J. Group Theory, Ahead of Print
DOI 10.1515/jgth-2025-0071

Certain residual properties of HNN-extensions with normal associated subgroups

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Communicated by Anton Klyachko

Abstract. Let $\mathbb{E}$ be the HNN-extension of a group B with subgroups H and K associated by an isomorphism $\varphi\colon H \to K$. Suppose that H and K are normal in B and $(H \cap K)\varphi = H \cap K$. Under these assumptions, we prove necessary and sufficient conditions for $\mathbb{E}$ to be residually a $\mathcal{C}$-group, where $\mathcal{C}$ is a class of groups closed under taking subgroups, quotient groups, and unrestricted wreath products. Among other things, these conditions give new facts on the residual finiteness and the residual p-finiteness of the group $\mathbb{E}$.

1 Introduction

Let $\mathcal{C}$ be a class of groups. Following [17], we say that a group X is *residually a $\mathcal{C}$-group* if any of its non-trivial elements is mapped to a non-trivial element by a suitable homomorphism of X onto a $\mathcal{C}$-group (i.e., a group from the class $\mathcal{C}$). Recall that if $\mathcal{C}$ is the class of all finite groups (or all solvable groups, or finite p-groups, where p is a prime), then a residually $\mathcal{C}$-group is also referred to as a *residually finite* (respectively, *residually solvable*, *residually p-finite*) group. We therefore use the term "*residual $\mathcal{C}$-ness*" along with the well-known notions of residual finiteness, residual p-finiteness, and residual solvability. Let us clarify that the residual $\mathcal{C}$-ness of a group X is the same as the property of X being residually a $\mathcal{C}$-group. In Section 1, for brevity, this term will also assume that $\mathcal{C}$ is a root class of groups.

According to [6, 16], a class of groups $\mathcal{C}$ is called a *root class* if it contains non-trivial groups, is closed under taking subgroups, and satisfies any of the following conditions, the equivalence of which is proved in [36]:

(1) for every group X and for every subnormal series $1 \leqslant Z \leqslant Y \leqslant X$ whose factors X/Y and Y/Z belong to $\mathcal{C}$, there exists a normal subgroup T of X such that $X/T \in \mathcal{C}$ and $T \leqslant Z$ (*Gruenberg's condition*);

The study was supported by the Russian Science Foundation grant No. 24-21-00307, http://rscf.ru/en/project/24-21-00307/.

(2) the class $\mathcal{C}$ is closed under taking unrestricted wreath products;

(3) the class $\mathcal{C}$ is closed under taking extensions and, together with any two groups X and Y, contains the unrestricted direct product $\prod_{y \in Y} X_y$, where X_y is an isomorphic copy of X for each $y \in Y$.

Examples of root classes are the classes of all finite groups, finite p-groups (where p is a prime), periodic $\mathfrak{P}$-groups of finite exponent (where $\mathfrak{P}$ is a non-empty set of primes), solvable groups, and torsion-free groups. It is also easy to see that the intersection of a family of root classes is again a root class if it contains a non-trivial group. The use of the concept of a root class turned out to be very productive in studying the residual properties of free constructions of groups: free and tree products, HNN-extensions, fundamental groups of graphs of group, etc. Clearly, it enables us to prove several statements at once instead of just one. But more importantly, the facts about residual $\mathcal{C}$-ness, where $\mathcal{C}$ is an *arbitrary* root class of groups, and the methods for finding them are well compatible with each other. This allows one to easily move from one free construction to another and quickly complicate the groups under consideration (see, e.g., [37, 38, 45, 46, 48, 50, 55]).

When studying the residual $\mathcal{C}$-ness of a group-theoretic construction, the main question is whether the construction inherits this property from the groups that compose it. As a rule, this question can only be answered by imposing various restrictions on the above-mentioned groups and their subgroups. The goal of this paper is to find necessary and sufficient conditions for the residual $\mathcal{C}$-ness of an HNN-extension whose associated subgroups are normal in the base group (here and below, we follow [25] in the use of terms related to HNN-extensions).

Considering the known results on the residual $\mathcal{C}$-ness of HNN-extensions, one can observe that most of them concern the case of residual finiteness [3–5, 8, 10, 21, 23, 27, 29, 33, 34, 58, 59]. The papers [2, 12, 28, 30–32] give some facts on the residual p-finiteness of this construction. In [15,40,42,45,46,50,51,53,55–57], the residual $\mathcal{C}$-ness of HNN-extensions is studied provided $\mathcal{C}$ is an arbitrary root class of groups, which possibly satisfies some additional restrictions. It should be noted that, at the time of writing, the results of the aforementioned papers generalize all known facts on the residual solvability and residual $\mathfrak{P}$-finiteness of HNN-extensions, where $\mathfrak{P}$ is a non-empty set of primes, as above.

The main method for studying the residual $\mathcal{C}$-ness of free constructions of groups is the so-called "filtration approach". It was originally proposed in [9] to study the residual finiteness of the free product of two groups with an amalgamated subgroup. After a number of generalizations and adaptations [8, 18, 24, 28, 35, 52], this approach was extended in [39] to the case of an arbitrary root class $\mathcal{C}$ and the fundamental group of an arbitrary graph of groups. The method includes two steps and, when applied to HNN-extensions, can be described as follows.

The first step is to find conditions for an HNN-extension $\mathbb{E}$ to have a homomorphism onto a $\mathcal{C}$-group that acts injectively on the base group. The existence of such a homomorphism is sometimes equivalent to the residual $\mathcal{C}$-ness of $\mathbb{E}$. For example, this equivalence holds if $\mathcal{C}$ consists of finite groups. But in general, this is not the case [41, 49].

We call the assertions proved at this step the *first-level results*. It should be noted that there are no general approaches to finding them and each new fact of this type is very valuable. Examples of such assertions are the theorem on the residual finiteness of an HNN-extension of a finite group [8] and the criteria for the residual p-finiteness of the same construction [2, 12, 28, 32]. When $\mathcal{C}$ is an arbitrary root class of groups closed under taking quotient groups, first-level results are found for

- an HNN-extension with coinciding associated subgroups that are normal in the base group [53];
- a number of HNN-extensions in which at least one of the associated subgroups lies in the center of the base group [40, 46, 50].

In this paper, we supplement this list and prove a criterion for the existence of a homomorphism with the properties described above provided the associated subgroups of an HNN-extension $\mathbb{E}$ are normal in the base group, while their intersection is normal in $\mathbb{E}$ (see Theorem 1 below).

Omitting technical details, we can say that the second step of the method consists in finding a sufficiently large number of homomorphisms mapping the HNN-extension $\mathbb{E}$ onto HNN-extensions satisfying the conditions of the previously obtained first-level results. This allows us to prove the residual $\mathcal{C}$-ness of $\mathbb{E}$ when the base group does not necessarily belong to the class $\mathcal{C}$. We call the sufficient conditions of the residual $\mathcal{C}$-ness found at this step the *second-level results*. As a rule, they can be proved only under restrictions stronger than those at Step 1. This is illustrated, in particular, by many years of studying the property of residual finiteness, during which a universal criterion for the residual finiteness of an HNN-extension of an arbitrary residually finite group was never found. In the present paper, the second-level results, Theorems 4–6, are also proved under certain assumptions supplementing the conditions of the criterion given by Theorem 1.

2 Statement of results

In what follows, the expression $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ means that $\mathbb{E}$ is the HNN-extension of a group B with a stable letter t and subgroups H and K associated by an isomorphism $\varphi\colon H \to K$. Let us say that the group $\mathbb{E}$ *satisfies* $(*)$ if

(1) the subgroups H and K are normal in B;

(2) the subgroup $L = H \cap K$ is φ-invariant, i.e., $\varphi|_L \in \operatorname{Aut} L$.

If X is a group and Y is a normal subgroup of X, then the restriction to Y of any inner automorphism of X is an automorphism of Y. The set of all such automorphisms is a subgroup of $\operatorname{Aut} Y$, which we denote below by $\operatorname{Aut}_X(Y)$. Let us note that if $\mathbb{E}$ satisfies $(*)$, then the following is defined:

(a) the subgroups $\mathfrak{H} = \operatorname{Aut}_B(H)$, $\mathfrak{K} = \varphi \operatorname{Aut}_B(K)\varphi^{-1}$, and $\mathfrak{U} = \operatorname{sgp}\{\mathfrak{H}, \mathfrak{K}\}$ of $\operatorname{Aut} H$;

(b) the subgroups $\mathfrak{L} = \operatorname{Aut}_B(L)$, $\mathfrak{F} = \operatorname{sgp}\{\varphi|_L\}$, and $\mathfrak{V} = \operatorname{sgp}\{\mathfrak{L}, \mathfrak{F}\}$ of $\operatorname{Aut} L$.

Throughout the paper, it is assumed that if $\mathbb{E} = \langle B, t;\, t^{-1}Ht = K,\, \varphi\rangle$, then the symbols L, $\mathfrak{H}$, $\mathfrak{K}$, $\mathfrak{U}$, $\mathfrak{L}$, $\mathfrak{F}$, and $\mathfrak{V}$ are defined as above.

Theorem 1. *Suppose that the group $\mathbb{E} = \langle B, t;\, t^{-1}Ht = K,\, \varphi\rangle$ satisfies $(*)$ and $\mathcal{C}$ is a root class of groups closed under taking quotient groups. If $B \in \mathcal{C}$, then the following statements are equivalent and any of them implies that $\mathbb{E}$ is residually a $\mathcal{C}$-group.*

(1) *There exists a homomorphism of $\mathbb{E}$ onto a group from $\mathcal{C}$ acting injectively on the subgroup B.*

(2) *The inclusions $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$ hold.*

Theorem 1 admits the following generalization.

Theorem 2. *Suppose that the group $\mathbb{E} = \langle B, t;\, t^{-1}Ht = K,\, \varphi\rangle$ satisfies $(*)$ and $\mathcal{C}$ is a root class of groups closed under taking quotient groups. If B has a homomorphism σ onto a group from $\mathcal{C}$ acting injectively on the subgroup HK, then the following statements hold.*

(1) *The condition $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$ is equivalent to the existence of a homomorphism of $\mathbb{E}$ onto a group from $\mathcal{C}$ that extends σ.*

(2) *If $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$ and B is residually a $\mathcal{C}$-group, then $\mathbb{E}$ is also residually a $\mathcal{C}$-group.*

Let us note that assertions like those in Theorem 2 can be useful for proving new first-level results (see, e.g., [50]).

Corollary 1. *Suppose that the group $\mathbb{E} = \langle B, t;\, t^{-1}Ht = K,\, \varphi\rangle$ satisfies $(*)$, $\mathcal{C}$ is a root class of groups closed under taking quotient groups, and the subgroups H and K are finite. Then $\mathbb{E}$ is residually a $\mathcal{C}$-group if and only if B is residually a $\mathcal{C}$-group and $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$.*

Theorem 1 requires that B belongs to $\mathcal{C}$. Theorem 2 removes this restriction, but still implicitly assumes that $H, K \in \mathcal{C}$. Now let us consider the case where B, H, and K are arbitrary residually $\mathcal{C}$-groups. We start with some necessary conditions for $\mathbb{E}$ to be residually a $\mathcal{C}$-group.

Theorem 3. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ satisfies $(*)$ and $\mathcal{C}$ is a class of groups closed under taking subgroups, quotient groups, and direct products of a finite number of factors. Suppose also that at least one of the following statements holds:*

(a) $\mathfrak{U} = \mathfrak{H}$ *or* $\mathfrak{U} = \mathfrak{K}$;

(b) *H and K satisfy a non-trivial identity;*

(c) *$\mathfrak{H}$ satisfies a non-trivial identity;*

(d) *$\mathfrak{K}$ satisfies a non-trivial identity.*

If $\mathbb{E}$ is residually a $\mathcal{C}$-group, then the quotient groups B/H and B/K have the same property.

Let us note that the papers [22, 39] contain several more necessary conditions for the residual $\mathcal{C}$-ness of HNN-extensions, which are similar to Theorem 3.

Given a class of groups $\mathcal{C}$ and a group X, we denote by $\mathcal{C}^*(X)$ the family of normal subgroups of X defined as follows: $N \in \mathcal{C}^*(X)$ if and only if $X/N \in \mathcal{C}$. Let us say that X is *$\mathcal{C}$-quasi-regular* with respect to its subgroup Y if, for each subgroup $M \in \mathcal{C}^*(Y)$, there exists a subgroup $N \in \mathcal{C}^*(X)$ such that $N \cap Y \leqslant M$. The property of $\mathcal{C}$-quasi-regularity is closely related to the classical notion of a $\mathcal{C}$-separable subgroup [26] and plays an important role in constructing the kernels of the homomorphisms that map a free construction of groups onto the groups from $\mathcal{C}$. Therefore, it is often a part of conditions sufficient for such a construction to be residually a $\mathcal{C}$-group (see, e.g., [21, 44, 50]). In [7, 42, 43], a number of situations are described in which a group turns out to be $\mathcal{C}$-quasi-regular with respect to its subgroup.

Theorem 4. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ satisfies $(*)$, $\mathcal{C}$ is a root class of groups closed under taking quotient groups, and at least one of the following statements holds:*

(α) $H/L \in \mathcal{C}$;

(β) *there exists a homomorphism of B onto a group from $\mathcal{C}$ acting injectively on L.*

Suppose also that $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$ and B is $\mathcal{C}$-quasi-regular with respect to HK. If B/H and B/K are residually $\mathcal{C}$-groups, then $\mathbb{E}$ and B are residually $\mathcal{C}$-groups simultaneously.

Let us note that if $\mathcal{C}$ is a root class of groups closed under taking quotient groups, while the HNN-extension $\mathbb{E}$ is residually a $\mathcal{C}$-group and satisfies $(*)$, then by Proposition 3.12 below, U and V are residually $\mathcal{C}$-groups and therefore $\mathfrak{H}$, $\mathfrak{K}$, $\mathfrak{L}$, $\mathfrak{F}$, $\mathfrak{U}$, and $\mathfrak{V}$ belong to $\mathcal{C}$ when they are finite. However, in general, neither the inclusions $\mathfrak{U} \in \mathcal{C}$ and $\mathfrak{V} \in \mathcal{C}$, which appear in Theorems 1, 2, and 4, nor the weaker conditions $\mathfrak{H} \in \mathcal{C}$, $\mathfrak{K} \in \mathcal{C}$, $\mathfrak{L} \in \mathcal{C}$, and $\mathfrak{F} \in \mathcal{C}$ are necessary for $\mathbb{E}$ to be residually a $\mathcal{C}$-group, as the following example shows.

Example. Suppose that

$$E = \langle a, b, c, t;\ [a, b] = [b, c] = [b, t] = 1,\ c^{-1}ac = t^{-1}at = ab \rangle,$$

$A = \mathrm{sgp}\{a, b\}$, and φ is the automorphism of A taking a to ab and b to b. Then E is the HNN-extension of the group

$$B = \langle a, b, c;\ [a, b] = [b, c] = 1,\ c^{-1}ac = ab \rangle$$

with the coinciding subgroups $H = A = K$ associated by φ. The group B, in turn, is the extension of the free abelian group A by the infinite cyclic group with the generator c, the conjugation by which acts on A as φ. Therefore,

$$\mathrm{Aut}_B(H) = \mathrm{Aut}_B(K) = \mathrm{Aut}_B(H \cap K) = \mathrm{sgp}\{\varphi|_{H \cap K}\}$$

is the infinite cyclic group generated by φ. At the same time, E splits as the generalized free product of the isomorphic polycyclic groups B and

$$D = \langle a, b, t;\ [a, b] = [b, t] = 1,\ t^{-1}at = ab \rangle$$

with the normal amalgamated subgroup A. Hence it is residually finite by [9, Theorem 9].

The next two theorems (and Propositions 6.5–6.8) describe some cases where the condition $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$ from Theorem 4 can be modified or weakened.

Theorem 5. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K, \varphi \rangle$ satisfies $(*)$, $\mathcal{C}$ is a root class of groups closed under taking quotient groups, and at least one of statements (α) and (β) from Theorem 4 holds. Suppose also that $\mathfrak{U} = \mathfrak{H}$ or $\mathfrak{U} = \mathfrak{K}$, $\mathfrak{V} \in \mathcal{C}$, and B is $\mathcal{C}$-quasi-regular with respect to HK. Then $\mathbb{E}$ is residually a $\mathcal{C}$-group if and only if the groups B, B/H, and B/K have the same property.*

Theorem 6. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K, \varphi \rangle$ satisfies $(*)$ and $\mathcal{C}$ is a root class of groups consisting only of periodic groups and closed under taking quotient groups. If H and K are locally cyclic groups, then the following statements hold.*

(1) *Let B/H and B/K be residually $\mathcal{C}$-groups. Then the group H/L is finite if and only if (α) holds.*

(2) *If B is residually a $\mathcal{C}$-group, then the group L is finite if and only if* (β) *holds.*

(3) *Suppose that H/L is finite, $\mathfrak{F} \in \mathcal{C}$, and B is $\mathcal{C}$-quasi-regular with respect to L. Then $\mathbb{E}$ is residually a $\mathcal{C}$-group if and only if the groups B, B/H, and B/K have the same property.*

(4) *Suppose that L is finite and B is $\mathcal{C}$-quasi-regular with respect to HK. Then $\mathbb{E}$ is residually a $\mathcal{C}$-group if and only if B, B/H, and B/K are residually $\mathcal{C}$-groups and $\mathfrak{F} \in \mathcal{C}$.*

Let us note that the condition "$\mathfrak{U} = \mathfrak{H}$ or $\mathfrak{U} = \mathfrak{K}$" holds if at least one of the subgroups H and K lies in the center of B. In this case, the subgroup L is certainly central in B, whence $\mathfrak{L} = 1$ and $\mathfrak{V} = \mathfrak{F}$. Therefore, Theorem 5 generalizes [50, Theorem 5]. As a comment to Theorem 6, we also note that if p is a prime and $\mathcal{F}_p$ is the class of finite p-groups, then every residually $\mathcal{F}_p$-group is $\mathcal{F}_p$-quasi-regular with respect to any of its locally cyclic subgroups [43, Theorem 3].

Given a class of groups $\mathcal{C}$ consisting only of periodic groups, let us denote by $\mathfrak{P}(\mathcal{C})$ the set of primes defined as follows: $p \in \mathfrak{P}(\mathcal{C})$ if and only if there exists a $\mathcal{C}$-group Z such that p divides the order of some element of Z. A subgroup Y of a group X is said to be $\mathfrak{P}(\mathcal{C})'$*-isolated* in this group if, for any element $x \in X$ and for any prime $q \notin \mathfrak{P}(\mathcal{C})$, it follows from the inclusion $x^q \in Y$ that $x \in Y$. Clearly, if $\mathfrak{P}(\mathcal{C})$ contains all prime numbers, then every subgroup is $\mathfrak{P}(\mathcal{C})'$-isolated.

Following [42], we say that

- an abelian group is $\mathcal{C}$*-bounded* if, for any quotient group B of A and for any $p \in \mathfrak{P}(\mathcal{C})$, the p-power torsion subgroup of B has a finite exponent and a cardinality not exceeding the cardinality of some $\mathcal{C}$-group;
- a nilpotent group is $\mathcal{C}$*-bounded* if it has a finite central series with $\mathcal{C}$-bounded abelian factors.

It is easy to see that if $\mathcal{C}$ is a root class of groups consisting only of periodic groups, then every finitely generated abelian group is $\mathcal{C}$-bounded abelian and therefore all finitely generated nilpotent groups are $\mathcal{C}$-bounded nilpotent.

Corollary 2. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ satisfies* $(*)$ *and $\mathcal{C}$ is a root class of groups consisting only of periodic groups and closed under taking quotient groups. Suppose also that B is a $\mathcal{C}$-bounded nilpotent group and at least one of statements* (α) *and* (β) *holds. Finally, suppose that at least one of the following statements hold:*

(a) $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$*;*

(b) $\mathfrak{U} = \mathfrak{H}$ *or* $\mathfrak{U} = \mathfrak{K}$*, and* $\mathfrak{V} \in \mathcal{C}$*;*

(c) *H and K are locally cyclic subgroups and $\mathfrak{F} \in \mathcal{C}$.*

Then $\mathbb{E}$ is residually a $\mathcal{C}$-group if and only if the subgroups $\{1\}$, H, and K are $\mathfrak{P}(\mathcal{C})'$-isolated in B.

Let us note that the known results on the residual finiteness and residual p-finiteness of HNN-extensions do not generalize the assertions that follow from Theorems 4–6 and Corollary 2 when $\mathcal{C}$ is the class of all finite groups or finite p-groups. Thus these assertions are also of interest. Proofs of the formulated theorems and corollaries are given in Sections 4–6.

3 Some auxiliary concepts and facts

We use the following notation throughout the paper:

- $\langle x \rangle$, the cyclic group generated by an element x;
- $\hat{x}$, the inner automorphism produced by an element x;
- $[x, y]$, the commutator of elements x and y, which is equal to $x^{-1}y^{-1}xy$;
- $[X : Y]$, the index of a subgroup Y in a group X;
- $\ker \sigma$, the kernel of a homomorphism σ;
- $\operatorname{Im} \sigma$, the image of a homomorphism σ.

Let $\mathcal{C}$ be a class of groups, and X a group. Following [26], we say that a subgroup Y of X is *$\mathcal{C}$-separable* in this group if, for each element $x \in X \setminus Y$, there exists a homomorphism σ of X onto a group from $\mathcal{C}$ such that $x\sigma \notin Y\sigma$.

Proposition 3.1 ([47, Proposition 3]). *Suppose that $\mathcal{C}$ is a class of groups closed under taking quotient groups, X is a group, and Y is a normal subgroup of X. Then Y is $\mathcal{C}$-separable in X if and only if X/Y is residually a $\mathcal{C}$-group.*

Proposition 3.2 ([47, Proposition 4]). *Suppose that $\mathcal{C}$ is a class of groups closed under taking subgroups, X is a group, Y is a subgroup of X, and $Z \in \mathcal{C}^*(X)$. Then $Y \cap Z \in \mathcal{C}^*(Y)$, and if X is residually a $\mathcal{C}$-group, then Y is also residually a $\mathcal{C}$-group.*

Proposition 3.3 ([47, Proposition 2]). *Suppose that $\mathcal{C}$ is a class of groups closed under taking subgroups and direct products of a finite number of factors. Then, for every group X, the following statements hold.*

(1) *The intersection of finitely many subgroups of the family $\mathcal{C}^*(X)$ is again a subgroup of this family.*

(2) *If X is residually a $\mathcal{C}$-group and Y is a finite subgroup of X, then there exists a subgroup $N \in \mathcal{C}^*(X)$ that meets Y trivially, whence $Y \in \mathcal{C}$.*

Proposition 3.4 ([54, Proposition 4]). *Suppose that $\mathcal{C}$ is a class of groups closed under taking quotient groups, X is a group, and Y is a normal subgroup of X. If there exists a homomorphism of X onto a group from $\mathcal{C}$ acting injectively on Y, then* $\mathrm{Aut}_X(Y) \in \mathcal{C}$.

Proposition 3.5. *If $\mathcal{C}$ is a class of groups closed under taking quotient groups, X is residually a $\mathcal{C}$-group, and Y is a normal subgroup of X, then* $\mathrm{Aut}_X(Y)$ *is also residually a $\mathcal{C}$-group.*

Proof. It is easy to see that $\mathrm{Aut}_X(Y) \cong X/\mathcal{Z}_X(Y)$, where $\mathcal{Z}_X(Y)$ is the centralizer of Y in X. By Proposition 3.1, it suffices to show that if $\mathcal{Z}_X(Y) \neq X$, then the subgroup $\mathcal{Z}_X(Y)$ is $\mathcal{C}$-separable in X.

Let $x \in X \setminus \mathcal{Z}_X(Y)$. Then $[x, y] \neq 1$ for some $y \in Y$. Since X is residually a $\mathcal{C}$-group, there exists a subgroup $N \in \mathcal{C}^*(X)$ which does not contain the commutator $[x, y]$. It follows that $x \notin \mathcal{Z}_X(Y)N$ and hence the subgroup $\mathcal{Z}_X(Y)$ is $\mathcal{C}$-separable. □

The proof of the following proposition is quite simple and is therefore omitted.

Proposition 3.6. *Suppose that X is a group, Y is a normal subgroup of X, and σ is a homomorphism of X. Suppose also that $\bar{\sigma}\colon$ $\mathrm{Aut}_X(Y) \to \mathrm{Aut}_{X\sigma}(Y\sigma)$ is the map taking $\hat{x}|_Y$ to $\widehat{x\sigma}|_{Y\sigma}$ for each $x \in X$. Then $\bar{\sigma}$ is a well-defined surjective homomorphism.*

Proposition 3.7. *Suppose that $\mathcal{C}$ is a class of groups closed under taking quotient groups, X is a group, and Y is a subgroup of X. If X is $\mathcal{C}$-quasi-regular with respect to Y and a subgroup $M \in \mathcal{C}^*(Y)$ is normal in X, then there exists a subgroup $N \in \mathcal{C}^*(X)$ such that $N \cap Y = M$.*

Proof. Suppose that a subgroup $M \in \mathcal{C}^*(Y)$ is normal in X. Since the latter is $\mathcal{C}$-quasi-regular with respect to Y, there exists a subgroup $T \in \mathcal{C}^*(X)$ such that $T \cap Y \leqslant M$. Let $N = MT$. Then N is normal in X and $N \cap Y = M$, as is easy to see. Since $\mathcal{C}$ is closed under taking quotient groups, it follows from the relations $X/N \cong (X/T)/(N/T)$ and $X/T \in \mathcal{C}$ that $X/N \in \mathcal{C}$. Thus N is the desired subgroup. □

Proposition 3.8. *If $\mathcal{C}$ is a root class of groups consisting only of periodic groups, then the following statements hold.*

(1) *Every $\mathcal{C}$-group is of finite exponent* [47, Proposition 17].

(2) *A finite solvable group belongs to $\mathcal{C}$ if and only if its order is a $\mathfrak{P}(\mathcal{C})$-number (i.e., each prime divisor of this order lies in $\mathfrak{P}(\mathcal{C})$)* [56, Proposition 8].

In what follows, the expression $\mathbb{P} = \langle A * B;\ H = K,\ \varphi\rangle$ means that $\mathbb{P}$ is the generalized free product of groups A and B with subgroups $H \leqslant A$ and $K \leqslant B$ amalgamated by an isomorphism $\varphi\colon H \to K$. According to [9], subgroups $R \leqslant A$ and $S \leqslant B$ are said to be *(H, K, φ)-compatible* if $(R \cap H)\varphi = S \cap K$. Suppose that R is normal in A, S is normal in B, and $\varphi_{R,S}\colon HR/R \to KS/S$ is the map taking an element hR, $h \in H$, to the element $(h\varphi)S$. It follows from the equality $(R \cap H)\varphi = S \cap K$ that $\varphi_{R,S}$ is a well-defined isomorphism and therefore we can consider the generalized free product

$$\mathbb{P}_{R,S} = \langle A/R * B/S;\ HR/R = KS/S,\ \varphi_{R,S}\rangle.$$

As is easy to see, the identity mapping of the generators of $\mathbb{P}$ into $\mathbb{P}_{R,S}$ defines a surjective homomorphism $\rho_{R,S}\colon \mathbb{P} \to \mathbb{P}_{R,S}$, whose kernel coincides with the normal closure of the set $R \cup S$ in $\mathbb{P}$. We note also that if H and K are normal in A and B, respectively, then H is normal in $\mathbb{P}$ and therefore the group $\mathrm{Aut}_{\mathbb{P}}(H)$ is defined. Clearly, this group is generated by its subgroups $\mathrm{Aut}_A(H)$ and $\varphi\,\mathrm{Aut}_B(K)\varphi^{-1}$.

Suppose that $x \in \mathbb{P}$ and

$$x = x_1 x_2 \dots x_n, \quad \text{where } n \geqslant 1 \text{ and } x_1, x_2, \dots, x_n \in A \cup B.$$

This product is called a *reduced form* of x if no two adjacent factors x_i and x_{i+1} lie simultaneously in A or B. The number n is said to be the *length* of this form. It is known that if an element $x \in \mathbb{P}$ has at least one reduced form of length greater than 1, then it is non-trivial (see, e.g., [25, Chapter IV, Theorem 2.6]).

Similar assertions hold for the HNN-extension $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$. A subgroup Q of B is said to be *(H, K, φ)-compatible* if $(Q \cap H)\varphi = Q \cap K$. When Q is normal in B, this equality ensures that the map $\varphi_Q\colon HQ/Q \to KQ/Q$ given by the rule $hQ \mapsto (h\varphi)Q$, $h \in H$, is a well-defined isomorphism. Therefore, the HNN-extension

$$\mathbb{E}_Q = \langle B/Q, t;\ t^{-1}(HQ/Q)t = KQ/Q,\ \varphi_Q\rangle$$

can be considered. As above, the identity mapping of the generators of $\mathbb{E}$ into $\mathbb{E}_Q$ defines a surjective homomorphism $\rho_Q\colon \mathbb{E} \to \mathbb{E}_Q$, whose kernel coincides with the normal closure of Q in $\mathbb{E}$.

Obviously, each element $x \in \mathbb{E}$ can be represented as a product

$$x = x_0 t^{\varepsilon_1} x_1 \dots x_{n-1} t^{\varepsilon_n} x_n,$$

where $n \geqslant 0$, $x_0, x_1, \dots, x_n \in B$, and $\varepsilon_1, \dots, \varepsilon_n \in \{1, -1\}$. This product is said to be a *reduced form* of x of *length* n if, for each $i \in \{1, \dots, n-1\}$, the equalities $-\varepsilon_i = 1 = \varepsilon_{i+1}$ imply that $x_i \notin H$, while the equalities $\varepsilon_i = 1 = -\varepsilon_{i+1}$ guarantee that $x_i \notin K$. Britton's lemma [11] states that if an element $x \in \mathbb{E}$ has a reduced form of non-zero length, then it is non-trivial. The next two propositions are special cases of [13, Theorem 4] and [41, Theorem 1].

Proposition 3.9. *Let* $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$*. If* N *is a normal subgroup of* $\mathbb{E}$ *and* $N \cap B = 1$*, then* N *is free.*

Proposition 3.10. *Let* $\mathcal{C}$ *be a root class of groups. If* $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$*,* B *is residually a* $\mathcal{C}$*-group, and there exists a homomorphism of* $\mathbb{E}$ *onto a group from* $\mathcal{C}$ *acting injectively on* H *and* K*, then* $\mathbb{E}$ *is residually a* $\mathcal{C}$*-group.*

In what follows, if $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$, then the expression

$$\mathbb{B} = \langle B * B;\ H = K,\ \varphi\rangle$$

means that $\mathbb{B}$ is the generalized free product of two isomorphic copies of B with the subgroups H and K amalgamated by the same isomorphism $\varphi\colon H \to K$. Let ζ_{gen} be the map of the generators of $\mathbb{B}$ into $\mathbb{E}$ given by the rule $x \mapsto t^{-1}xt$, $y \mapsto y$, where x and y are generators of the first and second instances of B, respectively. Clearly, when extended to a mapping of words, ζ_{gen} takes all defining relations of $\mathbb{B}$ to the equalities valid in $\mathbb{E}$ and therefore induces a homomorphism $\zeta\colon \mathbb{B} \to \mathbb{E}$. It is also easy to see that if $x_1 \dots x_n$ is a reduced form of an element $x \in \mathbb{B} \setminus \{1\}$, then the product $x_1\zeta \dots x_n\zeta$ is a reduced form of the element $x\zeta$ and $x\zeta \neq 1$. Hence ζ is injective. It can also be noted that if $\mathbb{E}$ satisfies $(*)$, then $\mathfrak{U} = \mathrm{Aut}_{\mathbb{B}}(H)$.

Proposition 3.11. *Suppose that the group* $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ *satisfies* $(*)$ *and* $\mathcal{C}$ *is a class of groups closed under taking subgroups and quotient groups. If there exists a homomorphism* σ *of* $\mathbb{E}$ *onto a group from* $\mathcal{C}$ *acting injectively on* H *and* K*, then* $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$*.*

Proof. It is obvious that $\mathfrak{V} = \mathrm{Aut}_{\mathbb{E}}(L)$. Therefore, the inclusion $\mathfrak{V} \in \mathcal{C}$ follows from Proposition 3.4. Let $\mathbb{B} = \langle B * B;\ H = K,\ \varphi\rangle$, and let $\zeta\colon \mathbb{B} \to \mathbb{E}$ be the homomorphism defined above. Since σ is injective on $K = K\zeta$ and $\mathcal{C}$ is closed under taking subgroups, $\mathbb{B}$ has a homomorphism onto a group from $\mathcal{C}$ acting injectively on H and K. Hence $\mathfrak{U} = \mathrm{Aut}_{\mathbb{B}}(H) \in \mathcal{C}$ by the same Proposition 3.4. □

Proposition 3.12. *Suppose that the group* $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ *satisfies* $(*)$ *and* $\mathcal{C}$ *is a root class of groups closed under taking quotient groups. If* $\mathbb{E}$ *is residually a* $\mathcal{C}$*-group, then* $\mathfrak{U}$ *and* $\mathfrak{V}$ *are also residually* $\mathcal{C}$*-groups and therefore the groups* $\mathfrak{H}$*,* $\mathfrak{K}$*,* $\mathfrak{L}$*,* $\mathfrak{F}$*,* $\mathfrak{U}$*, and* $\mathfrak{V}$ *belong to* $\mathcal{C}$ *when they are finite.*

Proof. As noted above, the group $\mathbb{B} = \langle B * B;\ H = K,\ \varphi\rangle$ can be embedded into the residually $\mathcal{C}$-group $\mathbb{E}$. Therefore, it is itself residually a $\mathcal{C}$-group by Proposition 3.2. Since $\mathfrak{U} = \mathrm{Aut}_{\mathbb{B}}(H)$ and $\mathfrak{V} = \mathrm{Aut}_{\mathbb{E}}(L)$, Proposition 3.5 implies that $\mathfrak{U}$ and $\mathfrak{V}$ are also residually $\mathcal{C}$-groups. The inclusions $\mathfrak{H}, \mathfrak{K}, \mathfrak{L}, \mathfrak{F}, \mathfrak{U}, \mathfrak{V} \in \mathcal{C}$ follow from Proposition 3.3. □

4 Proof of Theorems 1–2 and Corollary 1

Proposition 4.1 ([54, Theorem 1]). *If $\mathcal{C}$ is a root class of groups,*

$$\mathbb{P} = \langle A * B;\ H = K,\ \varphi\rangle,$$

H is normal in A, K is normal in B, and $A, B, A/H, B/K, \mathrm{Aut}_{\mathbb{P}}(H) \in \mathcal{C}$, then there exists a homomorphism of $\mathbb{P}$ onto a group from $\mathcal{C}$ acting injectively on A and B.

Let the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ satisfy $(*)$ and

$$\mathbb{B} = \langle B * B;\ H = K,\ \varphi\rangle.$$

Then the subgroup K of the first free factor of $\mathbb{B}$ and the subgroup H of the second are (H, K, φ)-compatible. Therefore, the generalized free product

$$\mathbb{B}_{K,H} = \langle (B/K) * (B/H);\ HK/K = KH/H,\ \varphi_{K,H}\rangle$$

and the group $\mathrm{Aut}_{\mathbb{B}_{K,H}}(HK/K)$ are defined.

Proposition 4.2. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ satisfies $(*)$, $\mathcal{C}$ is a root class of groups, and $H \cap K = 1$. If*

$$B/H, B/K, B/HK, \mathrm{Aut}_{\mathbb{B}_{K,H}}(HK/K) \in \mathcal{C},$$

then there exists a homomorphism of $\mathbb{E}$ onto a group from $\mathcal{C}$ acting injectively on B.

Proof. If $\mathcal{C}$ contains non-periodic groups, we denote by $\mathcal{I}$ the additive group of the ring $\mathbb{Z}$. Otherwise, let $\mathcal{I}$ be the additive group of the ring $\mathbb{Z}_n$, where n is the order of some $\mathcal{C}$-group and $n \geqslant 4$. Since $\mathcal{C}$ is closed under taking subgroups and extensions, the number n with the indicated properties exists and, in both cases, $\mathcal{I} \in \mathcal{C}$.

For each $i \in \mathcal{I}$, let B_i denote an isomorphic copy of B. Let also $\beta_i\colon B \to B_i$ be the corresponding isomorphism, $H_i = H\beta_i$, and $K_i = K\beta_i$. Consider the group

$$\mathfrak{P} = \langle B_i;\ H_i = K_{i-1}\ (i \in \mathcal{I})\rangle$$

whose generators are the generators of the groups B_i, $i \in \mathcal{I}$, and whose defining relations are those of B_i, $i \in \mathcal{I}$, and all possible relations of the form $h\varphi\beta_{i-1} = h\beta_i$, where $h \in H$ and $i \in \mathcal{I}$. It is easy to see that if $\mathcal{I}$ is infinite, then $\mathfrak{P}$ is the tree product of the groups B_i, $i \in \mathcal{I}$, that corresponds to an infinite chain. Otherwise, $\mathfrak{P}$ is the polygonal product of the same groups. Theorem 1 from [20] says that, in the first case, the identity mappings of the generators of B_i, $i \in \mathcal{I}$, into $\mathfrak{P}$ can be extended to injective homomorphisms. It follows from the relations $H \cap K = 1$ and $n \geqslant 4$ that the same statement holds in the second case [1].

Let α_{gen} be the mapping of the generators of $\mathfrak{P}$ acting as the isomorphisms $\beta_i^{-1}\beta_{i+1}$, $i \in \mathcal{I}$. Clearly, α_{gen} defines an automorphism α of $\mathfrak{P}$, whose order is equal to the order of $\mathcal{I}$. Let $\mathfrak{Q}$ denote the splitting extension of $\mathfrak{P}$ by the cyclic group $\langle\alpha\rangle$. Consider the mapping λ_{gen} of the generators of $\mathbb{E}$ into $\mathfrak{Q}$ that acts on the generators of B as β_0 and takes t to α (here and below, we identify the groups B, B_i, $i \in \mathcal{I}$, and $\mathfrak{P}$ with the corresponding subgroups of $\mathbb{E}$, $\mathfrak{P}$, and $\mathfrak{Q}$, respectively). It is easy to see that, when extended to a mapping of words, λ_{gen} takes all defining relations of $\mathbb{E}$ to the equalities valid in $\mathfrak{Q}$. Therefore, it induces a homomorphism $\lambda\colon \mathbb{E} \to \mathfrak{Q}$, which acts on B as β_0. Since $\mathfrak{Q}$ is obviously generated by the set $\{\alpha\} \cup \{b\beta_0 \mid b \in B\}$, the homomorphism λ is surjective.

Let $i \in \mathcal{I}$. It is clear that β_{i+1} and β_i induce an isomorphism γ_i of $\mathbb{B}_{K,H}$ onto the generalized free product

$$\mathfrak{P}_i = \langle (B_{i+1}/K_{i+1}) * (B_i/H_i);\ H_{i+1}K_{i+1}/K_{i+1} = K_i H_i/H_i,\ \varphi_i\rangle,$$

where the isomorphism $\varphi_i\colon H_{i+1}K_{i+1}/K_{i+1} \to K_i H_i/H_i$ is given by the rule

$$(h\beta_{i+1})K_{i+1} \mapsto (h\varphi\beta_i)H_i, \quad h \in H.$$

The relations

$$(B/H)/(KH/H) \cong B/HK \cong (B/K)/(HK/K)$$

and $B/HK \in \mathcal{C}$ mean that all conditions of Proposition 4.1 hold for the group $\mathbb{B}_{K,H}$. As $(B/K)\gamma_i = B_{i+1}/K_{i+1}$ and $(B/H)\gamma_i = B_i/H_i$, it follows that there exists a homomorphism η_i of $\mathfrak{P}_i$ onto a group from $\mathcal{C}$ satisfying the equalities

$$\ker\eta_i \cap B_i/H_i = 1 = \ker\eta_i \cap B_{i+1}/K_{i+1}.$$

Consider the mapping of the generators of $\mathfrak{P}$ into $\mathfrak{P}_i$ which acts identically on the elements of B_i and B_{i+1}, and takes the generators of other free factors to 1. It is easy to see that this mapping defines a surjective homomorphism $\theta_i\colon \mathfrak{P} \to \mathfrak{P}_i$. Therefore, if M_i denotes the subgroup $\ker\theta_i\eta_i$, we have

$$M_i \in \mathcal{C}^*(\mathfrak{P}), \quad M_i \cap B_i = H_i, \quad \text{and} \quad M_i \cap B_{i+1} = K_{i+1}.$$

Let $M = M_0 \cap M_{-1}$. It follows from Proposition 3.3 and the above that

$$M \cap B_0 = (M_0 \cap B_0) \cap (M_{-1} \cap B_0) = H_0 \cap K_0 = (H \cap K)\beta_0 = 1$$

and $M \in \mathcal{C}^*(\mathfrak{P})$. Since $\langle \alpha \rangle \cong \mathcal{J} \in \mathcal{C}$, we have that the factors of the subnormal sequence $M \leqslant \mathfrak{P} \leqslant \mathfrak{Q}$ belong to $\mathcal{C}$. Hence, by Gruenberg's condition, there exists a subgroup $N \in \mathcal{C}^*(\mathfrak{Q})$ lying in M. It now follows from the relations $B_0 = B\lambda$ and $N \cap B_0 \leqslant M \cap B_0 = 1$ that the composition of λ and the natural homomorphism $\mathfrak{Q} \to \mathfrak{Q}/N$ is the desired mapping. □

Proposition 4.3. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi \rangle$ satisfies $(*)$ and $\mathcal{C}$ is a root class of groups. If*

$$L, B/H, B/K, B/HK, \mathrm{Aut}_{\mathbb{E}}(L), \mathrm{Aut}_{\mathbb{B}_{K,H}}(HK/K) \in \mathcal{C},$$

then there exists a homomorphism of $\mathbb{E}$ onto a group from $\mathcal{C}$ acting injectively on B.

Proof. Since L is a normal subgroup of B and $(L \cap H)\varphi = L\varphi = L = L \cap K$, the HNN-extension

$$\mathbb{E}_L = \langle B/L, t;\ t^{-1}(H/L)t = K/L,\ \varphi_L \rangle$$

is defined. Consider the following groups:

$$\mathbb{B}_{K,H} = \langle (B/K) * (B/H);\ HK/K = KH/H,\ \varphi_{K,H} \rangle,$$
$$\mathbb{B}_{K/L,H/L} = \langle (B/L)/(K/L) * (B/L)/(H/L);\ (H/L)(K/L)/(K/L) = (K/L)(H/L)/(H/L),\ \varphi_{K/L,H/L} \rangle.$$

It is clear that the identity mapping of the generators of $\mathbb{B}_{K/L,H/L}$ into $\mathbb{B}_{K,H}$ defines an isomorphism, which takes the subgroup $(H/L)(K/L)/(K/L)$ onto HK/K. Therefore,

$$\mathrm{Aut}_{\mathbb{B}_{K/L,H/L}}\big((H/L)(K/L)/(K/L)\big) \cong \mathrm{Aut}_{\mathbb{B}_{K,H}}(HK/K) \in \mathcal{C}.$$

Since

$$(B/L)/(H/L) \cong B/H \in \mathcal{C}, \qquad (B/L)/(K/L) \cong B/K \in \mathcal{C},$$
$$(B/L)/(H/L)(K/L) \cong B/HK \in \mathcal{C}, \qquad H/L \cap K/L = 1,$$

the HNN-extension $\mathbb{E}_L$ satisfies the conditions of Proposition 4.2. Hence there exists a homomorphism τ_L of $\mathbb{E}_L$ onto a group from $\mathcal{C}$ which is injective on B/L. By Proposition 3.9, the kernel of τ_L is a free group.

Since L is normal in $\mathbb{E}$, the equality $L = \ker \rho_L$ holds. Therefore, the subgroup $U = \ker \rho_L \tau_L$ is an extension of L by a free group. It is well known that such an extension is splittable, i.e., U has a free subgroup F satisfying the relations $U = LF$ and $L \cap F = 1$.

Let $\xi\colon \mathbb{E} \to \operatorname{Aut} L$ be the homomorphism taking an element $x \in \mathbb{E}$ to the automorphism $\hat{x}|_L$. Its kernel obviously coincides with the centralizer $\mathcal{Z}_{\mathbb{E}}(L)$ of L in $\mathbb{E}$. Since $\mathcal{C}$ is closed under taking subgroups and extensions, it follows from this fact and the relations

$$U = LF, \quad L \cap F = 1, \quad L \leqslant B,$$
$$F\mathcal{Z}_{\mathbb{E}}(L)/\mathcal{Z}_{\mathbb{E}}(L) \leqslant \mathbb{E}/\mathcal{Z}_{\mathbb{E}}(L), \quad \operatorname{Im}\xi = \operatorname{Aut}_{\mathbb{E}}(L) \in \mathcal{C}$$

that the subgroup $V = \mathcal{Z}_{\mathbb{E}}(L) \cap F$ is normal in U,

$$F/V \cong F\mathcal{Z}_{\mathbb{E}}(L)/\mathcal{Z}_{\mathbb{E}}(L) \in \mathcal{C}, \quad LV/V \cong L/L \cap V \cong L \in \mathcal{C},$$
$$U/LV = LF/LV \cong F/V(L \cap F) = F/V \in \mathcal{C},$$

and the quotient group U/V belongs to $\mathcal{C}$ as an extension of LV/V by a group isomorphic to U/LV. In addition, $U \in \mathcal{C}^*(\mathbb{E})$, the definition of τ_L. Thus we can apply Gruenberg's condition to the subnormal series $1 \leqslant V \leqslant U \leqslant \mathbb{E}$ and find a subgroup $W \in \mathcal{C}^*(\mathbb{E})$ lying in V.

Since τ_L acts injectively on B/L, the equality $U \cap B = L$ holds. It now follows from the inclusions $W \leqslant V \leqslant F \leqslant U$ that $W \cap B \leqslant F \cap (U \cap B) = F \cap L = 1$. Hence the natural homomorphism $\mathbb{E} \to \mathbb{E}/W$ is the desired one. □

Proof of Theorem 1. The implication (1) ⇒ (2) and the residual $\mathcal{C}$-ness of $\mathbb{E}$ follow from Propositions 3.11 and 3.10, respectively. To prove (2) ⇒ (1), it is sufficient to show that all conditions of Proposition 4.3 hold if $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$.

Indeed, since $\mathcal{C}$ is closed under taking subgroups and quotient groups, the inclusions $L, B/H, B/K, B/HK \in \mathcal{C}$ follow from the condition $B \in \mathcal{C}$. Let

$$\mathbb{B} = \langle B * B;\ H = K, \varphi\rangle.$$

By Proposition 3.6, the relations

$$\operatorname{Aut}_{\mathbb{B}}(H) = \mathfrak{U} \in \mathcal{C}, \quad \mathbb{B}_{K,H} = \mathbb{B}\rho_{K,H}, \quad \text{and} \quad HK/K = H\rho_{K,H}$$

imply that $\operatorname{Aut}_{\mathbb{B}_{K,H}}(HK/K) \in \mathcal{C}$. It remains to note that $\operatorname{Aut}_{\mathbb{E}}(L) = \mathfrak{V} \in \mathcal{C}$. □

Proposition 4.4. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K, \varphi\rangle$ satisfies $(*)$, Q is a normal (H, K, φ)-compatible subgroup of B, and*

$$\mathbb{E}_Q = \langle B/Q, t;\ t^{-1}(HQ/Q)t = KQ/Q, \varphi_Q\rangle.$$

Suppose also that the symbols $\mathfrak{H}_Q$, $\mathfrak{K}_Q$, and $\mathfrak{L}_Q$ denote the subgroups

$$\operatorname{Aut}_{B/Q}(HQ/Q), \quad \varphi_Q \operatorname{Aut}_{B/Q}(KQ/Q)\varphi_Q^{-1}, \quad \textit{and} \quad \operatorname{Aut}_{B/Q}(LQ/Q),$$

respectively. Then the following statements hold.

(1) *There exists a homomorphism of* $\mathfrak{U}$ *onto the group* $\mathfrak{U}_Q = \mathrm{sgp}\{\mathfrak{H}_Q, \mathfrak{K}_Q\}$ *which maps the subgroups* $\mathfrak{H}$ *and* $\mathfrak{K}$ *onto* $\mathfrak{H}_Q$ *and* $\mathfrak{K}_Q$, *respectively.*

(2) *The subgroup* LQ/Q *is* φ_Q*-invariant and there exists a homomorphism of* $\mathfrak{V}$ *onto the group* $\mathfrak{V}_Q = \mathrm{sgp}\{\mathfrak{L}_Q, \varphi_Q|_{LQ/Q}\}$ *which maps* $\mathfrak{L}$ *and* $\mathfrak{F}$ *onto the subgroups* $\mathfrak{L}_Q$ *and* $\mathfrak{F}_Q = \langle \varphi_Q|_{LQ/Q}\rangle$, *respectively.*

Proof. (1) Let $\mathbb{B} = \langle B * B;\ H = K,\ \varphi\rangle$. Since Q is (H, K, φ)-compatible, the generalized free product

$$\mathbb{B}_{Q,Q} = \langle B/Q * B/Q;\ HQ/Q = KQ/Q,\ \varphi_{Q,Q}\rangle$$

is defined. It follows from Proposition 3.6 that the map

$$\overline{\rho_{Q,Q}}\colon \mathrm{Aut}_{\mathbb{B}}(H) \to \mathrm{Aut}_{\mathbb{B}_{Q,Q}}(H\rho_{Q,Q})$$

given by the rule $\hat{x}|_H \mapsto \widehat{x\rho_{Q,Q}}|_{H\rho_{Q,Q}}$, $x \in \mathbb{B}$, is a surjective homomorphism. Clearly,

$$\mathfrak{U}_Q = \mathrm{Aut}_{\mathbb{B}_{Q,Q}}(HQ/Q), \quad \mathfrak{U} = \mathrm{Aut}_{\mathbb{B}}(H),$$
$$H\rho_{Q,Q} = HQ/Q, \quad \mathfrak{H}\overline{\rho_{Q,Q}} = \mathfrak{H}_Q, \quad \text{and} \quad \mathfrak{K}\overline{\rho_{Q,Q}} = \mathfrak{K}_Q.$$

Therefore, the homomorphism $\overline{\rho_{Q,Q}}$ is as desired.

(2) The equality $L\varphi = L$ and the definition of φ_Q imply that

$$(LQ/Q)\varphi_Q = LQ/Q.$$

Since φ_Q is an isomorphism, this relation means that $\varphi_Q|_{LQ/Q} \in \mathrm{Aut}\, LQ/Q$. The existence of the desired homomorphism is ensured by Proposition 3.6 due to the equalities $\mathfrak{V}_Q = \mathrm{Aut}_{\mathbb{E}_Q}(LQ/Q)$, $\mathfrak{V} = \mathrm{Aut}_{\mathbb{E}}(L)$, and $LQ/Q = L\rho_Q$. □

Proof of Theorem 2. Statement (2) follows from (1) and Proposition 3.10. Let us prove statement (1). If there exists a homomorphism of $\mathbb{E}$ onto a group from $\mathcal{C}$ that extends σ, then $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$ by Proposition 3.11. It remains to show that the converse also holds.

Let $Q = \ker\sigma$. Then $B/Q \in \mathcal{C}$ and it follows from the equality $Q \cap HK = 1$ that $Q \cap H = 1 = Q \cap K$ and $HQ/Q \cap KQ/Q = LQ/Q$. Hence the groups $\mathbb{E}_Q$, $\mathfrak{U}_Q$ and $\mathfrak{V}_Q$ can be defined as in Proposition 4.4. By this proposition, the subgroup LQ/Q is φ_Q-invariant. Since $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$ and $\mathcal{C}$ is closed under taking quotient groups, Proposition 4.4 also implies that $\mathfrak{U}_Q, \mathfrak{V}_Q \in \mathcal{C}$. By Theorem 1, it follows from these relations and the equality $HQ/Q \cap KQ/Q = LQ/Q$ that there exists a homomorphism τ of the group $\mathbb{E}_Q$ which satisfies the conditions $\ker\tau \cap B/Q = 1$ and $\mathrm{Im}\,\tau \in \mathcal{C}$. Since ρ_Q extends σ, the composition $\rho_Q\tau$ is the desired mapping. □

Proof of Corollary 1. *Necessity.* Proposition 3.3 states that there exists a homomorphism of $\mathbb{E}$ onto a group from $\mathcal{C}$ acting injectively on the finite subgroup HK. Hence $\mathfrak{U}, \mathfrak{V} \in \mathcal{C}$ by Proposition 3.11. The residual $\mathcal{C}$-ness of B is ensured by Proposition 3.2.

Sufficiency. As above, if B is residually a $\mathcal{C}$-group, it has a homomorphism onto a group from $\mathcal{C}$ acting injectively on HK. Therefore, we can use Theorem 2 (2) to prove the residual $\mathcal{C}$-ness of $\mathbb{E}$. □

5 Proof of Theorem 3

Proposition 5.1 ([47, Proposition 9]). *Let* $\mathbb{P} = \langle A * B;\ H = K,\ \varphi\rangle$. *Suppose also that* H *is normal in* A, K *is normal in* B, *and the group* $\mathrm{Aut}_{\mathbb{P}}(H)$ *coincides with one of its subgroups* $\mathrm{Aut}_A(H)$ *and* $\varphi\,\mathrm{Aut}_B(H)\varphi^{-1}$. *If* $\mathcal{C}$ *is a class of groups closed under taking subgroups and* $\mathbb{P}$ *is residually a* $\mathcal{C}$*-group, then the following statements hold.*

(1) *If* $K \neq B$, *then* H *is* $\mathcal{C}$*-separable in* A.

(2) *If* $H \neq A$, *then* K *is* $\mathcal{C}$*-separable in* B.

Proposition 5.2 ([22, Theorem 1]). *Let* $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$. *Suppose also that* $\mathcal{C}$ *is a class of groups, the symbols* $\overline{H}$ *and* $\overline{K}$ *denote the subgroups*

$$\bigcap_{N \in \mathcal{C}^*(\mathbb{E})} H(N \cap B) \quad \text{and} \quad \bigcap_{N \in \mathcal{C}^*(\mathbb{E})} K(N \cap B),$$

respectively, and at least one of the following statements holds:

(a) *the subgroups* H *and* K *coincide and satisfy a non-trivial identity;*

(b) *the subgroups* H *and* K *are properly contained in a subgroup* D *of* B *satisfying a non-trivial identity.*

If $\mathbb{E}$ *is residually a* $\mathcal{C}$*-group, then* $\overline{H} = H$ *and* $\overline{K} = K$.

Proof of Theorem 3. Suppose that the subgroups $\overline{H}$ and $\overline{K}$ are defined as in Proposition 5.2. Since $(H \cap K)\varphi = H \cap K$, the relations $H \leqslant K$, $H = K$, and $H \geqslant K$ are equivalent. Therefore, only two cases are possible: $H = K$ and $H \neq HK \neq K$. If $H = B = K$, then the residual $\mathcal{C}$-ness of B/H and B/K is obvious. Hence we can further assume that $H \neq B \neq K$.

By Proposition 3.1, to complete the proof, it suffices to show that H and K are $\mathcal{C}$-separable in B. If $\mathfrak{U} = \mathfrak{H}$ or $\mathfrak{U} = \mathfrak{K}$, then the group $\mathbb{B} = \langle B * B;\ H = K,\ \varphi\rangle$ satisfies all the conditions of Proposition 5.1, which ensures the required separability.

Suppose that H and K satisfy a non-trivial identity. Then the group HK has the same property since it is an extension of K by a group isomorphic to HK/K and $HK/K \cong H/H \cap K$. Thus the conditions of Proposition 5.2 hold, and we get the equalities $\overline{H} = H$ and $\overline{K} = K$. It remains to note that if $N \in \mathcal{C}^*(\mathbb{E})$, then $N \cap B \in \mathcal{C}^*(B)$ by Proposition 3.2, and therefore these equalities imply the $\mathcal{C}$-separability of H and K in B.

Now suppose that $\mathfrak{H}$ satisfies a non-trivial identity. Then, by [34, Lemma 2], it satisfies a non-trivial identity of the form

$$\omega(y, x_1, x_2) = \omega_0(x_1, x_2) y^{\varepsilon_1} \omega_1(x_1, x_2) \dots y^{\varepsilon_n} \omega_n(x_1, x_2),$$

where $n \geqslant 1$, $\varepsilon_1, \dots, \varepsilon_n = \pm 1$, and

$$\omega_0(x_1, x_2), \dots, \omega_n(x_1, x_2) \in \{x_1^{\pm 1}, x_2^{\pm 1}, (x_1 x_2^{-1})^{\pm 1}\}.$$

Let us assume that there exist elements $u_1, u_2, v \in \mathbb{E}$ with the following properties:

(i) the commutator $[\omega(v, u_1, u_2), v]$ has a reduced form of non-zero length;

(ii) for each subgroup $N \in \mathcal{C}^*(\mathbb{E})$, the inclusions $u_1, u_2 \in BN$ and $v \in HN$ hold.

Then $\mathbb{E}$ is not residually a $\mathcal{C}$-group, contrary to the conditions of the theorem.

Indeed, since the element $g = [\omega(v, u_1, u_2), v]$ has a reduced form of non-zero length, it is not equal to 1. At the same time, if $N \in \mathcal{C}^*(\mathbb{E})$, then

$$g \equiv [\omega(h, b_1, b_2), h] \pmod N \quad \text{for some } b_1, b_2 \in B \text{ and } h \in H.$$

The restriction to H of the conjugation by $\omega(h, b_1, b_2)$ coincides with the element $\omega(\hat{h}, \widehat{b_1}|_H, \widehat{b_2}|_H)$ of $\mathfrak{H}$, which is the identity mapping because $\mathfrak{H}$ satisfies ω. Therefore, $[\omega(h, b_1, b_2), h] = 1$, $g \equiv 1 \pmod N$, and $\mathbb{E}$ is not residually a $\mathcal{C}$-group since N is chosen arbitrarily.

As above, to prove the $\mathcal{C}$-separability of H and K in B, it suffices to show that $\overline{H} = H$ and $\overline{K} = K$. Arguing by contradiction, we consider four cases and, in each of them, find elements $u_1, u_2, v \in \mathbb{E}$ satisfying (i) and (ii).

Case 1: *$\overline{H} \neq H$ and $[B : H] \geqslant 3$.* Let $b_1 \in \overline{H} \setminus H$. As $[B : H] \geqslant 3$ and $K \neq B$, there exist elements $b_2, c \in B$ such that $b_2, b_1 b_2^{-1} \notin H$ and $c \notin K$. Let us put $u_1 = b_1$, $u_2 = b_2$, and $v = tc^{-1}t^{-1}b_1tct^{-1}$.

Since $b_1^{\pm 1}, b_2^{\pm 1}, (b_1 b_2^{-1})^{\pm 1} \notin H$ and $c \notin K$, the element $[\omega(v, u_1, u_2), v]$ has a reduced form of length $8(n + 1)$. At the same time, if $N \in \mathcal{C}^*(\mathbb{E})$, then it follows from the inclusion $b_1 \in \overline{H}$ that $b_1 \in HN$, $t^{-1}b_1t \in KN$, $c^{-1}t^{-1}b_1tc \in KN$ (because K is normal in B), and $tc^{-1}t^{-1}b_1tct^{-1} \in HN$. Thus the elements u_1, u_2, and v satisfy (i) and (ii).

Case 2: $\overline{H} \neq H$ *and* $[B : H] = 2$. Let us fix some elements $b \in \overline{H} \setminus H$ and $c \in B \setminus K$, and put $u_1 = t^{-1}bt$, $u_2 = t^{-2}bt^2$, and $v = c$. Then

$$u_1 u_2^{-1} = t^{-1}bt^{-1}b^{-1}t^2$$

and the element $[\omega(v, u_1, u_2), v]$ has a reduced form of length at least $4(n+1)$. The relations $[B : H] = 2$ and $\overline{H} \neq H$ mean that $B = \overline{H}$. Therefore, $b \in HN$ and $BN = HN$ for any $N \in \mathcal{C}^*(\mathbb{E})$. It follows that

$$u_1 = t^{-1}bt \in KN \leqslant BN = HN, \quad u_2 = t^{-2}bt^2 = t^{-1}u_1 t \in KN,$$

and $v = c \in BN = HN$, as required.

Case 3: $\overline{K} \neq K$ *and* $[B : H] \geqslant 3$. Suppose that we have $c \in \overline{K} \setminus K$ and elements $b_1, b_2 \in B \setminus H$ are such that $b_1 b_2^{-1} \notin H$. Let us put

$$u_1 = b_1, \quad u_2 = b_2, \quad \text{and} \quad v = tct^{-1}.$$

Then the element $[\omega(v, u_1, u_2), v]$ has a reduced form of length $4(n+1)$ and, for each subgroup $N \in \mathcal{C}^*(\mathbb{E})$, the inclusions $b_1, b_2 \in BN$, $c \in KN$, $tct^{-1} \in HN$ hold.

Case 4: $\overline{K} \neq K$ *and* $[B : H] = 2$. It follows from the relations

$$K/L = K/H \cap K \cong KH/H \leqslant B/H$$

that $[K : L] \leqslant 2$. If $K = L$, then $K \leqslant H$ and hence $K = H$, as noted at the beginning of the proof. The last equality means that $\overline{H} = \overline{K}$ and $\overline{H} \neq H$, which is impossible due to Cases 1 and 2 considered above. Thus $[K : L] = 2$. Let us put $\overline{L} = \bigcap_{N \in \mathcal{C}^*(\mathbb{E})} L(N \cap B)$ and show that $\overline{L} \neq L$.

Suppose, on the contrary, that $\overline{L} = L$, and fix some elements $c \in \overline{K} \setminus K$ and $k \in K \setminus L$. Since $[K : L] = 2$ and $c \notin K$, the relations

$$K = L \cup kL \quad \text{and} \quad c, k^{-1}c \notin L = \overline{L}$$

hold. Hence we have $c \notin L(N_1 \cap B)$ and $k^{-1}c \notin L(N_2 \cap B)$ for suitable subgroups $N_1, N_2 \in \mathcal{C}^*(\mathbb{E})$. Let $N = N_1 \cap N_2$. Then $N \in \mathcal{C}^*(\mathbb{E})$ by Proposition 3.3 and $c, k^{-1}c \notin L(N \cap B)$. It follows that

$$c \notin L(N \cap B) \cup kL(N \cap B) = K(N \cap B)$$

and therefore $c \notin \overline{K}$ despite the choice of c.

So $\overline{L} \neq L$. Let us fix some elements $b \in \overline{L} \setminus L$ and $d \in B \setminus H$, and put

$$u_1 = tbt^{-1}, \quad u_2 = t^2bt^{-2}, \quad \text{and} \quad v = dtbt^{-1}d^{-1}.$$

Then $u_1 u_2^{-1} = tbtb^{-1}t^{-2}$. Since $L \leqslant H$, the relations $\overline{L} \leqslant \overline{H} = H$ and $b \in H$ hold. Therefore, $b \notin K$ and the element $[\omega(v, u_1, u_2), v]$ has a reduced form of length not less than $8(n+1)$. At the same time, since $b \in \overline{L}$ and L is normal in $\mathbb{E}$, the inclusions $b \in LN$, $tbt^{-1} \in LN$, $t^2bt^{-2} \in LN$, and $dtbt^{-1}d^{-1} \in LN$ hold for any $N \in \mathcal{C}^*(\mathbb{E})$.

Thus $\overline{H} = H$ and $\overline{K} = K$. When $\mathfrak{K}$ satisfies a non-trivial identity, the proof is similar. □

6 Proof of Theorems 4–6 and Corollary 2

If $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K, \varphi\rangle$, Q is a normal (H, K, φ)-compatible subgroup of B, and $\mathcal{C}$ is a class of groups, then we say that Q is

(a) *$\mathcal{C}$-admissible* if there exists a homomorphism of the group

$$\mathbb{E}_Q = \langle B/Q, t;\ t^{-1}(HQ/Q)t = KQ/Q, \varphi_Q\rangle$$

onto a group from $\mathcal{C}$ acting injectively on B/Q;

(b) *pre-$\mathcal{C}$-admissible* if $B/Q \in \mathcal{C}$ and $HQ/Q \cap KQ/Q = LQ/Q$.

The next proposition follows from [39, Theorems 1 and 3].

Proposition 6.1. *Let $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K, \varphi\rangle$, and let $\mathcal{C}$ be a root class of groups. Suppose also that H and K are $\mathcal{C}$-separable in B and each subgroup of $\mathcal{C}^*(B)$ contains a $\mathcal{C}$-admissible subgroup. Then $\mathbb{E}$ is residually a $\mathcal{C}$-group if and only if B has the same property.*

Proposition 6.2. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K, \varphi\rangle$ satisfies $(*)$ and $\mathcal{C}$ is a root class of groups closed under taking quotient groups. Suppose also that $\mathfrak{F} \in \mathcal{C}$ and the following conditions hold:*

(†) *at least one of the subgroups $\mathfrak{H}$ and $\mathfrak{K}$ is normal in $\mathfrak{U}$ or $\mathfrak{U} \in \mathcal{C}$;*

(‡) *at least one of the subgroups $\mathfrak{L}$ and $\mathfrak{F}$ is normal in $\mathfrak{V}$ or $\mathfrak{V} \in \mathcal{C}$.*

If B/H and B/K are residually $\mathcal{C}$-groups and each subgroup of $\mathcal{C}^*(B)$ contains a pre-$\mathcal{C}$-admissible subgroup, then $\mathbb{E}$ and B are residually $\mathcal{C}$-groups simultaneously.

Proof. In view of Propositions 3.1 and 6.1, it suffices to show that every pre-$\mathcal{C}$-admissible subgroup Q is $\mathcal{C}$-admissible.

Indeed, let the groups $\mathbb{E}_Q$, $\mathfrak{H}_Q$, $\mathfrak{K}_Q$, $\mathfrak{L}_Q$, $\mathfrak{F}_Q$, $\mathfrak{U}_Q$, and $\mathfrak{V}_Q$ be defined as in Proposition 4.4. By the latter, there exists a homomorphism of $\mathfrak{U}$ onto the group $\mathfrak{U}_Q$ mapping $\mathfrak{H}$ and $\mathfrak{K}$ onto $\mathfrak{H}_Q$ and $\mathfrak{K}_Q$, respectively. Therefore, if $\mathfrak{H}$ is normal in $\mathfrak{U}$, then $\mathfrak{H}_Q$ is normal in $\mathfrak{U}_Q$ and hence the latter is an extension of $\mathfrak{H}_Q$ by the group $\mathfrak{U}_Q/\mathfrak{H}_Q = \mathfrak{K}_Q\mathfrak{H}_Q/\mathfrak{H}_Q \cong \mathfrak{K}_Q/\mathfrak{H}_Q \cap \mathfrak{K}_Q$. Since $B/Q \in \mathcal{C}$, it follows from Proposition 3.4 that $\mathrm{Aut}_{B/Q}(HQ/Q) \in \mathcal{C}$ and $\mathrm{Aut}_{B/Q}(KQ/Q) \in \mathcal{C}$. But $\mathrm{Aut}_{B/Q}(KQ/Q)$ is isomorphic to $\mathfrak{K}_Q$. Hence $\mathfrak{H}_Q, \mathfrak{K}_Q \in \mathcal{C}$ and $\mathfrak{U}_Q \in \mathcal{C}$ because $\mathcal{C}$ is closed under taking quotient groups and extensions. If $\mathfrak{K}$ is normal in $\mathfrak{U}$, then the relation $\mathfrak{U}_Q \in \mathcal{C}$ is proved in exactly the same way, while if $\mathfrak{U} \in \mathcal{C}$, this relation follows from the fact that $\mathcal{C}$ is closed under taking quotient groups.

The inclusion $\mathfrak{V}_Q \in \mathcal{C}$ can be verified in a similar way; the only difference is that the relation $\mathfrak{F}_Q \in \mathcal{C}$ is ensured by the condition $\mathfrak{F} \in \mathcal{C}$. Hence it follows that Q is $\mathcal{C}$-admissible due to Theorem 1. □

Proposition 6.3 ([42, Propositions 5.2, 6.1, and 6.3, Theorem 2.2]). *If $\mathcal{C}$ is a root class of groups consisting only of periodic groups, then the following statements hold.*

(1) *The class of $\mathcal{C}$-bounded nilpotent groups is closed under taking subgroups and quotient groups.*

(2) *If the exponent of a $\mathcal{C}$-bounded nilpotent group is finite and is a $\mathfrak{P}(\mathcal{C})$-number, then this group belongs to $\mathcal{C}$.*

(3) *Every $\mathcal{C}$-bounded nilpotent group is $\mathcal{C}$-quasi-regular with respect to any of its subgroups.*

(4) *A subgroup of a $\mathcal{C}$-bounded nilpotent group X is $\mathcal{C}$-separable in this group if and only if it is $\mathfrak{P}(\mathcal{C})'$-isolated in X.*

Proposition 6.4. *If the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ satisfies $(*)$ and $\mathcal{C}$ is a root class of groups, then the following statements hold.*

(1) *If $\mathfrak{V} \in \mathcal{C}$, then for each subgroup $R \in \mathcal{C}^*(L)$, there is a subgroup $S \in \mathcal{C}^*(L)$ lying in R and such that $S\mathfrak{v} = S$ for any automorphism $\mathfrak{v} \in \mathfrak{V}$.*

(2) *If $\mathfrak{U} \in \mathcal{C}$, then for each subgroup $R \in \mathcal{C}^*(H)$, there is a subgroup $S \in \mathcal{C}^*(H)$ lying in R and such that $S\mathfrak{u} = S$ for any automorphism $\mathfrak{u} \in \mathfrak{U}$.*

(3) *Suppose that $\mathcal{C}$ consists only of periodic groups and N is a subgroup of B that is locally cyclic or $\mathcal{C}$-bounded nilpotent. Then, for each subgroup $R \in \mathcal{C}^*(N)$, there exists a subgroup $S \in \mathcal{C}^*(N)$ lying in R and such that $S\mathfrak{a} = S$ for any automorphism $\mathfrak{a} \in \mathrm{Aut}\, N$.*

Proof. (1) Let $S = \bigcap_{\mathfrak{v}\in\mathfrak{V}} R\mathfrak{v}$. By Remak's theorem (e.g., [19, Theorem 4.3.9]) the quotient group L/S can be embedded to the unrestricted direct product of the groups $L/R\mathfrak{v}$, $\mathfrak{v} \in \mathfrak{V}$, each of which is isomorphic to the $\mathcal{C}$-group L/R. Therefore, it follows from the condition $\mathfrak{V} \in \mathcal{C}$ and the root class definition that $L/S \in \mathcal{C}$. It is also clear that $S\mathfrak{v} = S$ for any $\mathfrak{v} \in \mathfrak{V}$.

(2) Let $S = \bigcap_{\mathfrak{u} \in \mathfrak{U}} R\mathfrak{u}$. As in the proof of statement (1), it follows from the inclusions $R \in \mathcal{C}^*(H)$ and $\mathfrak{U} \in \mathcal{C}$ that $S \in \mathcal{C}^*(H)$. The equality $S\mathfrak{u} = S$ is obvious for any $\mathfrak{u} \in \mathfrak{U}$.

(3) By Proposition 3.8, the exponent q of the $\mathcal{C}$-group N/R is finite. Consider the subgroup $S = \operatorname{sgp}\{x^q \mid x \in N\}$. Clearly, $S \leqslant R$ and $S\mathfrak{a} = S$ for any automorphism $\mathfrak{a} \in \operatorname{Aut} N$. It is also obvious that q is a $\mathfrak{P}(\mathcal{C})$-number and is equal to the exponent of N/S. Therefore, if N is a locally cyclic group, then N/S is a finite cyclic group, which belongs to $\mathcal{C}$ by Proposition 3.8. When N is a $\mathcal{C}$-bounded nilpotent group, N/S is also a $\mathcal{C}$-bounded nilpotent group and $N/S \in \mathcal{C}$ due to Proposition 6.3 (1) and (2). □

Proposition 6.5. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ satisfies $(*)$, $\mathcal{C}$ is a root class of groups closed under taking quotient groups, $H/L \in \mathcal{C}$, and $\mathfrak{F} \in \mathcal{C}$. Suppose also that at least one of the following statements holds:*

(a) *the group B is $\mathcal{C}$-quasi-regular with respect to HK, (†) holds, and $\mathfrak{V} \in \mathcal{C}$;*

(b) *the class $\mathcal{C}$ consists only of periodic groups, H and K are locally cyclic subgroups, and the group B is $\mathcal{C}$-quasi-regular with respect to HK;*

(c) *the class $\mathcal{C}$ consists only of periodic groups, (†) and (‡) hold, and B is a $\mathcal{C}$-bounded nilpotent group.*

If B/H and B/K are residually $\mathcal{C}$-groups, then $\mathbb{E}$ and B are residually $\mathcal{C}$-groups simultaneously.

Proof. First of all, let us note that, by Proposition 6.3, a $\mathcal{C}$-bounded nilpotent group is $\mathcal{C}$-quasi-regular with respect to any of its subgroups. It is also known that the automorphism group of a locally cyclic group is abelian (see, e.g., [14, §113, Exercise 4]). Therefore, (†), (‡), and the $\mathcal{C}$-quasi-regularity of B with respect to HK hold under any of statements (a)–(c). This fact and Proposition 6.2 imply that, to complete the proof, it suffices to fix a subgroup $M \in \mathcal{C}^*(B)$ and show that it contains a pre-$\mathcal{C}$-admissible subgroup.

Let $R = M \cap L$. Then $R \in \mathcal{C}^*(L)$ by Proposition 3.2. If H and K are locally cyclic groups, then L is also locally cyclic. By Proposition 6.3, if B is a $\mathcal{C}$-bounded nilpotent group, then L has the same property. Therefore, it follows from Proposition 6.4 (1) and (3) that there exists a subgroup $S \in \mathcal{C}^*(L)$ lying in R and satisfying the equality $S\mathfrak{v} = S$ for any automorphism $\mathfrak{v} \in \mathfrak{V}$. Since $\mathfrak{V} = \operatorname{Aut}_{\mathbb{E}}(L)$, the subgroup S turns out to be normal in $\mathbb{E}$ and, in particular, is φ-invariant. The quotient group HK/S is an extension of the $\mathcal{C}$-group L/S by a group isomorphic to HK/L. The latter, in turn, is an extension of the $\mathcal{C}$-group H/L by a group isomorphic to HK/H. The equalities $H\varphi = K$ and $L\varphi = L$ imply that $K/L \cong H/L$.

Since $HK/H \cong K/H \cap K = K/L$ and the class $\mathcal{C}$ is closed under taking extensions, it follows that $HK/S \in \mathcal{C}$. The $\mathcal{C}$-quasi-regularity of B with respect to HK and Proposition 3.7 guarantee the existence of a subgroup $N \in \mathcal{C}^*(B)$ such that $N \cap HK = S$. Let us show that the subgroup $Q = M \cap N$ is pre-$\mathcal{C}$-admissible and is therefore the desired one.

Indeed, it follows from the inclusions $M, N \in \mathcal{C}^*(B)$ and Proposition 3.3 that $Q \in \mathcal{C}^*(B)$. The relations

$$S \leqslant R = L \cap M = H \cap K \cap M$$

and $S\varphi = S$ imply that $Q \cap HK = M \cap (N \cap HK) = S$ and therefore

$$(Q \cap H)\varphi = (Q \cap H \cap HK)\varphi = S\varphi = S = Q \cap K \cap HK = Q \cap K.$$

If $x \in HQ/Q \cap KQ/Q$ and $x = hQ = kQ$ for some $h \in H$ and $k \in K$, then

$$h^{-1}k \in Q \cap HK = S \leqslant H \cap K.$$

Hence $h, k \in H \cap K = L$ and $x \in LQ/Q$. Thus $HQ/Q \cap KQ/Q \leqslant LQ/Q$ and, as the opposite inclusion is obvious, the subgroup Q is pre-$\mathcal{C}$-admissible. □

Proposition 6.6. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ satisfies $(*)$, $\mathcal{C}$ is a root class of groups closed under taking quotient groups, $\mathfrak{F} \in \mathcal{C}$, and there exists a homomorphism σ of B onto a group from $\mathcal{C}$ acting injectively on L. Suppose also that at least one of the following statements holds:*

(a) *the group B is $\mathcal{C}$-quasi-regular with respect to HK, (‡) holds, and $\mathfrak{U} \in \mathcal{C}$;*

(b) *the class $\mathcal{C}$ consists only of periodic groups, H and K are locally cyclic subgroups, and the group B is $\mathcal{C}$-quasi-regular with respect to HK;*

(c) *the class $\mathcal{C}$ consists only of periodic groups, (†) and (‡) hold, and B is a $\mathcal{C}$-bounded nilpotent group;*

(d) *the group B is $\mathcal{C}$-quasi-regular with respect to HK, (‡) holds, and the group $\mathfrak{U}$ coincides with one of its subgroups $\mathfrak{H}$ and $\mathfrak{K}$.*

If B/H and B/K are residually $\mathcal{C}$-groups, then $\mathbb{E}$ and B are residually $\mathcal{C}$-groups simultaneously.

Proof. Replacing, if necessary, H with K, φ with φ^{-1}, and t with t^{-1}, we can further assume that, if statement (d) holds, then $\mathfrak{U} = \mathfrak{H}$. As in the proof of Proposition 6.5, (†), (‡), and the $\mathcal{C}$-quasi-regularity of B with respect to HK hold under any of statements (a)–(d). Therefore, it suffices to show that each subgroup $M \in \mathcal{C}^*(B)$ contains some pre-$\mathcal{C}$-admissible subgroup.

Let $P = M \cap \ker\sigma$ and $R = (P \cap H) \cap (P \cap K)\varphi^{-1}$. It follows from the inclusions $M, \ker\sigma \in \mathcal{C}^*(B)$ and Propositions 3.2 and 3.3 that

$$P \in \mathcal{C}^*(B), \quad P \cap H \in \mathcal{C}^*(H), \quad P \cap K \in \mathcal{C}^*(K), \quad (P \cap K)\varphi^{-1} \in \mathcal{C}^*(H),$$

and $R \in \mathcal{C}^*(H)$. Let us show that there exists a subgroup $S \in \mathcal{C}^*(H)$ lying in R and satisfying the equality $S\mathfrak{u} = S$ for any automorphism $\mathfrak{u} \in \mathfrak{U}$.

If B is a $\mathcal{C}$-bounded nilpotent group, then H has the same property by Proposition 6.3. Therefore, when one of statements (a)–(c) holds, the desired subgroup S exists by Proposition 6.4 (2) and (3). Let statement (d) hold. Since the class $\mathcal{C}$ is closed under taking quotient groups, it follows from the relations

$$HK/RK \cong H/R(H \cap K) \cong (H/R)/(RL/R)$$

that $RK \in \mathcal{C}^*(HK)$. The $\mathcal{C}$-quasi-regularity of B with respect to HK guarantees the existence of a subgroup $T \in \mathcal{C}^*(B)$ such that $T \cap HK \leqslant RK$. Let

$$S = P \cap T \cap H.$$

Then $S \leqslant RK$ and $S \in \mathcal{C}^*(H)$ by Propositions 3.2 and 3.3. If $s \in S$, then $s = rk$ for suitable $r \in R$ and $k \in K$, and it follows from the inclusions $R, S \leqslant P \cap H$ that $k \in P \cap H \cap K \leqslant \ker\sigma \cap L = 1$. Hence $S \leqslant R$. Since P, T, and H are normal in B, the subgroup S has the same property. Therefore, $S\mathfrak{h} = S$ for each $\mathfrak{h} \in \mathfrak{H}$, and $S\mathfrak{u} = S$ for each $\mathfrak{u} \in \mathfrak{U}$ because $\mathfrak{U} = \mathfrak{H}$.

Thus it follows that a subgroup S with the required properties always exists. Since $\mathrm{Aut}_B(H) = \mathfrak{H} \leqslant \mathfrak{U}$ and $\varphi\,\mathrm{Aut}_B(K)\varphi^{-1} = \mathfrak{K} \leqslant \mathfrak{U}$, the equalities $S\mathfrak{h} = S$, $S\varphi\mathfrak{k}\varphi^{-1} = S$, and $(S\varphi)\mathfrak{k} = S\varphi$ hold for all $\mathfrak{h} \in \mathrm{Aut}_B(H)$, $\mathfrak{k} \in \mathrm{Aut}_B(K)$. Hence S and $S\varphi$ are normal in B. It follows from the relations

$$S \leqslant R \leqslant P \cap H, \quad S\varphi \leqslant R\varphi \leqslant P \cap K, \quad P \cap H \cap K \leqslant \ker\sigma \cap L = 1$$

that $S \cap K = 1 = S\varphi \cap H$. Therefore, $S \cdot S\varphi \cap H = S$ and $S \cdot S\varphi \cap K = S\varphi$.

The group $HK/S \cdot S\varphi$ is an extension of $SK/S \cdot S\varphi$ by a group isomorphic to HK/SK, and the class $\mathcal{C}$ is closed under taking extensions and quotient groups. Therefore, it follows from the relations

$$SK/S \cdot S\varphi \cong K/S\varphi(K \cap S) \cong (K/S\varphi)/(S\varphi(K \cap S)/S\varphi),$$
$$HK/SK \cong H/S(H \cap K) \cong (H/S)/(SL/S),$$

and $K/S\varphi \cong H/S \in \mathcal{C}$ that $HK/S \cdot S\varphi \in \mathcal{C}$. Since $S \cdot S\varphi$ is normal in B and the latter is $\mathcal{C}$-quasi-regular with respect to HK, Proposition 3.7 implies the existence of a subgroup $N \in \mathcal{C}^*(B)$ such that $N \cap HK = S \cdot S\varphi$. Let $Q = P \cap N$.

Then we have $Q \leqslant P \leqslant M$, and it follows from Proposition 3.3 and the inclusions $P, N \in \mathcal{C}^*(B)$ that $Q \in \mathcal{C}^*(B)$. Let us show that Q is pre-$\mathcal{C}$-admissible.

The relations $S \leqslant P \cap H$ and $S\varphi \leqslant P \cap K$ imply that

$$Q \cap HK = P \cap (N \cap HK) = S \cdot S\varphi.$$

Since $S \cdot S\varphi \cap H = S$ and $S \cdot S\varphi \cap K = S\varphi$, as proven above, the equalities

$$(Q \cap H)\varphi = (Q \cap H \cap HK)\varphi = S\varphi = Q \cap K \cap HK = Q \cap K$$

hold. If $x \in HQ/Q \cap KQ/Q$ and $x = hQ = kQ$ for some $h \in H$ and $k \in K$, then $h^{-1}k \in Q \cap HK = S \cdot S\varphi$ and $h^{-1}k = ss'$ for suitable $s \in S$ and $s' \in S\varphi$. Therefore, $hs = k(s')^{-1} \in H \cap K = L$, $h \in LS \leqslant LQ$, and $x \in LQ/Q$. Thus $HQ/Q \cap KQ/Q = LQ/Q$ and hence Q is pre-$\mathcal{C}$-admissible. □

Obviously, if $\mathcal{C}$ is a root class of groups, then the inclusion $\mathfrak{F} \in \mathcal{C}$ is guaranteed by the condition $\mathfrak{V} \in \mathcal{C}$. Therefore, Theorem 4 is a special case of Proposition 6.7 below, which, in turn, follows from Propositions 6.5 and 6.6. To anticipate possible questions from the reader, we note that Propositions 6.7 and 6.8 use statements (α) and (β) from Theorem 4.

Proposition 6.7. *Suppose that the group $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ satisfies $(*)$ and $\mathcal{C}$ is a root class of groups closed under taking quotient groups. Suppose also that B is $\mathcal{C}$-quasi-regular with respect to HK and at least one of the following statements holds:*

(a) $\mathfrak{V} \in \mathcal{C}$ *and* (α) *and* (†) *hold;*

(b) $\mathfrak{U}, \mathfrak{F} \in \mathcal{C}$ *and* (β) *and* (‡) *hold.*

If B/H and B/K are residually $\mathcal{C}$-groups, then $\mathbb{E}$ and B are residually $\mathcal{C}$-groups simultaneously.

Theorem 5 follows from Propositions 3.2, 6.5, 6.6 and Theorem 3.

Proof of Theorem 6. (1) Since B/H and B/K are residually $\mathcal{C}$-groups, the subgroups H and K are $\mathcal{C}$-separable in B by Proposition 3.1. It easily follows that L is also $\mathcal{C}$-separable in B and, again by Proposition 3.1, B/L is residually a $\mathcal{C}$-group. Hence, if the subgroup H/L is finite, then it belongs to $\mathcal{C}$ due to Proposition 3.3. Conversely, if the locally cyclic group H/L belongs to $\mathcal{C}$, then it has a finite exponent by Proposition 3.8 and is therefore finite.

(2) If (β) holds, then the locally cyclic group L can be embedded in a $\mathcal{C}$-group. As above, this implies its finiteness. The opposite statement follows from Proposition 3.3.

(3) Necessity is ensured by Proposition 3.2 and Theorem 3. To prove sufficiency, let us show that B is $\mathcal{C}$-quasi-regular with respect to HK. Then the residual $\mathcal{C}$-ness of $\mathbb{E}$ will follow from statement (1) of this theorem and Proposition 6.5.

As noted in the proof of the latter, the quotient group HK/L is an extension of H/L by a group isomorphic to H/L and is therefore finite. By the arguments used to verify statement (1), the residual $\mathcal{C}$-ness of B/H and B/K implies the residual $\mathcal{C}$-ness of B/L. By Proposition 3.3, it follows that there exists a subgroup $S/L \in \mathcal{C}^*(B/L)$ satisfying the condition $S/L \cap HK/L = 1$. Clearly, $S \in \mathcal{C}^*(B)$ and $S \cap HK \leqslant L$.

Now, if $M \in \mathcal{C}^*(HK)$ and $Q = M \cap L$, then $Q \in \mathcal{C}^*(L)$ by Proposition 3.2. Since B is $\mathcal{C}$-quasi-regular with respect to L, there exists a subgroup $R \in \mathcal{C}^*(B)$ such that $R \cap L \leqslant Q$. Let $N = R \cap S$. Then $N \in \mathcal{C}^*(B)$ by Proposition 3.3 and

$$N \cap HK = R \cap S \cap HK \leqslant R \cap L \leqslant Q \leqslant M.$$

Thus the group B is $\mathcal{C}$-quasi-regular with respect to HK, as required.

(4) Sufficiency follows from Proposition 3.3, which ensures that (β) holds, and Proposition 6.6. Let us prove necessity.

By Proposition 3.3, since $\mathbb{E}$ is residually a $\mathcal{C}$-group, it has a homomorphism onto a group from $\mathcal{C}$ acting injectively on the finite subgroup L. This fact and Proposition 3.4 imply that $\mathfrak{V} = \mathrm{Aut}_{\mathbb{E}}(L) \in \mathcal{C}$ and $\mathfrak{F} \in \mathcal{C}$. As above, the residual $\mathcal{C}$-ness of the groups B, B/H, and B/K is ensured by Proposition 3.2 and Theorem 3. □

Corollary 2 can be deduced either from Theorems 3–6 and Propositions 3.1, 3.2, and 6.3, or from Proposition 6.8 below. The second method uses the fact that the automorphism group of a locally cyclic group is abelian, which is already mentioned in the proof of Proposition 6.5.

Proposition 6.8. *Suppose that the group* $\mathbb{E} = \langle B, t;\ t^{-1}Ht = K,\ \varphi\rangle$ *satisfies* $(*)$ *and* $\mathcal{C}$ *is a root class of groups consisting only of periodic groups and closed under taking quotient groups. Suppose also that* $\mathfrak{F} \in \mathcal{C}$ *and* B *is a* $\mathcal{C}$*-bounded nilpotent group. Finally, let* (†), (‡), *and at least one of statements* (α) *and* (β) *hold. Then* $\mathbb{E}$ *is residually a* $\mathcal{C}$*-group if and only if the subgroups* $\{1\}$, H, *and* K *are* $\mathfrak{P}(\mathcal{C})'$*-isolated in* B.

Proof. First of all, let us note that, by Proposition 6.3, the subgroups $\{1\}$, H, and K are $\mathfrak{P}(\mathcal{C})'$-isolated in B if and only if they are $\mathcal{C}$-separable in this group. By Proposition 3.1, the latter property is equivalent to the residual $\mathcal{C}$-ness of the groups B, B/H, and B/K. Therefore, necessity follows from Proposition 3.2 and Theorem 3, while sufficiency can be deduced from Propositions 6.5 and 6.6. □

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Received May 5, 2025; revised August 24, 2025

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