq 0$. If K is embedded in another division ring then the division subring generated by K is isomorphic to $Q(K)$. This implies that if K is both left and right Ore domain then the left and right division rings of fractions coincide.

We will use the embeddedness into division rings of fractions of integral group rings. Suppose that the group ring $\mathbb{Z}G$ is a right Ore domain. Then $\mathbb{Z}G$ is also a left Ore domain. It has right division ring (which is also left division ring) of fractions henceforth denoted by $Q(G)$. It is known that the integral group ring of a torsion-free solvable group G is an Ore domain, which is hence embedded into a division ring of fractions $Q(G)$.

We say that a right module T over a ring K torsion-free if the condition $0 \neq t \in T, \ 0 \neq u \in K$ implies $tu \neq 0$. If K is a right Ore domain, the torsion elements of T form a submodule, which is called *the torsion submodule*, with the corresponding factor module having no torsion.

Let T be a right torsion-free unitary module over a right Ore domain K. Then T can be embedded in the right vector space $T \cdot Q(K) = T \otimes_K Q(K)$. *The rank of* T is the cardinality of any maximal linearly independent over K system of elements. The rank of T coincides with the dimension of the vector space $T \cdot Q(K)$.

Definition

m-rigid group G*: there is a normal series*

$$G = G_1 > G_2 > \ldots > G_m > G_{m+1} = 1,$$

G_i/G_{i+1} *are abelian and considering as right* $\mathbb{Z}[G/G_i]$*-modules have no torsion.*

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The rigid series is unique for given rigid group G and so we can use notation $G_i = \rho_i(G)$. Given group G is solvable of derived length exactly m.

Examples of rigid groups.
1) Free solvable groups are rigid, rigid series consists of commutator subgroups.
2) $A_m \wr (A_{m-1} \wr (\ldots \wr A_1)\ldots)$, where A_i are torsion free abelian groups.

Subgroups of rigid groups are rigid too: $G \geqslant H$, take $H_i = H \cap G_i$ and remove repetitions.

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Rigid group G is called divisible if any factor G_i/G_{i+1} is a divisible module over the ring $\mathbb{Z}[G/G_i]$ or, in other words, G_i/G_{i+1} is a vector space over skew field of fractions $Q(G/G_i)$.

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Fix some nonzero cardinalities $\alpha_1, \ldots, \alpha_m$ and construct a group $M(\alpha_1, \ldots, \alpha_m)$ by induction. $M(\alpha_1)$ is a direct sum of α_1 copies of $\mathbb{Q}$. $A = M(\alpha_1, \ldots, \alpha_{m-1})$. Let T be a vector space with a basis of cardinality α_m over the skew field $Q(A)$. Then set

$$M(\alpha_1, \ldots, \alpha_m) = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}.$$

We call such group M splittable divisible rigid group, M is a semidirect product $A_1 A_2 \ldots A_m$ of abelian groups A_i and $\rho_i(M) = A_i \ldots A_m$.

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$G \geqslant H,\ H_i = H \cap G_i,\ H_i/H_{i+1} \leqslant G_i/G_{i+1},\ \mathbb{Z}[H/H_i] \leqslant \mathbb{Z}[G/G_i].$
We say for m-rigid groups that H is embedded into G independently, if any system elements of H_i/H_{i+1} linear independent over the ring $\mathbb{Z}[H/H_i]$ has to be linear independent over the ring $\mathbb{Z}[G/G_i]$.

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Theorem 3 (R, 2008)

Let G be a m-rigid subgroup of divisible rigid group D. Then there is a minimal divisible subgroup containing G, let it be $\overline{G}$ = divisible closure of G in D. This subgroup $\overline{G}$ is m-rigid and $\overline{G}_i/\overline{G}_{i+1}$ is generated by the set G_i/G_{i+1} as a vector space over $Q(\overline{G}/\overline{G}_i)$.

Natural question: Let G_1 and G_2 be two divisible closures of G (in different groups), are they G-isomorphic?
NO, in general case, but YES with adding condition:

Theorem 4 (R, 2008)

For given m-rigid group G there is such divisible closure $\widehat{G}$ that G is embedded into $\widehat{G}$ independently. We call $\widehat{G}$ divisible completion of G. Any two divisible completions of G are G-isomorphic.

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For the class Σ_m of rigid groups of derived length $\leq m$ we define algebraic closed objects: G is called algebraic closed if for any independent embedding $G \hookrightarrow H$ in this class any system of equations over $x_1, \ldots, x_n$ with coefficients from G has a solution in $G^n \iff$ it has a solution in H^n. G is called existential closed if for any independent embedding $G \hookrightarrow H$ in this class any $\exists$-formula is true on $G \iff$ it is true on H.

Theorem 5 (R, 2017). *Divisible (splittable) m-rigid groups = algebraic closed objects in Σ_m = existential closed objects in Σ_m.*

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Theorem 5 (R, 2017). *Divisible (splittable) m-rigid groups = algebraic closed objects in Σ_m = existential closed objects in Σ_m.*

Then we study elementary theories of divisible m-rigid groups. We construct a system of axioms in group theory signature which define exactly all divisible m-rigid groups. Denote by $\mathfrak{T}_m$ corresponding theory.

Fix also some countable divisible m-rigid group M. We prove that this group is constructible. Extend the signature of group theory by constants from M. We add some recursive system of axioms which means that M is embedded into given rigid group independently. Denote corresponding theory by $\mathfrak{T}_m(M)$.

Theorem 6 (R, 2017). *The theories $\mathfrak{T}_m$ and $\mathfrak{T}_m(M)$ are complete and therefore decidable.*

So, any two divisible m-rigid groups have the same elementary theory and this theory is decidable,

Corollary. *Let $G \leqslant H$ be divisible m-rigid groups. Then the embedding $G \to H$ is elementary if and only if it is linear independent.*

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Theorem 7 (MR, 2018). *The theories $\mathfrak{T}_m$ and $\mathfrak{T}_m(M)$ are ω-stable.*

Note, that if a group $M(\alpha_1, \ldots, \alpha_m)$ is uncountable then its cardinality coincides with the maximal α_i. For m-rigid group G we can define a rank, it is a tuple $r(G) = (r_1(G), \ldots, r_m(G))$, where $r_i(G)$ denotes the rank of the module $\rho_i(G)/\rho_{i+1}(G)$. For the case when G is independent subgroup of H we have $r_i(G) \leqslant r_i(H)$ and we can talk about the corank of H over G, it is also m-tuple of some cardinalities. For divisible m-rigid group $G = M(\alpha_1, \ldots, \alpha_m)$ we have $r(G) = (\alpha_1, \ldots, \alpha_m)$.

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Theorem 8 (MR. 2018). *Let λ be infinite cardinality.*
1) *$M(\beta_1, \ldots, \beta_m)$ is λ-saturated if and only if $\lambda \leq \beta_i$ for all indexes.*
2) *$M(\beta_1, \ldots, \beta_m)$ is saturated if and only if $\beta_1 = \ldots = \beta_m$ is infinite cardinality.*
3) *A countable model of the theory $\mathfrak{T}_m(M)$ is saturated if and only if its corank over M equal $(\omega, \ldots, \omega) = \omega^m$.*
4) *Let $\lambda > \omega$. A model of cardinality λ of the theory $\mathfrak{T}_m(M)$ is saturated if and only if it is $M(\lambda, \ldots, \lambda) = M(\lambda^m)$.*

An important object here is the divisible m-rigid group $M(\omega,\ldots,\omega) = M(\omega^m)$, which is a countable saturated model of the theory $\mathfrak{T}_m$. We prove that it will be a limit group of the Fraisse system of all finitely generated m-rigid groups. Let us give adapted to our case definition.
For given m-rigid group G denote by $\mathrm{age}(G)$ the set of all finitely generated independent m-rigid subgroups, and by $\overline{\mathrm{age}}(G)$ corresponding class of groups. Let also $\mathcal{K}_m$ denote the class of all finitely generated m-rigid groups. We know that any finitely generated m-rigid group can be independently embedded into some divisible m-rigid group of finite rank and so into $M(\omega^m)$, therefore $\overline{\mathrm{age}}(M(\omega^m)) = \mathcal{K}_m$. We call m-rigid group a limit of the class $\mathcal{K}_m$, if it has following properties:
(i) countable;
(ii) $\overline{\mathrm{age}}(G) = \mathcal{K}_m$;
(iii) homogeneity: if $U, V \in \mathrm{age}(G)$ and $\varphi : U \to V$ is some isomorphism then it can be extented to an automorphism of G.

Theorem 9 (MR, 2018). *Limit group for the class $\mathcal{K}_m$ exists, unique and isomorphic $M(\omega^m)$.*

We also study intersections of elementary submodels in models of the theories $\mathfrak{T}_m$ and $\mathfrak{T}_m(M)$.

Theorem 10 (MR, 2018). 1) *An intersections of some set of elementary submodels in a model of the theory $\mathfrak{T}_m$ is elementary submodel if and only if its class of solvability is equal exactly m.*
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Next statement is associated with quantifier elimination of discussed theories.

Theorem 11 (MR 2018). *Any formula of the theory* $\mathfrak{T}_m$ *or* $\mathfrak{T}_m(M)$ *is equivalent to a Boolean combination of* $\forall\exists$*-formulas.*

Theorem 12 (R 2018). *Let* G *be a divisible rigid group. If for two tuples of elements of* G $tp_{\exists}(a_1 \ldots, a_n) = tp_{\exists}(b_1 \ldots, b_n)$ *then these tuples are conjugate by some automorphism of* G.

Corollary. *Any formula of the theory* $\mathfrak{T}_m$ *is equivalent to a Boolean combination of* $\forall$*-formulas.*

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Corollary. *Any formula of the theory* $\mathfrak{T}_m$ *is equivalent to a Boolean combination of* $\forall$*-formulas.*

Problem 1. *To calculate the Morley rank of the group* $M(\alpha_1, \ldots, \alpha_m)$.

Problem 2. *To describe the formula subgroups (with constants) of the group* $M(\alpha_1, \ldots, \alpha_m)$.

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