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ABSTRACT

Suppose that G is a group, H and K are proper isomorphic central subgroups of G, and $\mathfrak{G}$ is an HNN-extension of G with the associated subgroups H and K. We prove necessary and sufficient conditions for $\mathfrak{G}$ to be residually a $\mathcal{C}$-group, where $\mathcal{C}$ is a class of groups closed under taking subgroups, extensions, homomorphic images, and Cartesian products of the form $\prod_{y \in Y} X_y$, where $X, Y \in \mathcal{C}$ and X_y is an isomorphic copy of X for each $y \in Y$.

1. Introduction: Statement of results

Let $\mathcal{C}$ be a class of groups. Recall that a group X is said to be *residually a $\mathcal{C}$-group* if, for any nontrivial element $x \in X$, there is a homomorphism σ of X onto a group from $\mathcal{C}$ such that $x\sigma \neq 1$.

The main question in the study of the $\mathcal{C}$-residuality of HNN-extensions is whether an HNN-extension of a residually $\mathcal{C}$-group is again residually a $\mathcal{C}$-group. All known answers to this question are obtained under various restrictions imposed on the base group, the associated subgroups, and/or the isomorphism between them (the terminology for HNN-extensions used here and below follows [9]). One such restriction is the centrality of the associated subgroups in the base group; the $\mathcal{C}$-residuality of HNN-extensions of this type (including the case of an abelian base group) is studied in [2, 6, 8, 11–15, 20, 21, 26, 27].

In [2, 14, 15], S. Andreadakis, E. Raptis, and D. Varsos give criteria for the residual finiteness and the residual nilpotence of an HNN-extension of a finitely generated abelian group and prove that such an extension is residually solvable. D. I. Moldavanskii [11, 12] significantly strengthens some of these results by generalizing them to the case of HNN-extension with central associated subgroups. He proposes an original approach to the study of the $\mathcal{C}$-residuality of such HNN-extensions, which he call *the method of descent and ascent of compatible subgroups*. In [11, 12], this method is applied under the assumption that $\mathcal{C}$ is the class of all finite groups or all finite p-groups, where p is a prime number. The aim of this article is to generalize the results obtained in [11, 12] to the case when $\mathcal{C}$ is an arbitrary root class of groups closed under taking quotient groups.

The notion of a root class was introduced by K. Gruenberg [7], and its equivalent definitions are given in [17]. In accordance with one of them, a class of groups $\mathcal{C}$ containing at least one non-trivial group is called *root* if it is closed under taking subgroups, extensions, and Cartesian products of the form $\prod_{y \in Y} X_y$, where $X, Y \in \mathcal{C}$ and X_y is an isomorphic copy of X for each $y \in Y$. The examples of root classes are the above-mentioned classes of all finite groups and all finite p-groups, as well as the classes of periodic π-groups of finite exponent (where π is a non-empty set of primes), all solvable groups, and all torsion-free groups. We note also that the intersection of any number of root classes is again a root class [17].

The notion of a root class turns out to be very useful in studying the residual properties of free constructions of groups allowing one to prove many statements at once using the same argument. The papers [3, 7] are the starting point for these studies, some of the latest results obtained in this area can be found in [21, 22, 25]. The root-class residuality of HNN-extensions is studied in [6, 20, 21, 23–25]. These papers deal mainly with the cases when the associated subgroups coincide or intersect trivially.

Let us call a sequence (X, Y, Z, ψ) an *HNN-tuple* if X is a group, Y and Z are isomorphic subgroups of X, and $\psi : Y \to Z$ is an isomorphism. If (X, Y, Z, ψ) is an HNN-tuple, then by $\mathrm{HNN}(X, Y, Z, \psi)$ we denote the HNN-extension $\langle X, t; t^{-1} Y t = Z, \psi \rangle$. Recall that $\mathrm{HNN}(X, Y, Z, \psi)$ is the group whose generators are the generators of X and the symbol t, and whose defining relations are the relations of X and all possible relations of the form $t^{-1} y t = y\psi$, where y and $y\psi$ are words in the generators of X defining an element $y \in Y$ and its image under ψ.

Throughout this section, we assume that (G, H, K, φ) is an HNN-tuple, H and K <u>lie in the center</u> of G, and $\mathfrak{G} = \mathrm{HNN}(G, H, K, \varphi)$. We put $K_0 = G$, $H_1 = H$, $K_1 = K$, and (if H_i and K_i are already defined) $H_{i+1} = H_i \cap K_i$, $K_{i+1} = H_{i+1}\varphi$. To simplify the notation, the restriction of φ to H_i ($i \geq 1$) or some other subgroup is denoted below by the same symbol φ.

Since, for any $i \geq 0$, the sequence $(K_i, H_{i+1}, K_{i+1}, \varphi)$ is an HNN-tuple, the group $\mathfrak{K}_i = \mathrm{HNN}(K_i, H_{i+1}, K_{i+1}, \varphi)$ is defined. Obviously, if $H_n = K_n$ for some $n \geq 1$, then $H_{n+1} = K_n = K_{n+1}$, and therefore $\mathfrak{K}_n$ is a split extension of K_n by the infinite cyclic group $\langle t \rangle$. It is also easy to see that the restriction of φ to H_n turns out to be an automorphism of this subgroup, and $\mathfrak{K}_n$ is isomorphic to the subgroup $E = \mathrm{sgp}\{H_n, t\}$ of G, which is a split extension of H_n by $\langle t \rangle$.

The method of descent and ascent of compatible subgroups essentially consists in proving that, under certain conditions, for each $i \geq 0$, the $\mathcal{C}$-residuality of G is equivalent to the $\mathcal{C}$-residuality of $\mathfrak{K}_i$. If $H_n = K_n$ for some $n \geq 1$, this allows us to reduce the question of the $\mathcal{C}$-residuality of G to the much simpler problem of finding conditions for the split extension $\mathfrak{K}_n$ to be residually a $\mathcal{C}$-group. Two criteria for the root-class residuality of split extensions are given at the end of Section 5.

We note that the equality $H_n = K_n$ may not hold for any n. To guarantee its fulfillment, the theorems given below impose a weaker condition on G: $H_n = H_{n+1}$ for some n. Corollaries 1–4 describe a number of situations in which the last relation certainly takes place.

Throughout the article, if $\mathcal{C}$ is a class of groups and X is a group, then $\mathcal{C}^*(X)$ denotes the family of normal subgroups of X such that $Y \in \mathcal{C}^*(X)$ whenever $X/Y \in \mathcal{C}$. If $\mathcal{C}$ consists only of periodic groups, then we denote by $\pi(\mathcal{C})$ the set of all the prime numbers that divide the orders of the elements of all possible $\mathcal{C}$-groups. Recall that if π is a set of primes, then a π-*number* is an integer all of whose prime divisors belong to π, and a π-*group* is a periodic group in which the order of each element is a π-number.

Theorem 1. *Suppose that $\mathcal{C}$ is a root class of groups closed under taking quotient groups, G is residually a $\mathcal{C}$-group, and there exists a subgroup $Q \in \mathcal{C}^*(G)$ satisfying at least one of the following conditions:*

(α) $H \cap Q = 1 = K \cap Q$,

(β) $Q \leq H \cap K$ and $Q\varphi = Q$.

I. If $\mathcal{C}$ contains non-periodic groups, then $\mathfrak{G}$ is residually a $\mathcal{C}$-group.
II. If $\mathcal{C}$ consists only of periodic groups, $H \neq G \neq K$, and $H_n = H_{n+1}$ for some $n \geq 1$, then $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if
1) $H_n = K_n$;
2)   the subgroup $E = \text{sgp}\{H_n, t\}$ is residually a $\mathcal{C}$-group.

Corollary 1. *If $\mathcal{C}$ is a root class of groups closed under taking quotient groups, G is residually a $\mathcal{C}$-group, H and K are finite, then $H_n = K_n$ for some $n \geq 1$ and the following statements hold.*

I. *If $\mathcal{C}$ contains non-periodic groups, then $\mathfrak{G}$ is residually a $\mathcal{C}$-group.*
II. *If $\mathcal{C}$ consists only of periodic groups, then $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if the order of the automorphism φ of H_n is a $\pi(\mathcal{C})$-number.*

Corollary 2. *Suppose that $\mathcal{C}$ is a root class of groups closed under taking quotient groups, H and K have finite index in G, and $H \neq G \neq K$.*

I. *If $\mathcal{C}$ contains non-periodic groups, then $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if G is residually a $\mathcal{C}$-group and there exists a subgroup $Q \in \mathcal{C}^*(G)$ satisfying the conditions $Q \leq H \cap K$ and $Q\varphi = Q$.*
II. *If $\mathcal{C}$ consists only of <u>finite</u> groups, then $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if*
1)   *$G/H \in \mathcal{C}$ and $G/K \in \mathcal{C}$;*
2)   *$H_n = K_n$ for some $n \geq 1$;*
3)   *the subgroup $E = \text{sgp}\{H_n, t\}$ is residually a $\mathcal{C}$-group.*

Let π be a set of primes. Following [16], we call an abelian group A *π-bounded* if, for any quotient group B of A and for any $p \in \pi$, the p-primary component of B is finite. A nilpotent (solvable) group is said to be *π-bounded* if it has a finite central (respectively, subnormal) series with abelian π-bounded factors. It is easy to see that, for any set π of prime numbers, a finitely generated nilpotent group is π-bounded nilpotent and a polycyclic group is π-bounded solvable. It is also known (see Proposition 6.7 below) that if a π-bounded solvable group is abelian, then it belongs to the class of π-bounded abelian groups. Therefore, we can say that an abelian group is π-bounded without specifying the class of π-bounded groups (abelian, nilpotent, or solvable) that we mean.

Throughout the article, we denote by π' the set of all primes that do not belong to the set π. Recall that a subgroup Y of a group X is said to be *π'-isolated* in this group if, for any $x \in X$ and for any $q \in \pi'$, it follows from the inclusion $x^q \in Y$ that $x \in Y$. Obviously, if π contains all primes, then any subgroup turns out to be π'-isolated.

It is easy to see that the intersection of any number of π'-isolated subgroups is again a π'-isolated subgroup. Therefore, for any subgroup $Y \leq X$, there exists the smallest π'-isolated subgroup containing it. We call this subgroup the *π'-isolator* of Y in X and denote it by $\mathcal{I}_{\pi'}(X, Y)$.

Let $\mathcal{C}$ be a class of groups. Recall that a subgroup Y is said to be *$\mathcal{C}$-separable* in a group X if, for any $x \in X \setminus Y$, there exists a homomorphism σ of X onto a group from $\mathcal{C}$ such that $x\sigma \notin Y\sigma$ [10]. Obviously, the $\mathcal{C}$-residuality of X is equivalent to the $\mathcal{C}$-separability of its trivial subgroup. It is also known (see Proposition 6.3 below) that if $\mathcal{C}$ consists only of periodic groups, then any $\mathcal{C}$-separable subgroup of X is $\pi(\mathcal{C})'$-isolated in this group. Thus, for such a class of groups $\mathcal{C}$, the main problem in the study of $\mathcal{C}$-separability is the search for conditions under which a $\pi(\mathcal{C})'$-isolated subgroup turns out to be $\mathcal{C}$-separable.

Theorem 2. *Suppose that $\mathcal{C}$ is a root class of groups consisting only of periodic groups and closed under taking quotient groups, G is residually a $\mathcal{C}$-group, H and K are $\pi(\mathcal{C})$-bounded, and $H \neq G \neq K$. Suppose also that $H_n = H_{n+1}$ for some $n \geq 1$ and, for any $i \in \{0, 1, \ldots, n-1\}$ and*

$N \in \mathcal{C}^*(H_{i+1}K_{i+1})$, the subgroup $\mathcal{I}_{\pi(\mathcal{C})'}(K_i, N)$ is $\mathcal{C}$-separable in K_i. Then $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if

1) $H_n = K_n$;
2) the subgroup $E = \mathrm{sgp}\{H_n, t\}$ is residually a $\mathcal{C}$-group;
3) H and K are $\pi(\mathcal{C})'$-isolated in G.

Corollary 3. *Suppose that $\mathcal{C}$ is a root class of groups consisting only of periodic groups and closed under taking quotient groups, G is $\pi(\mathcal{C})$-bounded nilpotent, $H \neq G \neq K$. Suppose also that there exists a number $m \geq 0$ such that H_{m+1} and K_{m+1} are finitely generated or π'-isolated in K_m for some finite subset π of $\pi(\mathcal{C})$. Then $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if*

1) $H_n = K_n$ for some $n \geq 1$;
2) the subgroup $E = \mathrm{sgp}\{H_n, t\}$ is residually a $\mathcal{C}$-group;
3) G has no $\pi(\mathcal{C})'$-torsion, H and K are $\pi(\mathcal{C})'$-isolated in G.

Corollary 4. *Suppose that $\mathcal{C}$ is a root class of groups consisting only of periodic groups and closed under taking quotient groups, $\pi(\mathcal{C})$ contains all primes, G is $\pi(\mathcal{C})$-bounded solvable, and $H \neq G \neq K$. Suppose also that there exists a number $m \geq 0$ such that H_{m+1} and K_{m+1} are finitely generated or π'-isolated in K_m for some finite set of primes π. Then $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if $H_n = K_n$ for some $n \geq 1$.*

Theorems 1 and 2 formulated above are in fact corollaries of Theorems 3 and 4, which are given below and use the notion of regularity of a group with respect to a subgroup.

Suppose that $\mathcal{C}$ is a class of groups, X is a group, and Y is a subgroup of X. We say that X is $\mathcal{C}$-regular with respect to Y if, for any subgroup $M \in \mathcal{C}^*(Y)$, there exists a subgroup $N \in \mathcal{C}^*(X)$ such that $M = N \cap Y$. The notion of regularity generalizes the classical notion of a *potent element* [1]: if $\mathcal{F}$ is the class of all finite groups, then an element $x \in X$ is potent if and only if X is $\mathcal{F}$-regular with respect to the cyclic subgroup $\langle x \rangle$.

Theorem 3. *Suppose that $\mathcal{C}$ is a root class of groups consisting only of periodic groups and closed under taking quotient groups, G is residually a $\mathcal{C}$-group, $H \neq G \neq K$, and $H_n = H_{n+1}$ for some $n \geq 1$. Suppose also that, for any $i \in \{0, 1, \ldots, n-1\}$, K_i is $\mathcal{C}$-regular with respect to $H_{i+1}K_{i+1}$ and $\mathcal{I}_{\pi(\mathcal{C})'}(K_i, H_{i+1}K_{i+1})$ is $\mathcal{C}$-separable in K_i. Then $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if*

1) $H_n = K_n$;
2) the subgroup $E = \mathrm{sgp}\{H_n, t\}$ is residually a $\mathcal{C}$-group;
3) H and K are $\pi(\mathcal{C})'$-isolated in G.

Theorem 4. *Suppose that $\mathcal{C}$ is a root class of groups containing non-periodic groups and closed under taking quotient groups, G is residually a $\mathcal{C}$-group, and, for some $n \geq 0$, there exists a subgroup $Q \in \mathcal{C}^*(K_n)$ satisfying at least one of the following conditions:*

(α) $H_{n+1} \cap Q = 1 = K_{n+1} \cap Q$,
(β) $Q \leq H_{n+1} \cap K_{n+1}$ and $Q\varphi = Q$.

Suppose also that, for any $i \in \{0, 1, \ldots, n-1\}$, K_i is $\mathcal{C}$-regular with respect to $H_{i+1}K_{i+1}$ and $H_{i+1}K_{i+1}$ is $\mathcal{C}$-separable in K_i. Then $\mathfrak{G}$ is residually a $\mathcal{C}$-group.

The above theorems and corollaries generalize the main results from [2, 6, 20] (as for [2], in the part concerning non-ascending HNN-extensions), as well as Theorem 1.1 from [14] and Theorem 3 from [21]. However, the results mentioned are easier to formulate due to the

additional restrictions imposed on the base group and the associated subgroups. The proofs of Theorems 1–4 and Corollaries 1–4 are given in Sections 2–6.

2. Generalized direct products of groups

Let $\Gamma = (V, E)$ be a non-empty connected undirected graph with a vertex set V and an edge set E. It is assumed that Γ is not necessarily finite, but has no multiple edges and loops. Let us assign to each vertex $v \in V$ a group G_v and to each edge $e = \{v, w\} \in E$ a group H_e and injective homomorphisms $\varphi_{e,v} : H_e \to G_v$, $\varphi_{e,w} : H_e \to G_w$. As a result, we get a *graph of groups* $\mathcal{G}(\Gamma)$. We call the groups G_v ($v \in V$), H_e ($e \in E$), the subgroups $H_e\varphi_{e,v}$, and the homomorphisms $\varphi_{e,v}$ ($e \in E$, $v \in e$) the *vertex* and *edge groups*, the *edge subgroups*, and the *edge homomorphisms*, respectively.

Consider the group

$$\mathrm{GDP}(\mathcal{G}(\Gamma)) = \langle G_\lambda \ (\lambda \in V); \ [G_\mu, G_\eta] = 1 \ (\mu, \eta \in V, \ \mu \neq \eta),$$
$$H_e\varphi_{e,v} = H_e\varphi_{e,w} \ (e = \{v, w\} \in E)\rangle,$$

whose generators are the generators of G_λ ($\lambda \in V$), and whose defining relations are the relations of G_λ ($\lambda \in V$) and all possible relations of the form

$$[g_\mu, g_\eta] = 1 \ (\mu, \eta \in V, \ \mu \neq \eta), \quad h_e\varphi_{e,v} = h_e\varphi_{e,w} \ (e = \{v, w\} \in E, \ h_e \in H_e),$$

where g_μ and g_η are arbitrary words in the generators of G_μ and G_η, respectively, $h_e\varphi_{e,v}$ and $h_e\varphi_{e,w}$ are some words in the generators of G_v and G_w that define (in these groups) the images of h_e under $\varphi_{e,v}$ and $\varphi_{e,w}$. Following [21], we call the group $\mathrm{GDP}(\mathcal{G}(\Gamma))$ *the generalized direct product associated with the graph of groups* $\mathcal{G}(\Gamma)$ if

(i)   for any $v \in V$, the identity mapping of the generators of G_v to $\mathrm{GDP}(\mathcal{G}(\Gamma))$ can be extended to an injective homomorphism, and therefore all the groups G_v ($v \in V$) can be considered subgroups of $\mathrm{GDP}(\mathcal{G}(\Gamma))$;

(ii)   for any $e = \{v, w\} \in E$, the equalities $H_e\varphi_{e,v} = G_v \cap G_w = H_e\varphi_{e,w}$ hold in $\mathrm{GDP}(\mathcal{G}(\Gamma))$.

We say that the generalized direct product associated with $\mathcal{G}(\Gamma)$ *exists* if $\mathrm{GDP}(\mathcal{G}(\Gamma))$ satisfies (i) and (ii). Some conditions for the existence of generalized direct products are found in [21]. In particular, the following proposition is proved.

Proposition 2.1. [21, Theorem 1] *If Γ is a tree and, for any $e \in E$ and $v \in e$, the subgroup $H_e\varphi_{e,v}$ lies in the center of G_v, then the generalized direct product associated with $\mathcal{G}(\Gamma)$ exists.*

The main aim of this section is to find conditions for the existence of certain generalized direct products associated with simple cycles.

Let (X, Y, Z, ψ) be an HNN-tuple, and let Γ be the simple cycle of length $n \geq 3$ with the vertex set $V = \mathbb{Z}_n$ and the edge set $E = \{\{i - 1, i\} \mid i \in \mathbb{Z}_n\}$. Suppose also that, for any $i \in \mathbb{Z}_n$, X_i is an isomorphic copy of X and $\sigma_i : X \to X_i$ is an isomorphism. We associate a vertex $i \in \mathbb{Z}_n$ with the group X_i, an edge $e = \{i - 1, i\} \in E$ with the group Y and the homomorphisms $\varphi_{e,i-1} = \sigma_{i-1}|_Y$, $\varphi_{e,i} = \psi\sigma_i|_Z$ (here and below, all indices are considered modulo n), and denote the resulting graph of groups by $\mathcal{G}_n(X, Y, Z, \psi)$.

Let us call a number $n \geq 3$ *admissible for an HNN-tuple (X, Y, Z, ψ) with a reserve r* ($0 \leq r \leq n - 2$) if the following conditions hold:

$(i)'$ for any $i \in \mathbb{Z}_n$, the identity mapping of the generators of X_i to $\mathrm{GDP}(\mathcal{G}_n(X, Y, Z, \psi))$ can be extended to an injective homomorphism;

$(ii)'$ for any $s \in \{0, \ 1, \ ..., \ r\}$, $q \in \mathbb{Z}_n$, $x_q \in X_q$, $x_{q+1} \in X_{q+1}$, $\ ..., \ x_{q+s+1} \in X_{q+s+1}$, it follows from the equality $x_q x_{q+1}...x_{q+s+1} = 1$ that $x_q \in Y\sigma_q$ and $x_{q+s+1} \in Z\sigma_{q+s+1}$.

We note that if a number n is admissible for a tuple (X, Y, Z, ψ) with a reserve $r \in \{0, \ldots, n-2\}$, then the generalized direct product associated with $\mathcal{G}_n(X, Y, Z, \psi)$ exists. Indeed, for any $e = \{i-1, i\} \in E$, if $x \in X_{i-1} \cap X_i$, $x_{i-1} = x$, and $x_i = x^{-1}$, then $x_{i-1}x_i = 1$ and, by the admissibility,

$$x_{i-1} \in Y\sigma_{i-1} = Y\varphi_{e,i-1}, \quad x_i \in Z\sigma_i = Y\varphi_{e,i}.$$

Thus, $X_{i-1} \cap X_i \leq Y\varphi_{e,i-1} \cap Y\varphi_{e,i}$ and therefore $Y\varphi_{e,i-1} = X_{i-1} \cap X_i = Y\varphi_{e,i}$.

Proposition 2.2. *Suppose that (G, H, K, φ) is an HNN-tuple, H and K lie in the center of G, $L = H \cap K$, and $M = L\varphi$. Then a number $n \geq 3$ is admissible for (G, H, K, φ) with a reserve $r \leq n - 3$ if and only if it is admissible for (K, L, M, φ) with the reserve $r + 1$.*

Proof. For each $i \in \mathbb{Z}_n$, let G_i denote an isomorphic copy of G, and let $\sigma_i : G \to G_i$ be an isomorphism. We put

$$H_i = H\sigma_i, \quad K_i = K\sigma_i, \quad L_i = L\sigma_i, \quad M_i = M\sigma_i, \quad \varphi_i = (\sigma_{i-1}|_{H_{i-1}})^{-1}\varphi\sigma_i|_K$$

(in Section 1, the symbols H_i and K_i correspond to other subgroups; the notation just introduced is valid only throughout this proof). Then φ_i is an isomorphism of H_{i-1} onto K_i, and its restriction to L_{i-1} is an isomorphism of L_{i-1} and M_i. If $\mathrm{GDP}(\mathcal{G}_n(G, H, K, \varphi))$ satisfies $(i)'$, then it can be considered containing H_i and K_i ($i \in \mathbb{Z}_n$), and it follows from the relations $h\sigma_{i-1} = h\varphi\sigma_i$ ($i \in \mathbb{Z}_n$, $h \in H$) that the equalities $h\varphi_i = h$ ($i \in \mathbb{Z}_n$, $h \in H_{i-1}$) hold in $\mathrm{GDP}(\mathcal{G}_n(G, H, K, \varphi))$. Similarly, if $\mathrm{GDP}(\mathcal{G}_n(K, L, M, \varphi))$ satisfies $(i)'$, then the equalities $h\varphi_i = h$ ($i \in \mathbb{Z}_n$, $h \in L_{i-1}$) hold in it.

Sufficiency. We fix a number $i \in \mathbb{Z}_n$ and define a mapping θ_i of the subgroup $K_iH_i \leq G_i$ to the subgroup $K_iK_{i+1} \leq \mathrm{GDP}(\mathcal{G}_n(K, L, M, \varphi))$ as follows: if $h \in H_i$ and $k \in K_i$, then $(kh)\theta_i = k(h\varphi_{i+1})$. Let us show that this mapping is well defined and is a subgroup isomorphism extending the identity mapping of K_i.

If $h_1, h_2 \in H_i$, $k_1, k_2 \in K_i$, and $k_1h_1 = k_2h_2$, then $k_2^{-1}k_1 = h_2h_1^{-1} \in H_i \cap K_i = L_i$ and therefore the equality $(h_2h_1^{-1})\varphi_{i+1} = h_2h_1^{-1}$ holds in $\mathrm{GDP}(\mathcal{G}_n(K, L, M, \varphi))$. Hence,

$$(k_1h_1)\theta_i = k_1(h_1\varphi_{i+1}) = k_2(k_2^{-1}k_1)((h_1h_2^{-1})h_2)\varphi_{i+1}$$
$$= k_2(k_2^{-1}k_1h_1h_2^{-1})(h_2\varphi_{i+1}) = k_2(h_2\varphi_{i+1}) = (k_2h_2)\theta_i.$$

Suppose that $h \in H_i$, $k \in K_i$, and $1 = (kh)\theta_i = k(h\varphi_{i+1})$. Since $k \in K_i$, $h\varphi_{i+1} \in K_{i+1}$, and n is admissible for (K, L, M, φ), then $k \in L_i$ and $h\varphi_{i+1} \in M_{i+1}$. Hence, $h \in L_i$ and $1 = k(h\varphi_{i+1}) = kh$ because the equality $h\varphi_{i+1} = h$ holds in $\mathrm{GDP}(\mathcal{G}_n(K, L, M, \varphi))$.

Thus, θ_i is well defined and injective, it is clear that it is homomorphic and surjective. Let Δ be the star graph with the vertex set $\{v_i \, (i \in \mathbb{Z}_n), w\}$. We associate its central vertex w with the group $\mathrm{GDP}(\mathcal{G}_n(K, L, M, \varphi))$, the leaf v_i ($i \in \mathbb{Z}_n$) with the group G_i, the edge $\{w, v_i\}$ ($i \in \mathbb{Z}_n$) with the subgroup K_iH_i of G_i and the homomorphisms, one of which is the identity mapping and the other coincides with θ_i. Let us denote the resulting graph by $\mathcal{G}(\Delta)$.

It follows from the definitions of $\mathcal{G}(\Delta)$, θ_i ($i \in \mathbb{Z}_n$), and φ_i ($i \in \mathbb{Z}_n$) that, for all $i \in \mathbb{Z}_n$ and $h \in H$, the equalities $h\sigma_{i-1} = (h\sigma_{i-1})\theta_{i-1} = (h\sigma_{i-1})\varphi_i = h\varphi\sigma_i$ hold in $\mathrm{GDP}(\mathcal{G}(\Delta))$. Therefore, the identity mapping of the generators of $\mathrm{GDP}(\mathcal{G}_n(G, H, K, \varphi))$ to $\mathrm{GDP}(\mathcal{G}(\Delta))$ defines a homomorphism, which we denote by λ.

Let, for any $i \in \mathbb{Z}_n$, $\alpha_i : G_i \to \mathrm{GDP}(\mathcal{G}(\Delta))$ and $\beta_i : G_i \to \mathrm{GDP}(\mathcal{G}_n(G, H, K, \varphi))$ be the homomorphisms defined by the identity mappings of the generators of G_i. Since the diagram

$$G_i \xrightarrow{\beta_i} \text{GDP}(\mathcal{G}_n(G, H, K, \varphi))$$
$$\| \qquad\qquad \downarrow \lambda$$
$$G_i \xrightarrow{\alpha_i} \quad \text{GDP}(\mathcal{G}(\Delta))$$

is commutative and α_i is injective by Proposition 2.1, then β_i is also injective and therefore $\text{GDP}(\mathcal{G}_n(G, H, K, \varphi))$ satisfies $(i)'$. Let us verify that this group also satisfies $(ii)'$.

Suppose that numbers $q \in \mathbb{Z}_n$, $s \in \{0, 1, ..., r\}$ and elements

$$g_q \in G_q, g_{q+1} \in G_{q+1}, ..., g_{q+s+1} \in G_{q+s+1}$$

are such that $g_q g_{q+1} \cdots g_{q+s+1} = 1$ in $\text{GDP}(\mathcal{G}_n(G, H, K, \varphi))$. Then this equality also holds in $\text{GDP}(\mathcal{G}(\Delta))$ by the definition of λ.

Let $j \in \{0, 1, ..., s+1\}$, and let $\mathcal{G}_{q+j}$ be the graph of groups obtained from $\mathcal{G}(\Delta)$ by deleting the vertex v_{q+j} and the edge $\{v_{q+j}, w\}$. Then $\text{GDP}(\mathcal{G}(\Delta))$ is the generalized direct product $\mathfrak{P}_j$ of the groups $\text{GDP}(\mathcal{G}_{q+j})$ and G_{q+j} with the amalgamated subgroups $K_{q+j} K_{q+j+1}$ and $K_{q+j} H_{q+j}$. It follows from the relation $r \leq n - 3$ that $s + 2 < n$ and therefore

$$g_{q+j-1}^{-1} g_{q+j-2}^{-1} \cdots g_q^{-1} g_{q+s+1}^{-1} \cdots g_{q+j+2}^{-1} g_{q+j+1}^{-1} \in \text{GDP}(\mathcal{G}_{q+j}),$$
$$g_{q+j} = g_{q+j-1}^{-1} g_{q+j-2}^{-1} \cdots g_q^{-1} g_{q+s+1}^{-1} \cdots g_{q+j+2}^{-1} g_{q+j+1}^{-1} \in \text{GDP}(\mathcal{G}_{q+j}) \cap G_{q+j}.$$

Since $\mathfrak{P}_j$ satisfies (ii) by Proposition 2.1, we have $g_{q+j} \in K_{q+j} H_{q+j}$.

For each $j \in \{0, 1, ..., s+1\}$, let us write the element g_{q+j} in the form $g_{q+j} = k_{q+j} h_{q+j}$, where $h_{q+j} \in H_{q+j}$, $k_{q+j} \in K_{q+j}$. Since $\text{GDP}(\mathcal{G}_n(G, H, K, \varphi))$ satisfies $(i)'$, then the relations $h_{q+j} = h_{q+j} \varphi_{q+j+1} \in K_{q+j+1}$ $(0 \leq j \leq s+1)$ hold in it and therefore

$$k_q \in K_q, h_q k_{q+1} \in K_{q+1}, ..., h_{q+s} k_{q+s+1} \in K_{q+s+1}, h_{q+s+1} \in K_{q+s+2}.$$

As noted above, the equality $g_q g_{q+1} \cdots g_{q+s+1} = 1$ holds in $\text{GDP}(\mathcal{G}(\Delta))$. We rewrite it in the form

$$k_q (h_q k_{q+1}) \cdots (h_{q+s} k_{q+s+1}) h_{q+s+1} = 1.$$

By Proposition 2.1, $\text{GDP}(\mathcal{G}_n(K, L, M, \varphi))$ is embedded into $\text{GDP}(\mathcal{G}(\Delta))$ by the identity mapping of the generators. Therefore, the last relation holds in $\text{GDP}(\mathcal{G}_n(K, L, M, \varphi))$, and since n is admissible for (K, L, M, φ) with the reserve $r + 1$, then $h_{q+s+1} \in M_{q+s+2}$ and $k_q \in L_q$. Hence, $h_{q+s+1} \varphi_{q+s+2}^{-1} \in L_{q+s+1} \leq K_{q+s+1}$ and

$$g_q = k_q h_q \in H_q, \quad g_{q+s+1} = k_{q+s+1} h_{q+s+1} = k_{q+s+1} (h_{q+s+1} \varphi_{q+s+2}^{-1}) \in K_{q+s+1},$$

as required.

Necessity. It is easy to see that the mapping of words acting on the generators of the groups K_i $(i \in \mathbb{Z}_n)$ as the natural embeddings $\iota_i : K_i \to G_i$ $(i \in \mathbb{Z}_n)$ takes all the defining relations of $\text{GDP}(\mathcal{G}_n(K, L, M, \varphi))$ to the equalities valid in $\text{GDP}(\mathcal{G}_n(G, H, K, \varphi))$ and therefore defines a homomorphism μ from the first group to the second.

Let $\beta_i : G_i \to \text{GDP}(\mathcal{G}_n(G, H, K, \varphi))$ and $\gamma_i : K_i \to \text{GDP}(\mathcal{G}_n(K, L, M, \varphi))$ be the homomorphisms defined by the identity mappings of the generators of G_i and K_i. Then, for any $i \in \mathbb{Z}_n$, the diagram

$$K_i \xrightarrow{\gamma_i} \text{GDP}(\mathcal{G}_n(K, L, M, \varphi))$$
$$\iota_i \downarrow \qquad\qquad\qquad \downarrow \mu$$
$$G_i \xrightarrow{\beta_i} \text{GDP}(\mathcal{G}_n(G, H, K, \varphi))$$

is commutative, and since the homomorphisms β_i $(i \in \mathbb{Z}_n)$ are injective, the homomorphisms γ_i $(i \in \mathbb{Z}_n)$ are also injective. Thus, the group $\mathrm{GDP}(\mathcal{G}_n(K, L, M, \varphi))$ satisfies $(i)'$.

Suppose that numbers $q \in \mathbb{Z}_n$, $s \in \{0, \ 1, \ ..., \ r+1\}$ and elements

$$k_q \in K_q, \ k_{q+1} \in K_{q+1}, \ ..., \ k_{q+s+1} \in K_{q+s+1}$$

are such that $k_q k_{q+1} ... k_{q+s+1} = 1$. It follows from the definition of μ that the last equality also holds in $\mathrm{GDP}(\mathcal{G}_n(G, H, K, \varphi))$. Since n is admissible for (G, H, K, φ) with the reserve r and

$$k_q \in G_q, \ ..., \ k_{q+s-1} \in G_{q+s-1}, \ k_{q+s} k_{q+s+1} = k_{q+s}(k_{q+s+1} \varphi_{q+s+1}^{-1}) \in G_{q+s},$$

then $k_q \in H_q$ and $k_{q+s}(k_{q+s+1} \varphi_{q+s+1}^{-1}) \in K_{q+s}$. The first relation means that $k_q \in H_q \cap K_q = L_q$, the second one implies that $k_{q+s+1} \varphi_{q+s+1}^{-1} \in K_{q+s}$ and therefore

$$k_{q+s+1} \varphi_{q+s+1}^{-1} \in H_{q+s} \cap K_{q+s} = L_{q+s}.$$

Hence, $k_{q+s+1} \in M_{q+s+1}$, as required. ☐

Proposition 2.3. *Suppose that (G, H, K, φ) is an HNN-tuple, G is an abelian group, and $H = G = K$. If the order q of the automorphism φ is finite and divides a number $n \geq 3$, then n is admissible for (G, H, K, φ) with any reserve $r \in \{0, \ 1, \ ..., \ n-2\}$.*

Proof. For each $i \in \mathbb{Z}_n$, let again G_i denote an isomorphic copy of G, and let $\sigma_i : G \to G_i$ be an isomorphism. Consider the star graph Δ with the vertex set $\{v_i \, (i \in \mathbb{Z}_n), w\}$ and associate its central vertex w with the group G, the leaf v_i $(i \in \mathbb{Z}_n)$ with the group G_i, and the edge $\{w, v_i\}$ $(i \in \mathbb{Z}_n)$ with the group G and the homomorphisms, one of which is the identity mapping and the other coincides with the isomorphism $\varphi^i \sigma_i$. Since $q|n$, it follows from the relation $x \equiv y \pmod{n}$ that $\varphi^x = \varphi^y$. Therefore, the notation φ^i is correct.

We denote the constructed graph of groups by $\mathcal{G}(\Delta)$. It follows from its definition that, for any $i \in \mathbb{Z}_n$ and $g' \in G$, the equality $g' = g' \varphi^i \sigma_i$ holds in $\mathrm{GDP}(\mathcal{G}(\Delta))$ and therefore $g\sigma_{i-1} = g\varphi^{-(i-1)} = g\varphi\sigma_i$ for all $i \in \mathbb{Z}_n, g \in G$. Hence, the identity mapping of the generators of $\mathrm{GDP}(\mathcal{G}_n(G, H, K, \varphi))$ to $\mathrm{GDP}(\mathcal{G}(\Delta))$ can be extended to a homomorphism

$$\lambda : \mathrm{GDP}(\mathcal{G}_n(G, H, K, \varphi)) \to \mathrm{GDP}(\mathcal{G}(\Delta)).$$

Let $\alpha_i : G_i \to \mathrm{GDP}(\mathcal{G}(\Delta))$ and $\beta_i : G_i \to \mathrm{GDP}(\mathcal{G}_n(G, H, K, \varphi))$ be the homomorphisms defined by the identity mappings of the generators of G_i $(i \in \mathbb{Z}_n)$. Then, for any $i \in \mathbb{Z}_n$, the diagram

$$
\begin{array}{ccc}
G_i & \xrightarrow{\ \beta_i\ } & \mathrm{GDP}(\mathcal{G}_n(G, H, K, \varphi)) \\
\| & & \downarrow{\lambda} \\
G_i & \xrightarrow{\ \alpha_i\ } & \mathrm{GDP}(\mathcal{G}(\Delta))
\end{array}
$$

is commutative. Since the homomorphisms α_i $(i \in \mathbb{Z}_n)$ are injective by Proposition 2.1, then the homomorphisms β_i $(i \in \mathbb{Z}_n)$ are also injective. Therefore, $\mathrm{GDP}(\mathcal{G}_n(G, H, K, \varphi))$ satisfies $(i)'$. It follows from the equalities $H = G = K$ that $(ii)'$ is satisfied trivially. ☐

Proposition 2.4. *Suppose that a number $n \geq 3$ is admissible for an HNN-tuple (G, H, K, φ) with a reserve $r \in \{0, \ 1, \ ..., \ n-2\}$ and C_n is a cyclic group of order n with a generator c. Suppose also that $\mathcal{C}$ is a class of groups closed under taking quotient groups and extensions. If $G \in \mathcal{C}$ and $C_n \in \mathcal{C}$, then there exists a homomorphism of the group $\mathfrak{G} = \mathrm{HNN}(G, H, K, \varphi)$ onto a group from $\mathcal{C}$ acting injectively on G.*

Proof. Let G_i and σ_i $(i \in \mathbb{Z}_n)$ be defined as above. Then the generalized direct product $P = \text{GDP}(\mathcal{G}_n(G, H, K, \varphi))$ has the representation

$$P = \langle G_i \, (i \in \mathbb{Z}_n); \, [G_i, G_j] = 1 \, (i, j \in \mathbb{Z}_n, \, i \neq j), \, H\sigma_{i-1} = K\sigma_i \, (i \in \mathbb{Z}_n) \rangle,$$

which shows that the map extending the isomorphisms $\sigma_i^{-1}\sigma_{i-1} : G_i \to G_{i-1}$ defines an automorphism α of this group. Obviously, the order of this automorphism divides n, and therefore there exists a split extension Q of P by C_n such that $\hat{c}|_P = \alpha$ (here and below, $\hat{x}$ denotes the inner automorphism defined by an element x). Since $G \in \mathcal{C}$, then P is a quotient group of the direct product of n $\mathcal{C}$-groups. Hence, $P \in \mathcal{C}$ and $Q \in \mathcal{C}$ because $\mathcal{C}$ is closed under taking quotient groups and extensions.

It is easy to see that the map $\rho : \mathfrak{G} \to Q$ extending the homomorphism $\sigma_0 : G \to G_0$ and mapping t to c takes all the defining relations of $\mathfrak{G}$ to the equalities valid in Q and therefore is a homomorphism. Since $c^i G_0 c^{-i} = G_i \, (i \in \mathbb{Z}_n)$, this homomorphism is surjective. It remains to note that, because n is admissible, ρ is injective on G and thus is the required mapping. $\qquad\square$

3. Compatible subgroups

Let (G, H, K, φ) be an HNN-tuple, and let $\mathfrak{G} = \text{HNN}(G, H, K, \varphi)$. Recall that a subgroup $N \leq G$ is said to be (H, K, φ)-*compatible* if $(N \cap H)\varphi = N \cap K$.

It is easy to verify that if a subgroup N is normal in G and is (H, K, φ)-compatible, then the mapping $\varphi_N : HN/N \to KN/N$ taking a coset hN $(h \in H)$ to $(h\varphi)N$ is well defined and is a subgroup isomorphism. Therefore, the sequence

$$(G/N, HN/N, KN/N, \varphi_N)$$

turns out to be an HNN-tuple. It is also easy to see that the map

$$\rho_N : \mathfrak{G} \to \text{HNN}(G/N, HN/N, KN/N, \varphi_N)$$

extending the natural homomorphism $G \to G/N$ and taking t to t is a surjective homomorphism, and its kernel coincides with the normal closure of N in $\mathfrak{G}$.

For every class of groups $\mathcal{C}$, consider three families of subgroups. Namely, suppose that

- $\mathcal{C}^*(G, H, K, \varphi)$ is the family of all (H, K, φ)-compatible subgroups from $\mathcal{C}^*(G)$;
- $\mathcal{C}_r^*(G, H, K, \varphi)$ $(r \geq 0)$ is the subset of $\mathcal{C}^*(G, H, K, \varphi)$ defined as follows: a subgroup $N \in \mathcal{C}^*(G, H, K, \varphi)$ belongs to $\mathcal{C}_r^*(G, H, K, \varphi)$ if and only if there exists a number $n \geq \max\{3, r + 2\}$ such that $\mathcal{C}$ contains a cyclic group of order n and n is admissible for $(G/N, HN/N, KN/N, \varphi_N)$ with the reserve r;
- $\mathcal{C}_\cap^*(G, H, K, \varphi) = \{U \cap G \,|\, U \in \mathcal{C}^*(\mathfrak{G})\}$.

Proposition 3.1. *If $\mathcal{C}$ is a class of groups closed under taking subgroups and direct products of a finite number of factors, and X is a group, then the following statements hold.*

1) *The intersection of a finite number of subgroups from the family $\mathcal{C}^*(X)$ again belongs to this family [22, Proposition 2];*
2) *If X is residually a $\mathcal{C}$-group and S is a finite subset of non-trivial elements of X, then there exists a subgroup $Y \in \mathcal{C}^*(X)$ such that $Y \cap S = \emptyset$ [25, Proposition 2].*

Proposition 3.2. *If $\mathcal{C}$ is a class of groups closed under taking subgroups and direct products of a finite number of factors, and (G, H, K, φ) is an HNN-tuple, then the families $\mathcal{C}^*(G, H, K, \varphi)$ and $\mathcal{C}_\cap^*(G, H, K, \varphi)$ are closed under taking the intersections of a finite number of subgroups.*

Proof. An obvious induction allows us to consider the intersection of only two subgroups.

If $N_1, N_2 \in \mathcal{C}^*(G, H, K, \varphi)$ and $N = N_1 \cap N_2$, then $N \in \mathcal{C}^*(G)$ by Proposition 3.1 and

$$(N \cap H)\varphi = ((N_1 \cap H) \cap (N_2 \cap H))\varphi = (N_1 \cap H)\varphi \cap (N_2 \cap H)\varphi$$
$$= (N_1 \cap K) \cap (N_2 \cap K) = N \cap K$$

because φ is injective. Therefore, $N \in \mathcal{C}^*(G, H, K, \varphi)$.

Suppose that $N_1, N_2 \in \mathcal{C}^*_\cap(G, H, K, \varphi)$, $\mathfrak{G} = \text{HNN}(G, H, K, \varphi)$, and $U_1, U_2 \in \mathcal{C}^*(\mathfrak{G})$ are subgroups such that $N_1 = U_1 \cap G$, $N_2 = U_2 \cap G$. If $N = N_1 \cap N_2$ and $U = U_1 \cap U_2$, then $N = U \cap G$ and, again by Proposition 3.1, $U \in \mathcal{C}^*(\mathfrak{G})$. Thus, $N \in \mathcal{C}^*_\cap(G, H, K, \varphi)$. □

Proposition 3.3. [21, Theorem 3] *Suppose that $\mathcal{C}$ is a root class of groups containing non-periodic groups and closed under taking quotient groups, (G, H, K, φ) is an HNN-tuple, and $\mathfrak{G} = \text{HNN}(G, H, K, \varphi)$. Suppose also that G is residually a $\mathcal{C}$-group, H and K lie in the center of G, and there exists a homomorphism ρ of G onto a group from $\mathcal{C}$ acting injectively on H and K. Then ρ can be extended to a homomorphism of $\mathfrak{G}$ onto a group from $\mathcal{C}$ and $\mathfrak{G}$ is residually a $\mathcal{C}$-group.*

Proposition 3.4. *Let $\mathcal{C}$ be a class of groups, and let (G, H, K, φ) be an HNN-tuple.*
  1. *If $\mathcal{C}$ is closed under taking subgroups, then $\mathcal{C}^*_\cap(G, H, K, \varphi) \subseteq \mathcal{C}^*(G, H, K, \varphi)$.*
  2. *If $\mathcal{C}$ is a root class of groups containing non-periodic groups and closed under taking quotient groups, H and K lie in the center of G, then $\mathcal{C}^*(G, H, K, \varphi) \subseteq \mathcal{C}^*_\cap(G, H, K, \varphi)$.*
  3. *If $\mathcal{C}$ is closed under taking quotient groups and extensions, then $\mathcal{C}^*_r(G, H, K, \varphi) \subseteq \mathcal{C}^*_\cap(G, H, K, \varphi)$ for any $r \geq 0$.*

Proof. 1. Suppose that $\mathfrak{G} = \text{HNN}(G, H, K, \varphi)$, $N \in \mathcal{C}^*_\cap(G, H, K, \varphi)$, and $U \in \mathcal{C}^*(\mathfrak{G})$ is a subgroup such that $N = U \cap G$. Then

$$G/N = G/(U \cap G) \cong GU/U \leq \mathfrak{G}/U$$

and $G/N \in \mathcal{C}$ because $\mathcal{C}$ is closed under taking subgroups. Since U is normal in $\mathfrak{G}$, then

$$(U \cap H)\varphi = t^{-1}(U \cap H)t \leq t^{-1}Ut \cap t^{-1}Ht = U \cap H\varphi = U \cap K$$

and similarly $(U \cap K)\varphi^{-1} \leq U \cap H$. Hence,

$$(N \cap H)\varphi = (U \cap G \cap H)\varphi = (U \cap H)\varphi = U \cap K = U \cap G \cap K = N \cap K$$

and therefore $N \in \mathcal{C}^*(G, H, K, \varphi)$.

2. Let $N \in \mathcal{C}^*(G, H, K, \varphi)$. Then the HNN-tuple $(G/N, HN/N, KN/N, \varphi_N)$, the group $\mathfrak{G}_N = \text{HNN}(G/N, HN/N, KN/N, \varphi_N)$, and the homomorphism $\rho_N : \mathfrak{G} \to \mathfrak{G}_N$ are defined. It follows from the definition of the family $\mathcal{C}^*(G, H, K, \varphi)$ that $G/N \in \mathcal{C}$. Hence, by Proposition 3.3, there exists a homomorphism σ of $\mathfrak{G}_N$ onto a group from $\mathcal{C}$ extending the identity mapping of G/N. Let $U_N = \ker\sigma$, and let U be the preimage of U_N under ρ_N. Then $U_N \in \mathcal{C}^*(\mathfrak{G}_N)$, $U_N \cap G/N = 1$, and $U \in \mathcal{C}^*(\mathfrak{G})$. Since ρ_N extends the natural homomorphism $G \to G/N$, then $\ker\rho_N \cap G = N$ and $U \cap G \leq \ker\rho_N$. Therefore, $U \cap G = N$ and $N \in \mathcal{C}^*_\cap(G, H, K, \varphi)$.

3. If $N \in \mathcal{C}^*_r(G, H, K, \varphi)$, then, by Proposition 2.4, there exists a homomorphism of the group $\text{HNN}(G/N, HN/N, KN/N, \varphi_N)$ onto a group from $\mathcal{C}$ acting injectively on G/N. Hence, we can use exactly the same argument as in the proof of Statement 2. □

Proposition 3.5. *Let $\mathcal{C}$ be a root class of groups consisting only of periodic groups and closed under taking quotient groups. If (G, H, K, φ) is an HNN-tuple, G is an abelian group, $H = G = K$, and N is a subgroup of G, then the following statements are equivalent.*

1) $N \in \mathcal{C}^*_\cap(G, H, K, \varphi)$.
2) $N \in \mathcal{C}^*_r(G, H, K, \varphi)$ *for any $r \geq 0$.*

3) $N\varphi = N$, $G/N \in \mathcal{C}$, and the order of the automorphism φ_N of G/N induced by the automorphism φ is finite and is a $\pi(\mathcal{C})$-number.

Proof. 1 $\Rightarrow$ 3. By Proposition 3.4, $N \in \mathcal{C}^*(G, H, K, \varphi)$. Hence, $G/N \in \mathcal{C}$, and since $H = G = K$, then

$$N\varphi = (N \cap H)\varphi = N \cap K = N.$$

Let $\mathfrak{G} = \mathrm{HNN}(G, H, K, \varphi)$. Because $N \in \mathcal{C}^*_\cap(G, H, K, \varphi)$, there exists a subgroup $U \in \mathcal{C}^*(\mathfrak{G})$ such that $N = U \cap G$. Since $\mathfrak{G}/U \in \mathcal{C}$ and $\mathcal{C}$ consists only of periodic groups, then the order n of t modulo U is finite and is a $\pi(\mathcal{C})$-number.

It follows from the equality $N = U \cap G$ that the mapping $\gamma : G/N \to \mathfrak{G}/U$ taking a coset gN to gU is well defined and is an injective homomorphism. For any $g \in G$, we have

$$(gN)\varphi_N^n = (g\varphi^n)N = (gN)\gamma(\widehat{t^n U})\gamma^{-1} = gN.$$

Hence, the order of φ_N divides n and therefore is a $\pi(\mathcal{C})$-number.

3 $\Rightarrow$ 2. We fix a number $r \geq 0$ and choose a $\pi(\mathcal{C})$-number $n > r + 3$ to be a multiple of the order of φ_N (it is possible to find a $\pi(\mathcal{C})$-number greater than $r + 3$ because $\mathcal{C}$ contains non-trivial groups and therefore $\pi(\mathcal{C}) \neq \emptyset$). Since $H = G = K$, then

$$(N \cap H)\varphi = N\varphi = N = N \cap K$$

and hence $N \in \mathcal{C}^*(G, H, K, \varphi)$. Therefore, the tuple $(G/N, HN/N, KN/N, \varphi_N)$ is defined and, by Proposition 2.3, n is admissible for this tuple with the reserve r. It remains to note that $\mathcal{C}$ contains a cyclic group of order n: this fact follows from the definition of $\pi(\mathcal{C})$ and the assumption that $\mathcal{C}$ is closed under taking subgroups and extensions.

2 $\Rightarrow$ 1. This implication is a consequence of Proposition 3.4. ☐

4. Descent and ascent of compatible subgroups

The proofs of the propositions in this section follow the ideas of [11, 12] and, in some places, repeat the arguments given in the above papers almost word for word.

Proposition 4.1. *Suppose that $\mathcal{C}$ is a class of groups closed under taking subgroups, (G, H, K, φ) is an HNN-tuple, $L = H \cap K$, $M = L\varphi$, and X is a subgroup of G containing L and M. Suppose also that N is a subgroup of G and $R = N \cap X$. Then the following statements hold.*

1) *If $N \in \mathcal{C}^*(G, H, K, \varphi)$, then $R \in \mathcal{C}^*(X, L, M, \varphi)$.*
2) *If $N \in \mathcal{C}^*_\cap(G, H, K, \varphi)$, then $R \in \mathcal{C}^*_\cap(X, L, M, \varphi)$.*

Proof. 1. We have $X/R \in \mathcal{C}$ because

$$X/R = X/(N \cap X) \cong XN/N \leq G/N \in \mathcal{C}$$

and $\mathcal{C}$ is closed under taking subgroups. Since φ is injective and N is (H, K, φ)-compatible, then

$$((N \cap H) \cap L)\varphi = (N \cap K) \cap M.$$

It follows from this equality and the relation $L \cup M \leq X$ that

$$\begin{aligned}(R \cap L)\varphi = ((N \cap X) \cap L)\varphi &= (N \cap L)\varphi = ((N \cap H) \cap L)\varphi \\ &= (N \cap K) \cap M = N \cap M = (N \cap X) \cap M = R \cap M.\end{aligned}$$

Thus, $R \in \mathcal{C}^*(X, L, M, \varphi)$.

2. If $\mathfrak{G} = \mathrm{HNN}(G, H, K, \varphi)$ and $\mathfrak{X} = \mathrm{HNN}(X, L, M, \varphi)$, then the map $\lambda : \mathfrak{X} \to \mathfrak{G}$ taking the generators of $\mathfrak{X}$ to the corresponding elements of $\mathfrak{G}$ defines a homomorphism, which acts

identically on X. Let $U \in \mathcal{C}^*(\mathfrak{G})$ be a subgroup such that $N = U \cap G$, and suppose that $\tilde{\mathfrak{X}} = \mathfrak{X}\lambda$, $\tilde{V} = U \cap \tilde{\mathfrak{X}}$, V is the full preimage of $\tilde{V}$ under λ. Then $\tilde{\mathfrak{X}}/\tilde{V}$ and $\mathfrak{X}/V$ are isomorphic groups, which belong to $\mathcal{C}$ because

$$\tilde{\mathfrak{X}}/\tilde{V} = \tilde{\mathfrak{X}}/(U \cap \tilde{\mathfrak{X}}) \cong \tilde{\mathfrak{X}}U/U \leq \mathfrak{G}/U \in \mathcal{C}$$

and $\mathcal{C}$ is closed under taking subgroups. Since λ acts identically on X, it follows from the equalities

$$\tilde{V} \cap X = U \cap \tilde{\mathfrak{X}} \cap X = U \cap X = U \cap G \cap X = N \cap X = R$$

that $V \cap X = R$. Therefore, $R \in \mathcal{C}_\cap^*(X, L, M, \varphi)$. □

Let X be a group, and let Y and Z be its subgroups. Recall that a family Ω of normal subgroups of X is said to be a *filtration* if $\cap_{N \in \Omega} N = 1$. A filtration Ω is called
  – a *Y-filtration* if $\cap_{N \in \Omega} YN = 1$;
  – a *(Y, Z)-filtration* if it is a Y-filtration and a Z-filtration.

Proposition 4.2. *Suppose that $\mathcal{C}$ is a class of groups closed under taking subgroups, (G, H, K, φ) is an HNN-tuple, $L = H \cap K$, $M = L\varphi$, and X is a subgroup of G containing L and M. Then the following statements hold.*

1) *If $\mathcal{C}_\cap^*(G, H, K, \varphi)$ is a filtration, then $\mathcal{C}_\cap^*(X, L, M, \varphi)$ is also a filtration.*
2) *Let X coincide with either G or K. If $\mathcal{C}_\cap^*(G, H, K, \varphi)$ is an (H, K)-filtration, then $\mathcal{C}_\cap^*(X, L, M, \varphi)$ is an (L, M)-filtration.*

Proof. By Proposition 4.1, if $N \in \mathcal{C}_\cap^*(G, H, K, \varphi)$, then $N \cap X \in \mathcal{C}_\cap^*(X, L, M, \varphi)$. Hence,

$$\bigcap_{R \in \mathcal{C}_\cap^*(X, L, M, \varphi)} R \leq \bigcap_{N \in \mathcal{C}_\cap^*(G, H, K, \varphi)} (N \cap X), \qquad \bigcap_{R \in \mathcal{C}_\cap^*(X, L, M, \varphi)} RL \leq \bigcap_{N \in \mathcal{C}_\cap^*(G, H, K, \varphi)} (N \cap X)L.$$

If $\mathcal{C}_\cap^*(G, H, K, \varphi)$ is a filtration, then

$$1 = \bigcap_{N \in \mathcal{C}_\cap^*(G, H, K, \varphi)} N = \bigcap_{N \in \mathcal{C}_\cap^*(G, H, K, \varphi)} (N \cap X)$$

and therefore $\mathcal{C}_\cap^*(X, L, M, \varphi)$ is also a filtration.

Let $\mathcal{C}_\cap^*(G, H, K, \varphi)$ be an (H, K)-filtration. Then

$$L \leq \bigcap_{R \in \mathcal{C}_\cap^*(X, L, M, \varphi)} RL \leq \bigcap_{N \in \mathcal{C}_\cap^*(G, H, K, \varphi)} NL = \bigcap_{N \in \mathcal{C}_\cap^*(G, H, K, \varphi)} N(H \cap K)$$

$$\leq \left(\bigcap_{N \in \mathcal{C}_\cap^*(G, H, K, \varphi)} NH \right) \cap \left(\bigcap_{N \in \mathcal{C}_\cap^*(G, H, K, \varphi)} NK \right) = H \cap K = L$$

and therefore $\mathcal{C}_\cap^*(X, L, M, \varphi)$ is an L-filtration. To prove that this family is an M-filtration, we take an arbitrary element $x \in X \setminus M$ and indicate a subgroup $R \in \mathcal{C}_\cap^*(X, L, M, \varphi)$ satisfying the condition $x \notin MR$.

If $x \notin K$, then, because $\mathcal{C}_\cap^*(G, H, K, \varphi)$ is a K-filtration, there exists a subgroup $N \in \mathcal{C}_\cap^*(G, H, K, \varphi)$ such that $x \notin KN$. It follows from Proposition 4.1 and the inclusion $M(N \cap X) \leq KN$ that $N \cap X \in \mathcal{C}_\cap^*(X, L, M, \varphi)$ and $x \notin M(N \cap X)$. Hence, $N \cap X$ is the required subgroup.

If $x \in K$, we put $y = x\varphi^{-1}$. Since $x \notin M$, then $y \in H \setminus L$ and therefore $y \notin K$. Let us consider two cases.

Case 1. $X = G$.

Let $\mathfrak{X} = \mathrm{HNN}(G, L, M, \varphi)$. As proved above, $\mathcal{C}_\cap^*(G, L, M, \varphi)$ is an L-filtration. Therefore, $y \notin LR$ for some subgroup $R \in \mathcal{C}_\cap^*(G, L, M, \varphi)$. By the definition of the family $\mathcal{C}_\cap^*(G, L, M, \varphi)$, there exists a subgroup $U \in \mathcal{C}^*(\mathfrak{X})$ such that $R = U \cap G$. If $y \in LU$ and $y = hu$ for some $h \in L$, $u \in U$,

then $u = h^{-1}y \in U \cap G = R$ and we get the inclusion $y \in LR$, which contradicts the choice of R. Therefore, $y \notin LU$ and

$$x = y\varphi = t^{-1}yt \notin t^{-1}LUt.$$

Since U is normal in $\mathfrak{X}$, then $t^{-1}LUt = MU$. Hence, $x \notin MU$ and $x \notin MR$ because $R \leq U$. Thus, R is the required subgroup.

Case 2. $X = K$.

Since $\mathcal{C}^*_\cap(G, H, K, \varphi)$ is a K-filtration, there exists a subgroup $N \in \mathcal{C}^*_\cap(G, H, K, \varphi)$ such that $y \notin KN$. By Proposition 3.4, N is (H, K, φ)-compatible and therefore

$$(M(N \cap K))\varphi^{-1} = L(N \cap H) \leq KN.$$

Hence, $y \notin (M(N \cap K))\varphi^{-1}$ and $x = y\varphi \notin M(N \cap K)$. It remains to note that, by Proposition 4.1, $N \cap K \in \mathcal{C}^*_\cap(K, L, M, \varphi)$ and thus $N \cap K$ is the required subgroup. $\square$

Let (G, H, K, φ) be an HNN-tuple. Following [11], we say that a subgroup $N \leq G$ is an *ascend* of a subgroup $R \leq K$ if $K \cap N = R$.

Let H and K lie in the center of G. We call an ascend $N \leq G$ of a subgroup $R \leq K$ *canonical* if $H \cap N = R\varphi^{-1}$ and $HK \cap N = (R\varphi^{-1})R$. It should be noted that a canonical ascend N is always an (H, K, φ)-compatible subgroup because $(H \cap N)\varphi = R = K \cap N$.

Proposition 4.3. *If $\mathcal{C}$ is a class of groups, (G, H, K, φ) is an HNN-tuple, H and K lie in the center of G, $L = H \cap K$, and $M = L\varphi$, then the following statements hold.*

1) *Suppose that $\mathcal{C}$ is closed under taking subgroups, quotient groups, and extensions, G is $\mathcal{C}$-regular with respect to the subgroup $P = HK$, and $R \in \mathcal{C}^*(K, L, M, \varphi)$. Then there exists a canonical ascend $N \in \mathcal{C}^*(G, H, K, \varphi)$ of R. Moreover, if X is a normal subgroup of G such that $P \leq X$ and G/X is residually a $\mathcal{C}$-group, then, for any finite set $S \subseteq G \setminus X$, there exists a canonical ascend $N \in \mathcal{C}^*(G, H, K, \varphi)$ of R satisfying the condition $S \cap XN = \emptyset$.*
2) *If $R \in \mathcal{C}^*_{r+1}(K, L, M, \varphi)$ for some $r \geq 0$ and $N \in \mathcal{C}^*(G, H, K, \varphi)$ is a canonical ascend of R, then $N \in \mathcal{C}^*_r(G, H, K, \varphi)$.*

Proof. 1. If $X = G$ and $S = \emptyset$, then $P \leq X$, G/X is residually a $\mathcal{C}$-group, and $S \subseteq G \setminus X$. Therefore, it suffices to prove only the second part of Statement 1. We put $Q = R\varphi^{-1}$ and show that $H \cap QR = Q$, $K \cap QR = R$.

Since $L, Q \leq H$, φ is injective, and R is (L, M, φ)-compatible, then

$$L \cap Q = M\varphi^{-1} \cap R\varphi^{-1} = (M \cap R)\varphi^{-1} = L \cap R.$$

If $g \in H \cap QR$ and $g = xy$ for some $x \in Q$, $y \in R$, then $g, x \in H$,

$$y \in H \cap R = H \cap K \cap R = L \cap R = L \cap Q,$$

and therefore $g \in Q$. Hence, $H \cap QR \subseteq Q$. The relation $K \cap QR \subseteq R$ is verified in the same way. Since the opposite inclusions are obvious, the required equalities are proved.

The quotient group P/QR is an extension of HR/QR by P/HR. We have

$$HR/QR \cong H/Q(R \cap H) \cong (H/Q)/(Q(R \cap H)/Q),$$
$$P/HR = HK/HR \cong K/R(H \cap K) \cong (K/R)/(R(H \cap K)/R).$$

Since $R \in \mathcal{C}^*(K, L, M, \varphi)$, then $K/R \in \mathcal{C}$ and therefore $H/Q \in \mathcal{C}$. Hence, $P/QR \in \mathcal{C}$ because $\mathcal{C}$ is closed under taking quotient groups and extensions. Thus, using the $\mathcal{C}$-regularity of G with respect to P, we can find a subgroup $U \in \mathcal{C}^*(G)$ such that $U \cap P = QR$.

By the hypothesis of the proposition, G/X is residually a $\mathcal{C}$-group and $\mathcal{C}$ is closed under taking subgroups and extensions. Hence, $\mathcal{C}$ is also closed under taking direct products of a finite number

of factors and, by Proposition 3.1, there exists a subgroup $V/X \in \mathcal{C}^*(G/X)$ such that $SX/X \cap V/X = \emptyset$. Then $V \in \mathcal{C}^*(G)$ and $S \cap V = \emptyset$.

Let $N = U \cap V$. Then $S \cap XN \subseteq S \cap XV = S \cap V = \emptyset$ and, again by Proposition 3.1, $N \in \mathcal{C}^*(G)$. Because $P \leq X \leq V$, we have

$$P \cap N = P \cap U \cap V = P \cap U = QR,$$
$$H \cap N = H \cap P \cap N = H \cap QR = Q,$$
$$K \cap N = K \cap P \cap N = K \cap QR = R.$$

Since $Q\varphi = R$, it follows from these equalities that N is an (H, K, φ)-compatible subgroup, which is the required canonical ascend of R.

2. The relation $R \in \mathcal{C}^*_{r+1}(K, L, M, \varphi)$ means that there exists a number $n \geq r + 3$ such that a cyclic group C_n of order n belongs to $\mathcal{C}$ and n is admissible for the HNN-tuple $(K/R, LR/R, MR/R, \varphi_R)$ with the reserve $r + 1$. Since $K \cap N = R$, the map of K/R into G/N taking a coset kR ($k \in K$) to kN is an isomorphism of K/R onto the subgroup KN/N of G/N. Under this isomorphism, the subgroups LR/R and MR/R are mapped onto the subgroups LN/N and MN/N, and the isomorphism φ_R corresponds to the isomorphism φ_N. Therefore, n is admissible with the reserve $r + 1$ for the HNN-tuple $(KN/N, LN/N, MN/N, \varphi_N)$. Let us show that $HN/N \cap KN/N = LN/N$.

If $gN \in HN/N \cap KN/N$, then $g \in HN \cap KN$ and $g = hx = ky$ for some $h \in H$, $k \in K$, and $x, y \in N$. Hence, $h^{-1}k = xy^{-1} \in HK \cap N$. Since N is a canonical ascend of R, then $HK \cap N = QR$ (where, as above, $Q = R\varphi^{-1}$) and therefore $h^{-1}k = h_1k_1$ for some $h_1 \in Q$, $k_1 \in R$. Thus, $hh_1 = kk_1^{-1} \in H \cap K = L$ and $h \in h_1^{-1}L$. Since $h_1 \in Q \leq N$, then $g = hx \in LN$ and $gN \in LN/N$. Therefore, $HN/N \cap KN/N \subseteq LN/N$ and, because the opposite inclusion is obvious, the required equality is proved.

Thus, $HN/N \cap KN/N = LN/N$ and $(LN/N)\varphi_N = MN/N$ by the definition of φ_N. It follows from these equalities and Proposition 2.2 that n is admissible for the HNN-tuple $(G/N, HN/N, KN/N, \varphi_N)$ with the reserve r. Since $C_n \in \mathcal{C}$, we have $N \in \mathcal{C}^*_r(G, H, K, \varphi)$, as required. □

Let X be a group, and let Y, Z be its subgroups. We call a family Ω of normal subgroups of X a *strong (Y, Z)-filtration* if, for any finite subset S of X, there exists a subgroup $N \in \Omega$ such that, for each $x \in S$, the following statements hold:

1) if $x \neq 1$, then $x \notin N$;
2) if $x \notin Y$, then $x \notin YN$;
3) if $x \notin Z$, then $x \notin ZN$.

It is easy to see that any strong (Y, Z)-filtration is a (Y, Z)-filtration and, if Ω is closed under taking finite intersections of subgroups, then any (Y, Z)-filtration is a strong (Y, Z)-filtration.

Proposition 4.4. *Suppose that $\mathcal{C}$ is a class of groups closed under taking subgroups, quotient groups, and extensions, (G, H, K, φ) is an HNN-tuple, H and K lie in the center of G and are π'-isolated in this group for some set of primes π, $L = H \cap K$, $M = L\varphi$, and G is $\mathcal{C}$-regular with respect to the subgroup $P = HK$. Suppose also that there exists a normal subgroup X of G such that $P \leq X$, X/P is a periodic π'-group, and G/X is residually a $\mathcal{C}$-group. Then the following statements hold.*

1) *If $\mathcal{C}^*(K, L, M, \varphi)$ is a strong (L, M)-filtration, then $\mathcal{C}^*(G, H, K, \varphi)$ is a strong (H, K)-filtration.*
2) *If, for some $r \geq 0$, $\mathcal{C}^*_{r+1}(K, L, M, \varphi)$ is a strong (L, M)-filtration, then $\mathcal{C}^*_r(G, H, K, \varphi)$ is a strong (H, K)-filtration.*

Proof. We prove Statements 1 and 2 simultaneously. Suppose that S is a finite subset of G and $g \in S$. Then:

1) if $g \in K \setminus H$, then $g \notin L$;
2) if $g \in H \setminus K$, then $g\varphi \notin M$;
3) if $g \in P \setminus (H \cup K)$ and $g = hk$, where $h \in H$, $k \in K$, then $k \notin L$, $h\varphi \notin M$;
4) if $g \in X \setminus P$, q is the order of g modulo P, and $g^q = hk$, where $h \in H$, $k \in K$, then $k \notin L$, $h\varphi \notin M$.

Indeed, if $g \in K \setminus H$, then $g \notin H \cap K = L$; if $g \in H \setminus K$, then $g \notin H \cap K = L$ and therefore $g\varphi \notin M$. If $g \in P \setminus (H \cup K)$ and $g = hk$ for some $h \in H$, $k \in K$, then $h \notin K$, $k \notin H$, and, as above, $k \notin L$, $h\varphi \notin M$. Suppose that $g \in X \setminus P$, q is the order of g modulo P, and $g^q = hk$, where $h \in H$, $k \in K$. Since $g \notin H \cup K$, H and K are π'-isolated in G, and, by the hypothesis of the proposition, q is a π'-number, then $hk \notin H \cup K$. Hence, using the above argument, we get $k \notin L$, $h\varphi \notin M$.

Thus, because the family $\mathcal{C}^*(K, L, M, \varphi)$ or $\mathcal{C}^*_{r+1}(K, L, M, \varphi)$ is a strong (L, M)-filtration, we can choose a subgroup R from this family so that, for each $g \in S$, the following conditions hold:

1) if $g \in L$ and $g \neq 1$, then $g \notin R$;
2) if $g \in K \setminus H$, then $g \notin LR$;
3) if $g \in H \setminus K$, then $g\varphi \notin MR$;
4) if $g \in P \setminus (H \cup K)$ and $g = hk$, where $h \in H$, $k \in K$, then $k \notin LR$, $h\varphi \notin MR$;
5) if $g \in X \setminus P$, q is the order of g modulo P, and $g^q = hk$, where $h \in H$, $k \in K$, then $k \notin LR$, $h\varphi \notin MR$.

It is sufficient for us that, for each $g \in S$, the fourth and fifth conditions hold for some fixed choice of h and k. However, if $h_1 k_1 = h_2 k_2$ for some $h_1, h_2 \in H$, $k_1, k_2 \in K$, then $h_1^{-1} h_2 = k_1 k_2^{-1} \in L$. Therefore, if $k_1 \notin LR$ and $h_1\varphi \notin MR$, then $k_2 \notin LR$ and $h_2\varphi \notin MR$.

Let $S_1 = S \setminus X$, and let $Q = R\varphi^{-1}$. By Proposition 4.3, there exists a canonical ascend $N \in \mathcal{C}^*(G, H, K, \varphi)$ of R such that $S_1 \cap XN = \emptyset$ and, if $R \in \mathcal{C}^*_{r+1}(K, L, M, \varphi)$, then $N \in \mathcal{C}^*_r(G, H, K, \varphi)$. To complete the proof, it suffices to show that, for each $g \in S$, the following statements hold:

1) if $g \neq 1$, then $g \notin N$;
2) if $g \notin H$, then $g \notin HN$;
3) if $g \notin K$, then $g \notin KN$.

Let us consider several cases.

Case 1: $g \in L$ and $g \neq 1$.

Since $R = N \cap K$, $g \in K$, and $g \notin R$ by the choice of R, then $g \notin N$.

Case 2: $g \in K \setminus H$.

If $g \in HN$ and $g = hx$ for some $h \in H$, $x \in N$, then $x = h^{-1}g \in N \cap P$. Since N is a canonical ascend of R, then $N \cap P = QR$ and $x = h_1 k$ for some $h_1 \in Q$, $k \in R$. Hence, $gk^{-1} = hh_1 \in H \cap K = L$, $g \in LR$, and we get a contradiction with the choice of R.

Case 3: $g \in H \setminus K$.

As in the previous case, if $g \in KN$ and $g = kx$ for some $k \in K$, $x \in N$, then $x = k^{-1}g \in N \cap P = RQ$ and therefore $x = k_1 h$ for some $h \in Q$, $k_1 \in R$. Hence, $gh^{-1} = kk_1 \in L$, $g = (kk_1)h \in LQ$, $g\varphi \in MR$, and we again get a contradiction with the choice of R.

Case 4: $g \in P \setminus (H \cup K)$.

Let $g = hk$, where $h \in H$, $k \in K$. If $g \in HN$ and $g = h_1 x$ for some $h_1 \in H$, $x \in N$, then it follows from the equality $hk = h_1 x$ that $x \in P$ and therefore $x \in N \cap P = QR$. Hence, $x = h_2 k_2$ for some $h_2 \in Q$, $k_2 \in R$, $hk = h_1 h_2 k_2$, and $kk_2^{-1} = h^{-1} h_1 h_2 \in L$. Thus, we get the relations $k = (h^{-1} h_1 h_2) k_2 \in LR$, which contradict the choice of R.

Similarly, if $g \in KN$ and $g = k_1 x$ for some $k_1 \in K$, $x \in N$, then again $x \in N \cap P = QR$ and $x = h_2 k_2$ for some $h_2 \in Q$, $k_2 \in R$. Hence, $hk = k_1 h_2 k_2$ and, since $[H, K] = 1$, then $hh_2^{-1} = k^{-1}k_1 k_2 \in L$, $h = (k^{-1}k_1 k_2)h_2 \in LQ$, and $h\varphi \in MR$. The last inclusion again contradicts the choice of R.

Case 5: $g \in X \setminus P$.

As already proved above, it follows from the relation $g \in X \setminus P$ that $g^q \notin H \cup K$, where q is the order of g modulo P. Hence, using the same argument as in Case 4, we get $g^q \notin HN \cup KN$. Then $g \notin HN \cup KN$, as required.

Case 6: $g \notin X$.

Since $g \notin X$, then $g \in S_1$, $g \notin XN$ by the choice of N, and $g \notin HN \cup KN$ because $H \cup K \subseteq X$. □

5. Necessary and sufficient conditions for the $\mathcal{C}$-residuality of HNN-extensions

Throughout this section, it is assumed that (G, H, K, φ) is an HNN-tuple and $\mathfrak{G} = \mathrm{HNN}(G, H, K, \varphi)$. Let us recall that any element $g \in \mathfrak{G}$ can be written in a *reduced form*:

$$g = g_0 t^{\varepsilon_1} g_1 \ldots t^{\varepsilon_n} g_n,$$

where $g_i \in G$, $\varepsilon_i = \pm 1$, and if $-\varepsilon_i = 1 = \varepsilon_{i+1}$, then $g_i \notin H$, if $\varepsilon_i = 1 = -\varepsilon_{i+1}$, then $g_i \notin K$. Britton's lemma (see, e. g., [9, Ch. IV, Section 2]) says that an element of $\mathfrak{G}$ is non-trivial if it has a reduced form containing at least one letter t or t^{-1}. As a consequence, all the reduced forms of an element $g \in \mathfrak{G}$ have the same number of occurrences of the letters t and t^{-1}. This number is called the *length* of g and is denoted in this article by $|g|$.

Proposition 5.1. *If $\mathcal{C}$ is a root class of groups, then the following statements hold.*

1) *Every free group is residually a $\mathcal{C}$-group [3, Theorem 1].*
2) *Any extension of a residually $\mathcal{C}$-group by a $\mathcal{C}$-group is residually a $\mathcal{C}$-group [7, Lemma 1.5].*

The following proposition is a generalization of Theorem 4.2 from [4] and is partially proved in [24].

Proposition 5.2. *Let $\mathcal{C}$ be a class of groups.*

1) *If $\mathfrak{G}$ is residually a $\mathcal{C}$-group, then $\mathcal{C}^*_\cap(G, H, K, \varphi)$ is a filtration.*
2) *If $\mathfrak{G}$ is residually a $\mathcal{C}$-group, H and K are proper central subgroups of G, then $\mathcal{C}^*_\cap(G, H, K, \varphi)$ is an (H, K)-filtration.*
3) *If $\mathcal{C}$ is a root class of groups and $\mathcal{C}^*_\cap(G, H, K, \varphi)$ is an (H, K)-filtration, then $\mathfrak{G}$ is residually a $\mathcal{C}$-group.*

Proof. 1. Since $\mathfrak{G}$ is residually a $\mathcal{C}$-group, then

$$1 = \bigcap_{U \in \mathcal{C}^*(\mathfrak{G})} U = \bigcap_{U \in \mathcal{C}^*(\mathfrak{G})} (U \cap G) = \bigcap_{N \in \mathcal{C}^*_\cap(G, H, K, \varphi)} N,$$

as required.

2. Suppose that $\mathcal{C}^*_\cap(G, H, K, \varphi)$ is not an H-filtration and

$$g_1 \in \left(\bigcap_{N \in \mathcal{C}^*_\cap(G, H, K, \varphi)} HN \right) \setminus H.$$

Because $K \neq G$, we can take an element $g_2 \in G \setminus K$ and put $g = [t^{-1}g_1 t, g_2]$. Then $|g| = 4$ and therefore $g \neq 1$. Since $\mathfrak{G}$ is residually a $\mathcal{C}$-group, there exists a subgroup $U \in \mathcal{C}^*(\mathfrak{G})$ such that $g \notin U$. By the choice of g_1, the inclusion $g_1 \in H(U \cap G)$ holds, and therefore $g_1 \equiv h \pmod{U}$ for some $h \in H$. Hence,

$$g \equiv \left[t^{-1}ht, g_2\right] = [h\varphi, g_2] \pmod{U}.$$

Since $h\varphi \in K$ and K is central in G, then $[h\varphi, g_2] = 1$, and we get a contradiction with the choice of U. The fact that $\mathcal{C}^*_\cap(G, H, K, \varphi)$ is a K-filtration is proved in a similar way.

3. Let $g \in \mathfrak{G} \setminus \{1\}$. We show that there exists a subgroup $N \in \mathcal{C}^*_\cap(G, H, K, \varphi)$ such that $g\rho_N \neq 1$.

If $g \in G$, then, because $\mathcal{C}^*_\cap(G, H, K, \varphi)$ is a filtration, g does not belong to some subgroup $N \in \mathcal{C}^*_\cap(G, H, K, \varphi)$. Since ρ_N extends the natural homomorphism $G \to G/N$, then $g\rho_N \neq 1$ and therefore N is the required subgroup.

Suppose that $g \notin G$ and $g = g_0 t^{\varepsilon_1} g_1 ... t^{\varepsilon_n} g_n$, where $g_i \in G$, $\varepsilon_i = \pm 1$, and if $-\varepsilon_i = 1 = \varepsilon_{i+1}$, then $g_i \notin H$, if $\varepsilon_i = 1 = -\varepsilon_{i+1}$, then $g_i \notin K$. It should be noted that, since $g \notin G$, then $n \geq 1$. For each $i \in \{0, 1, ..., n\}$, we define a subgroup $N_i \in \mathcal{C}^*_\cap(G, H, K, \varphi)$ as follows.

If $1 \leq i \leq n-1$ and $-\varepsilon_i = 1 = \varepsilon_{i+1}$, then $g_i \notin H$ and, since $\mathcal{C}^*_\cap(G, H, K, \varphi)$ is an (H, K)-filtration, we can find a subgroup $N_i \in \mathcal{C}^*_\cap(G, H, K, \varphi)$ such that $g_i \notin HN_i$. Similarly, if $1 \leq i \leq n-1$ and $\varepsilon_i = 1 = -\varepsilon_{i+1}$, we choose a subgroup $N_i \in \mathcal{C}^*_\cap(G, H, K, \varphi)$ so that $g_i \notin KN_i$. For all other $i \in \{0, 1, ..., n\}$, we put $N_i = G$. The last choice is possible because $\mathcal{C}$ is root and therefore the trivial group belongs to $\mathcal{C}$, $\mathfrak{G} \in \mathcal{C}^*(\mathfrak{G})$, and $G \in \mathcal{C}^*_\cap(G, H, K, \varphi)$.

Let

$$N = \bigcap_{i=0}^{n} N_i.$$

Then $N \in \mathcal{C}^*_\cap(G, H, K, \varphi)$ by Proposition 3.2 and, for any $i \in \{1, 2, ..., n-1\}$, if $-\varepsilon_i = 1 = \varepsilon_{i+1}$, then $g_i \notin HN$, if $\varepsilon_i = 1 = -\varepsilon_{i+1}$, then $g_i \notin KN$. Hence,

$$g\rho_N = (g_0 N) t^{\varepsilon_1} (g_1 N) ... t^{\varepsilon_n} (g_n N)$$

is a reduced form of $g\rho_N$ of length $n \geq 1$ and, in particular, $g\rho_N \neq 1$. Thus, N is the required subgroup.

Let us now show that $\mathfrak{G}_N = \mathrm{HNN}(G/N, HN/N, KN/N, \varphi_N)$ is residually a $\mathcal{C}$-group and therefore ρ_N can be extended to a homomorphism of $\mathfrak{G}$ onto a group from $\mathcal{C}$ taking g to a non-trivial element. Suppose that $U \in \mathcal{C}^*(\mathfrak{G})$ is a subgroup satisfying the equality $N = U \cap G$ and $\sigma : \mathfrak{G}_N \to \mathfrak{G}/U$ is the mapping defined by the rule: $(x\rho_N)\sigma = xU$ $(x \in \mathfrak{G})$. Since the kernel of ρ_N coincides with the normal closure of N in $\mathfrak{G}$, it is contained in U and therefore σ is well defined. It is also easy to see that σ is a surjective homomorphism. Since $G\rho_N = G/N$ and $N = U \cap G$, then $\ker\sigma \cap G/N = 1$. Hence, $\ker\sigma$ is free (see, e. g., [5]) and $\mathfrak{G}_N$ is an extension of the free group $\ker\sigma$ by the $\mathcal{C}$-group $\mathfrak{G}/U$. Such an extension is residually a $\mathcal{C}$-group by Proposition 5.1.

Proposition 5.3. *Suppose that $\mathcal{C}$ is a class of groups consisting only of periodic groups, H is a central subgroup of G, and $H \leq K \neq G$. If $\mathfrak{G}$ is residually a $\mathcal{C}$-group, then $H = K$.*

Proof. Suppose that $H \neq K$ and $k \in K \setminus H$. Suppose also that $g \in G \setminus K$ and $x = [t^{-1}kt, g]$. Then $|x| = 4$ and therefore $x \neq 1$. Since $\mathfrak{G}$ is residually a $\mathcal{C}$-group, there exists a subgroup $U \in \mathcal{C}^*(\mathfrak{G})$ such that $x \notin U$. Because $\mathcal{C}$ consists only of periodic groups, t has a finite order modulo U and therefore $t^{-1} \equiv t^m \pmod{U}$ for some $m > 0$. It follows from the last relation and the inclusion $H \leq K$ that

$$t^{-1}kt \equiv t^m k t^{-m} = k\varphi^{-m} \in H \pmod{U}.$$

Since H is central in G, then $[k\varphi^{-m}, g] = 1$. Thus, $x \in U$, and we get a contradiction with the choice of U. $\qquad\square$

Proposition 5.4. *Suppose that $\mathcal{C}$ is a class of groups closed under taking subgroups and direct products of a finite number of factors. Suppose also that G is residually a $\mathcal{C}$-group and there exists a subgroup $Q \in \mathcal{C}^*(G)$ satisfying at least one of the following conditions:*

(α) $H \cap Q = 1 = K \cap Q$,
(β) $Q \leq H \cap K$ and $Q\varphi = Q$.

Then H and K are C-separable in G.

Proof. We take an element $g \in G \setminus H$ and find a subgroup $N \in \mathcal{C}^*(G)$ such that $g \notin HN$.

If $Q \leq H$ or $g \notin HQ$, then Q is the required subgroup. Therefore, we can assume that $H \cap Q = 1$ and $g = hx$ for some $h \in H$, $x \in Q$. Since $g \notin H$, then $x \neq 1$ and, because G is residually a $\mathcal{C}$-group, there exists a subgroup $M \in \mathcal{C}^*(G)$ that does not contain x. We put $N = M \cap Q$. Then $N \in \mathcal{C}^*(G)$ by Proposition 3.1. If $g = h_1 x_1$ for some $h_1 \in H$, $x_1 \in N$, then $xx_1^{-1} = h^{-1}h_1 \in H \cap Q = 1$. Hence, $x = x_1 \in N \leq M$, and we get a contradiction with the choice of M. Therefore, $g \notin HN$ and N is the required subgroup.

Thus, H is $\mathcal{C}$-separable in G. The $\mathcal{C}$-separability of K is proved in a similar way. □

Proposition 5.5. *Suppose that $\mathcal{C}$ is a root class of groups containing infinite groups, G is residually a $\mathcal{C}$-group, H and K are central subgroups of G. Suppose also that there exists a subgroup $Q \in \mathcal{C}^*(G)$ satisfying at least one of the following conditions:*

(α) $H \cap Q = 1 = K \cap Q$,
(β) $Q \leq H \cap K$ and $Q\varphi = Q$.

Then $\mathcal{C}^*(G, H, K, \varphi)$ is an (H, K)-filtration.

Proof. We only need to show that every subgroup from $\mathcal{C}^*(G)$ contains some subgroup from $\mathcal{C}^*(G, H, K, \varphi)$. Since G is residually a $\mathcal{C}$-group and H, K are $\mathcal{C}$-separable in G by Proposition 5.4, then it will follow from this fact that $\mathcal{C}^*(G, H, K, \varphi)$ is an (H, K)-filtration.

So, let $L \in \mathcal{C}^*(G)$. By Proposition 3.1, the subgroup $M = L \cap Q$ belongs to $\mathcal{C}^*(G)$. If (α) holds, then

$$(H \cap M)\varphi = (H \cap L \cap Q)\varphi = 1 = K \cap L \cap Q = K \cap M$$

and therefore $M \in \mathcal{C}^*(G, H, K, \varphi)$. If ($\beta$) holds, then the subgroup

$$N = \bigcap_{i \in \mathbb{Z}} M\varphi^i$$

belongs to the same family.

Indeed, N lies in the center of G, is φ-invariant, and therefore is (H, K, φ)-compatible. Since $Q/M \leq G/M$ and $\mathcal{C}$ is closed under taking subgroups, then $M \in \mathcal{C}^*(Q)$. Because Q is φ-invariant, the restriction of φ to this subgroup is its automorphism and therefore $M\varphi^i \in \mathcal{C}^*(Q)$. Hence, Q/N is embedded into the Cartesian product of countably many isomorphic $\mathcal{C}$-groups $Q/M\varphi^i$. Since $\mathcal{C}$ is root and contains infinite groups, then it also contains the indicated Cartesian product, its subgroup Q/N, and the group G/N, which is an extension of Q/N by the $\mathcal{C}$-group G/Q. Therefore, $N \in \mathcal{C}^*(G)$. □

Proposition 5.6. *Suppose that $\mathcal{C}$ is a class of groups closed under taking subgroups and direct products of a finite number of factors, H and K are proper central subgroups of finite index of G. If $\mathfrak{G}$ is residually a $\mathcal{C}$-group, then there exists a subgroup $Q \in \mathcal{C}^*(G)$ such that $Q \leq H \cap K$ and $Q\varphi = Q$.*

Proof. Because H and K are of finite index in G, the subgroup $H \cap K$ has the same property. Let $1 = g_1, g_2, ..., g_n$ be a complete set of cosets representatives of this subgroup in G, and let $S = \{g_2, ..., g_n\}$. Since $\mathfrak{G}$ is residually a $\mathcal{C}$-group, then, by Proposition 5.2, $\mathcal{C}^*_\cap(G, H, K, \varphi)$ is an (H, K)-filtration. Hence,

$$H \cap K \leq \bigcap_{N \in \mathcal{C}^*_\cap(G,H,K,\varphi)} (H \cap K)N \leq \left(\bigcap_{N \in \mathcal{C}^*_\cap(G,H,K,\varphi)} HN \right) \cap \left(\bigcap_{N \in \mathcal{C}^*_\cap(G,H,K,\varphi)} KN \right) = H \cap K$$

and therefore, for each $s \in S$, there exists a subgroup $Q_s \in \mathcal{C}^*_\cap(G,H,K,\varphi)$ such that $s \notin (H \cap K)Q_s$. Let us show that

$$Q = \bigcap_{s \in S} Q_s$$

is the required subgroup.

Indeed, if $g \in G \setminus (H \cap K)$ and $g = xs$ for suitable $s \in S$, $x \in H \cap K$, then $x^{-1}g = s \notin (H \cap K)Q_s$ and therefore $x^{-1}g \notin (H \cap K)Q$. Hence, $g \notin (H \cap K)Q$ and, because g is chosen arbitrarily, $(H \cap K)Q \leq H \cap K$. Thus, $Q \leq H \cap K$. It remains to note that $Q \in \mathcal{C}^*_\cap(G,H,K,\varphi)$ by Proposition 3.2 and $\mathcal{C}^*_\cap(G,H,K,\varphi) \subseteq \mathcal{C}^*(G,H,K,\varphi)$ by Proposition 3.4. Therefore, $Q \in \mathcal{C}^*(G)$ and $Q\varphi = (Q \cap H)\varphi = Q \cap K = Q$. □

We conclude this section with two criteria for the root-class residuality of split extensions.

Proposition 5.7. *Suppose that $\mathcal{C}$ is a root class of groups consisting only of periodic groups and closed under taking quotient groups. Suppose also that G is an abelian group, $H = G = K$, and Ω is the family of subgroups of G defined as follows: $N \in \Omega$ if and only if $N \in \mathcal{C}^*(G)$, $N\varphi = N$, and the automorphism φ_N of the group G/N induced by the automorphism φ has a finite order, which is a $\pi(\mathcal{C})$-number. Then $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if $\cap_{N \in \Omega} N = 1$.*

Proof. Since $H = G = K$, then any filtration is an (H, K)-filtration. Therefore, by Proposition 5.2, $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if $\mathcal{C}^*_\cap(G,H,K,\varphi)$ is a filtration. It remains to note that, by Proposition 3.5, a subgroup N belongs to $\mathcal{C}^*_\cap(G,H,K,\varphi)$ if and only if $N\varphi = N$, $G/N \in \mathcal{C}$, and the order of φ_N is finite and is a $\pi(\mathcal{C})$-number. □

Proposition 5.8. *Suppose that $\mathcal{C}$ is a root class of groups consisting only of periodic groups and closed under taking quotient groups, G is an abelian residually $\mathcal{C}$-group, and $H = G = K$. If the order q of the automorphism φ is finite, then $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if q is a $\pi(\mathcal{C})$-number.*

Proof. Necessity. Because q is the order of φ, for any $i \in \{1, \ldots, q-1\}$, there exists an element $g_i \in G$ such that $g_i\varphi^i \neq g_i$. Let

$$S = \{g_i\varphi^i g_i^{-1} \mid 1 \leq i \leq q-1\}.$$

Since $\mathfrak{G}$ is residually a $\mathcal{C}$-group, then, by Proposition 3.1, there exists a subgroup $U \in \mathcal{C}^*(\mathfrak{G})$ such that $U \cap S = \emptyset$. If $N = U \cap G$, then $N \in \mathcal{C}^*_\cap(G,H,K,\varphi)$ and, by Proposition 3.5, N is φ-invariant, the automorphism φ_N of G/N induced by φ has a finite order q_N, and this order is a $\pi(\mathcal{C})$-number. It remains to note that $N \cap S = \emptyset$ and therefore $q_N = q$.

Sufficiency. Suppose that Ω is the family of subgroups defined in Proposition 5.7, $M \in \mathcal{C}^*(G)$, and

$$N = \bigcap_{i=0}^{q-1} M\varphi^i.$$

Then $N \leq M$, $N\varphi = N$, and, by Proposition 3.1, $N \in \mathcal{C}^*(G)$. The order of the automorphism φ_N induced by φ divides q and therefore is a $\pi(\mathcal{C})$-number. Hence, $N \in \Omega$.

Thus, any subgroup from $\mathcal{C}^*(G)$ contains a subgroup from Ω and, since G is residually a $\mathcal{C}$-group, Ω is a filtration. Therefore, $\mathfrak{G}$ is residually a $\mathcal{C}$-group by Proposition 5.7. □

6. Proofs of Theorems 1–4 and Corollaries 1–4

Throughout this section, it is assumed that (G, H, K, φ) is an HNN-tuple, H and K lie in the center of G, and $\mathfrak{G} = \text{HNN}(G, H, K, \varphi)$. It is also assumed that

$$K_0 = G, \; H_1 = H, \; K_1 = K, \; H_{i+1} = H_i \cap K_i, \; K_{i+1} = H_{i+1}\varphi, \; P_i = H_i K_i \; (i \geq 1).$$

Proposition 6.1. *Suppose that $\mathcal{C}$ is a root class of groups and $H \neq G \neq K$. If $\mathfrak{G}$ is residually a $\mathcal{C}$-group, then, for any $n \geq 1$, the HNN-extensions $\mathfrak{G}_{n-1} = \text{HNN}(G, H_n, K_n, \varphi)$ and $\mathfrak{K}_{n-1} = \text{HNN}(K_{n-1}, H_n, K_n, \varphi)$ are also residually $\mathcal{C}$-groups.*

Proof. Since $\mathfrak{G}$ is residually a $\mathcal{C}$-group, then the family $\mathcal{C}_\cap^*(G, H, K, \varphi)$ is an (H, K)-filtration by Proposition 5.2. Repeatedly using Proposition 4.2, we see that $\mathcal{C}_\cap^*(G, H_n, K_n, \varphi)$ and $\mathcal{C}_\cap^*(K_{n-1}, H_n, K_n, \varphi)$ are (H_n, K_n)-filtrations, and, again by Proposition 5.2, $\mathfrak{G}_{n-1}$ and $\mathfrak{K}_{n-1}$ are residually $\mathcal{C}$-groups. $\qquad\square$

Proposition 6.2. *The following statements hold.*

1) *Let π be a set of primes. If H_n and K_n are π'-isolated in G for some $n \geq 1$, then H_i and K_i are π'-isolated in this group for all $i \geq n$.*
2) *If a number $n \geq 1$ and a subgroup Q are such that $Q \leq H_n \cap K_n$ and $Q\varphi = Q$, then $Q \leq H_i \cap K_i$ for all $i \geq n$.*
3) *Let $\mathcal{C}$ be a class of groups closed under taking subgroups and extensions. If $H \in \mathcal{C}^*(G)$ and $K \in \mathcal{C}^*(G)$, then $H_i \in \mathcal{C}^*(G)$ and $K_i \in \mathcal{C}^*(G)$ for all $i \geq 1$.*

Proof. Let us use induction on i and note that, for all three statements, the induction base is obvious.

1) Suppose that H_i and K_i are π'-isolated in G for some $i \geq n$. If an element $g \in G$ and a number $q \in \pi'$ are such that $g^q \in H_{i+1}$, then $g^q \in H_i$ and $g^q \in K_i$. Since H_i and K_i are π'-isolated, then $g \in H_i \cap K_i = H_{i+1}$ and therefore H_{i+1} is π'-isolated in G. If, for some $g \in G$ and $q \in \pi'$, the inclusion $g^q \in K_{i+1}$ holds, then $g \in K_i$ because K_i is π'-isolated. Hence, the element $g\varphi^{-1}$ is defined and $(g\varphi^{-1})^q \in H_{i+1}$. Since H_{i+1} is π'-isolated, it follows from the last inclusion that $g\varphi^{-1} \in H_{i+1}$ and $g \in K_{i+1}$. Thus, K_{i+1} is also π'-isolated in G.
2) Let $Q \leq H_i \cap K_i$ for some $i \geq n$. Then

$$Q \leq H_{i+1} = H_i \cap K_i, \quad Q\varphi \leq H_{i+1}\varphi = K_{i+1},$$

and since $Q\varphi = Q$, then $Q \leq H_{i+1} \cap K_{i+1}$.
3) Let $\mathcal{C}^*(G)$ contain H_i and K_i for some $i \geq 1$. Then $H_{i+1} = H_i \cap K_i \in \mathcal{C}^*(G)$ by Proposition 3.1. Since $\mathcal{C}$ is closed under taking subgroups, then $H_{i+1} \in \mathcal{C}^*(H_i)$ and it follows from the equalities $H_i\varphi = K_i$, $H_{i+1}\varphi = K_{i+1}$ that $K_{i+1} \in \mathcal{C}^*(K_i)$. Hence, the quotient group G/K_{i+1} is an extension of the $\mathcal{C}$-group K_i/K_{i+1} by the $\mathcal{C}$-group G/K_i, and $K_{i+1} \in \mathcal{C}^*(G)$ because $\mathcal{C}$ is closed under taking extensions. $\qquad\square$

Proposition 6.3. [19, Proposition 5] *Suppose that $\mathcal{C}$ is a class of groups consisting only of periodic groups, X is a group, and Y is a subgroup of X. If Y is $\mathcal{C}$-separable in X, then it is $\pi(\mathcal{C})'$-isolated in this group. In particular, if X is residually a $\mathcal{C}$-group, then it has no $\pi(\mathcal{C})'$-torsion.*

If π is a set of primes, X is a group, and Y is a subgroup of X, then we denote by $\mathcal{R}_{\pi'}(X, Y)$ the set of elements of X defined as follows: $x \in \mathcal{R}_{\pi'}(X, Y)$ if and only if $x^q \in Y$ for some π'-number q.

Obviously, $\mathcal{R}_{\pi'}(X, Y) \subseteq \mathcal{I}_{\pi'}(X, Y)$ and the equality $\mathcal{I}_{\pi'}(X, Y) = \mathcal{R}_{\pi'}(X, Y)$ holds if and only if $\mathcal{R}_{\pi'}(X, Y)$ is a subgroup. It is easy to see that, if X is an abelian group, then $\mathcal{R}_{\pi'}(X, Y)$ is always a subgroup.

Proposition 6.4. *Suppose that $\mathcal{C}$ is a class of groups closed under taking quotient groups, X is residually a $\mathcal{C}$-group, Y is a central subgroup of X. Then the $\pi(\mathcal{C})'$-isolator $\mathcal{I}_{\pi(\mathcal{C})'}(X, Y)$ is contained in the center of X and coincides with the set $\mathcal{R}_{\pi(\mathcal{C})'}(X, Y)$.*

Proof. Let Z be the center of X. We show that Z is $\mathcal{C}$-separable in X.

If $x \in X \setminus Z$, then $[x, y] \neq 1$ for some $y \in X$ and, because X is residually a $\mathcal{C}$-group, there exists a subgroup $N \in \mathcal{C}^*(X)$ such that $[x, y] \notin N$. Hence, $[xN, yN] \neq N$ and therefore xN does not belong to the center $\mathcal{Z}(X/N)$ of X/N. It is easy to see that $ZN/N \leq \mathcal{Z}(X/N)$. Thus, $xN \notin ZN/N$ and $x \notin ZN$. It remains to note that $X/ZN \cong (X/N)/(ZN/N) \in \mathcal{C}$ because $\mathcal{C}$ is closed under taking quotient groups.

Now, it follows from Proposition 6.3 that Z is $\pi(\mathcal{C})'$-isolated in X, and since $Y \leq Z$, then $\mathcal{I}_{\pi(\mathcal{C})'}(X, Y) \leq Z$. Hence, $\mathcal{I}_{\pi(\mathcal{C})'}(X, Y) = \mathcal{I}_{\pi(\mathcal{C})'}(Z, Y), \mathcal{R}_{\pi(\mathcal{C})'}(X, Y) = \mathcal{R}_{\pi(\mathcal{C})'}(Z, Y)$, and $\mathcal{I}_{\pi(\mathcal{C})'}(Z, Y) = \mathcal{R}_{\pi(\mathcal{C})'}(Z, Y)$ by the above remark. $\square$

Proposition 6.5. [22, Proposition 3] *Suppose that $\mathcal{C}$ is a class of groups closed under taking quotient groups, X is a group, and Y is a normal subgroup of X. Then Y is $\mathcal{C}$-separable in X if and only if X/Y is residually a $\mathcal{C}$-group.*

Proof of Theorem 3. Necessity. Since $\mathfrak{G}$ is residually a $\mathcal{C}$-group and $\mathcal{C}$ is closed under taking subgroups, then E is residually a $\mathcal{C}$-group. By Proposition 6.1, the HNN-extension $\mathfrak{G}_{n-1} = \mathrm{HNN}(G, H_n, K_n, \varphi)$ is also residually a $\mathcal{C}$-group. Since $H_n = H_{n+1} = H_n \cap K_n$, then $H_n \leq K_n$, and we get the equality $H_n = K_n$ by applying Proposition 5.3 to $\mathfrak{G}_{n-1}$.

It follows from Proposition 5.2 that $\mathcal{C}_\cap^*(G, H, K, \varphi)$ is an (H, K)-filtration. By Proposition 3.4, $\mathcal{C}_\cap^*(G, H, K, \varphi) \subseteq \mathcal{C}^*(G)$, and therefore $\mathcal{C}^*(G)$ is also an (H, K)-filtration. Hence, H and K are $\mathcal{C}$-separable in G and $\pi(\mathcal{C})'$-isolated in this group by Proposition 6.3.

Sufficiency. Since

$$H_{n+1} = H_n = K_n, \quad K_{n+1} = H_{n+1}\varphi = H_n\varphi = K_n,$$

then $\mathfrak{K}_n = \mathrm{HNN}(K_n, H_{n+1}, K_{n+1}, \varphi)$ is a split extension of K_n by the infinite cyclic group generated by t. Therefore, the mapping of the generators of $\mathfrak{K}_n$ to the corresponding elements of $\mathfrak{G}$ can be extended to an injective homomorphism taking $\mathfrak{K}_n$ onto E. Hence, $\mathfrak{K}_n$ is residually a $\mathcal{C}$-group, and $\mathcal{C}_\cap^*(K_n, H_{n+1}, K_{n+1}, \varphi)$ is a filtration by Proposition 5.2. Because $H_{n+1} = K_n = K_{n+1}$, this family is actually an (H_{n+1}, K_{n+1})-filtration. By Proposition 3.2, it is closed under taking finite intersections and therefore is a strong (H_{n+1}, K_{n+1})-filtration. It follows from Proposition 3.5 that $\mathcal{C}_\cap^*(K_n, H_{n+1}, K_{n+1}, \varphi) = \mathcal{C}_n^*(K_n, H_{n+1}, K_{n+1}, \varphi)$. Hence, $\mathcal{C}_n^*(K_n, H_{n+1}, K_{n+1}, \varphi)$ is also a strong (H_{n+1}, K_{n+1})-filtration.

Since G is residually a $\mathcal{C}$-group and $\mathcal{C}$ is closed under taking subgroups, then K_i $(i \geq 0)$ are also residually $\mathcal{C}$-groups. For any $i \in \{0, 1, ..., n-1\}$, P_{i+1} is central in K_i. Hence, by Proposition 6.4, the $\pi(\mathcal{C})'$-isolator $X_i = \mathcal{I}_{\pi(\mathcal{C})'}(K_i, P_{i+1})$ lies in the center of K_i and coincides with the set $\mathcal{R}_{\pi(\mathcal{C})'}(K_i, P_{i+1})$. Therefore, X_i/P_{i+1} is a periodic $\pi(\mathcal{C})'$-group.

By the hypothesis of the theorem, X_i is $\mathcal{C}$-separable in K_i for any $i \in \{0, 1, ..., n-1\}$, and therefore K_i/X_i is residually a $\mathcal{C}$-group by Proposition 6.5. Again by the hypothesis, K_i is $\mathcal{C}$-regular with respect to P_{i+1}. By Proposition 6.2, H_{i+1} and K_{i+1} are $\pi(\mathcal{C})'$-isolated in G and therefore in K_i. Hence, we can successively apply Proposition 4.4 to the HNN-tuples $(K_i, H_{i+1}, K_{i+1}, \varphi)$, $i = n-1, ..., 1, 0$. As a result, we see that $\mathcal{C}_0^*(G, H, K, \varphi)$ is a strong (H, K)-filtration. By

Proposition 3.4, $\mathcal{C}_0^*(G, H, K, \varphi) \subseteq \mathcal{C}_\cap^*(G, H, K, \varphi)$. Therefore, $\mathcal{C}_\cap^*(G, H, K, \varphi)$ is an (H, K)-filtration, and $\mathfrak{G}$ is residually a $\mathcal{C}$-group by Proposition 5.2. □

Proof of Theorem 4. Since G is residually a $\mathcal{C}$-group and $\mathcal{C}$ is closed under taking subgroups, then K_n is also residually a $\mathcal{C}$-group and, by Proposition 5.5, $\mathcal{C}^*(K_n, H_{n+1}, K_{n+1}, \varphi)$ is an (H_{n+1}, K_{n+1})-filtration. By Proposition 3.2, this family is closed under taking finite intersections and therefore turns out to be a strong (H_{n+1}, K_{n+1})-filtration. By the hypothesis of the theorem, P_{i+1} is $\mathcal{C}$-separable in K_i for each $i \in \{0, 1, \ldots, n-1\}$. Hence, K_i/P_{i+1} is residually a $\mathcal{C}$-group by Proposition 6.5. If we take the set of all primes as π and successively apply Proposition 4.4 to the HNN-tuples $(K_i, H_{i+1}, K_{i+1}, \varphi)$ and the subgroups $X_{i+1} = P_{i+1}$ $(i = n-1, \ldots, 1, 0)$, then we see that $\mathcal{C}^*(G, H, K, \varphi)$ is a strong (H, K)-filtration. Since $\mathcal{C}^*(G, H, K, \varphi) = \mathcal{C}_\cap^*(G, H, K, \varphi)$ by Proposition 3.4, then $\mathcal{C}_\cap^*(G, H, K, \varphi)$ is an (H, K)-filtration. Thus, $\mathfrak{G}$ is residually a $\mathcal{C}$-group by Proposition 5.2. □

Proof of Theorem 1. Statement I follows from Theorem 4. Let us prove Statement II.

We take a number $i \geq 0$ and show that $K_i/P_{i+1} \in \mathcal{C}$.

If $Q \leq H \cap K$ and $Q\varphi = Q$, then $Q \leq H_{i+1} \cap K_{i+1}$ by Proposition 6.2 and $Q \leq P_{i+1}$. Hence, $G/P_{i+1} \cong (G/Q)/(P_{i+1}/Q) \in \mathcal{C}$ and $K_i/P_{i+1} \in \mathcal{C}$ because $\mathcal{C}$ is closed under taking subgroups and quotient groups. If $H \cap Q = 1 = K \cap Q$, then $K_i \cap Q = 1$ and $K_i/P_{i+1} \cong K_i/P_{i+1}(K_i \cap Q) \cong K_iQ/P_{i+1}Q$. Since $Q \leq P_{i+1}Q$, then, as above, $K_iQ/P_{i+1}Q \in \mathcal{C}$.

Thus, $K_i/P_{i+1} \in \mathcal{C}$. It follows that P_{i+1} is $\mathcal{C}$-separable in K_i and, by Proposition 6.3, is $\pi(\mathcal{C})'$-isolated in this group. Therefore, $P_{i+1} = \mathcal{I}_{\pi(\mathcal{C})'}(K_i, P_{i+1})$. We note also that if $N \in \mathcal{C}^*(P_{i+1})$, then K_i/N is an extension of the $\mathcal{C}$-group P_{i+1}/N by the $\mathcal{C}$-group K_i/P_{i+1} and $K_i/N \in \mathcal{C}$ because $\mathcal{C}$ is closed under taking extensions. Hence, $\mathcal{C}^*(P_{i+1}) \subseteq \mathcal{C}^*(K_i)$ and therefore K_i is $\mathcal{C}$-regular with respect to P_{i+1}.

Thus, the statement to be proved follows from Theorem 3. We only need to note that H and K are $\mathcal{C}$-separable in G by Proposition 5.4 and are $\pi(\mathcal{C})'$-isolated in this group by Proposition 6.3. □

Proof of Corollary 1. Because H is finite, the equalities $H_n = H_{n+1} = H_n \cap K_n$ hold for some $n \geq 1$. Therefore, $H_n \leq K_n$. But the subgroups H_n and K_n are finite and isomorphic, hence, $H_n = K_n$. Since G is residually a $\mathcal{C}$-group, then, by Proposition 3.1, there exists a subgroup $Q \in \mathcal{C}^*(G)$ such that $H \cap Q = 1 = K \cap Q$. Therefore, Statement I follows from Theorem 1. Let us prove Statement II.

Since $H \cap Q = 1$, then H_n is embedded into the $\mathcal{C}$-group G/Q and belongs to $\mathcal{C}$ because this class is closed under taking subgroups. The restriction of φ to the finite subgroup H_n has a finite order q. Hence, by Proposition 5.8, the subgroup $E = \text{sgp}\{H_n, t\}$ of $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if q is a $\pi(\mathcal{C})$-number. If $H = G = K$, then $\mathfrak{G} = E$, and the required statement is proved. Otherwise, the inequalities $G \neq H$ and $G \neq K$ hold simultaneously because H and K are finite and isomorphic. Therefore, the statement of the corollary follows from Theorem 1. □

Proof of Corollary 2. I. The conditions of the statement are necessary for $\mathfrak{G}$ to be residually a $\mathcal{C}$-group by Proposition 5.6 and because $\mathcal{C}$ is closed under taking subgroups. Their sufficiency follows from Theorem 1.

II. *Necessity.* By Propositions 5.6 and 6.2, there exists a subgroup $Q \in \mathcal{C}^*(G)$ such that $Q\varphi = Q$ and $Q \leq H_i \cap K_i$ for all $i \geq 1$. Since $\mathcal{C}$ consists of finite groups, Q has a finite index in G and therefore $H_n = H_{n+1}$ for some n. Because $\mathcal{C}$ is closed under taking subgroups, G is residually a $\mathcal{C}$-group. Therefore, Conditions 2 and 3 follow from Theorem 1. Since $\mathcal{C}$ is closed under taking quotient groups, Condition 1 follows from the relations

$$G/H \cong (G/Q)/(H/Q), \quad G/K \cong (G/Q)/(K/Q).$$

Sufficiency. By Proposition 6.2, $H_i \in \mathcal{C}^*(G)$ and $K_i \in \mathcal{C}^*(G)$ for any $i \geq 1$. Hence, $H_n \in \mathcal{C}^*(G)$, $H_n \leq H \cap K$, and $H_n\varphi = H_n$. Since E is residually a $\mathcal{C}$-group and $\mathcal{C}$ is closed under taking subgroups, then H_n is also residually a $\mathcal{C}$-group. Therefore, G is an extension of the residually $\mathcal{C}$-group H_n by the $\mathcal{C}$-group G/H_n. Such an extension is residually a $\mathcal{C}$-group by Proposition 5.1. Thus, it follows from Theorem 1 that $\mathfrak{G}$ is residually a $\mathcal{C}$-group. □

Proposition 6.6. [22, Proposition 18] *If $\mathcal{C}$ is a root class of groups consisting only of periodic groups and closed under taking quotient groups, then any $\pi(\mathcal{C})$-bounded solvable $\mathcal{C}$-group is finite.*

Proposition 6.7. [16, Propositions 1, 2, 3] *If π is a non-empty set of primes, then the following statements hold.*

1) *Any π-bounded abelian group is of finite rank.*
2) *The classes of π-bounded abelian, π-bounded nilpotent, and π-bounded solvable groups are closed under taking subgroups and quotient groups.*
3) *If a π-bounded solvable group is abelian, then it belongs to the class of π-bounded abelian groups.*

Proof of Theorem 2. Let us show that, for any $i \in \{0, 1, \ldots, n-1\}$, K_i is $\mathcal{C}$-regular with respect to P_{i+1}. Then the required statement will follow from Theorem 3.

Suppose that $i \in \{0, 1, \ldots, n-1\}$ and $M \in \mathcal{C}^*(P_{i+1})$. We need to find a subgroup $N \in \mathcal{C}^*(K_i)$ such that $N \cap P_{i+1} = M$.

Let us denote the $\pi(\mathcal{C})'$-isolator $\mathcal{I}_{\pi(\mathcal{C})'}(K_i, M)$ by $\mathfrak{I}$ for brevity. Since G is residually a $\mathcal{C}$-group and $\mathcal{C}$ is closed under taking subgroups, then K_i is also residually a $\mathcal{C}$-group. Therefore, by Proposition 6.4, $\mathfrak{I}$ lies in the center of K_i and coincides with the set $\mathcal{R}_{\pi(\mathcal{C})'}(K_i, M)$. It follows from the equality $\mathfrak{I} = \mathcal{R}_{\pi(\mathcal{C})'}(K_i, M)$ that, if $x \in \mathfrak{I} \cap P_{i+1}$, then $x^q \in M$ for some $\pi(\mathcal{C})'$-number q. But the quotient group P_{i+1}/M belongs to $\mathcal{C}$ and therefore has no $\pi(\mathcal{C})'$-torsion. Hence, $xM = 1$ and $x \in M$. Thus, $\mathfrak{I} \cap P_{i+1} \leq M$ and, because the opposite inclusion is obvious, $\mathfrak{I} \cap P_{i+1} = M$.

Since H and K are $\pi(\mathcal{C})$-bounded, then it follows from Propositions 6.6 and 6.7 that the $\mathcal{C}$-group P_{i+1}/M is $\pi(\mathcal{C})$-bounded and therefore finite. Hence, the subgroup

$$P_{i+1}\mathfrak{I}/\mathfrak{I} \cong P_{i+1}/(P_{i+1} \cap \mathfrak{I}) = P_{i+1}/M$$

is also finite. By the hypothesis of the theorem, $\mathfrak{I}$ is $\mathcal{C}$-separable in K_i. Therefore, by Proposition 6.5, $K_i/\mathfrak{I}$ is residually a $\mathcal{C}$-group and, by Proposition 3.1, there exists a subgroup $N/\mathfrak{I} \in \mathcal{C}^*(K_i/\mathfrak{I})$ such that $N/\mathfrak{I} \cap P_{i+1}\mathfrak{I}/\mathfrak{I} = 1$. Then $N \in \mathcal{C}^*(K_i)$ and, as it is easy to see, $N \cap P_{i+1} = M$. Thus, N is the required subgroup. □

Proposition 6.8. *Suppose that $\mathcal{C}$ is a root class of groups consisting only of periodic groups and closed under taking quotient groups, $H \neq G \neq K$, and there exists $m \geq 0$ such that at least one of the following conditions holds:*

(α) *H_{m+1} and K_{m+1} are finitely generated;*
(β) *H_{m+1} and K_{m+1} are $\pi(\mathcal{C})$-bounded and π'-isolated in K_m for some finite subset π of $\pi(\mathcal{C})$.*

If $\mathfrak{G}$ is residually a $\mathcal{C}$-group, then $H_n = H_{n+1}$ for some n.

Proof. Because $\mathcal{C}$ contains non-trivial groups, $\pi(\mathcal{C})$ is non-empty. Hence, the rank $\operatorname{rk} H_{m+1}$ of H_{m+1} is finite by Proposition 6.7, and since $\operatorname{rk} H_{i+1} \leq \operatorname{rk} H_i$ for any $i \geq 1$, then $\operatorname{rk} H_l = \operatorname{rk} H_{l+1}$

for some $l \geq m + 1$. It follows from the relations $H_l \cong K_l$ and $H_{l+1} \cong K_{l+1}$ that

$$\operatorname{rk} K_l = \operatorname{rk} H_l = \operatorname{rk} H_{l+1} = \operatorname{rk} K_{l+1}$$

and therefore the quotient groups K_l/H_{l+1} and K_l/K_{l+1} are periodic. If H_{m+1} and K_{m+1} are finitely generated, then K_l/H_{l+1} and K_l/K_{l+1} are finite. Let us show that this is also true if (β) holds.

Since H_{m+1} and K_{m+1} are π'-isolated in K_m, then, by Proposition 6.2 applied to the HNN-tuple $(K_m, H_{m+1}, K_{m+1}, \varphi)$, H_{l+1} and K_{l+1} are also π'-isolated in this group. Hence, K_l/H_{l+1} and K_l/K_{l+1} are periodic π-groups, and each of them has a finite number of primary components because π is finite. By Proposition 6.7, these groups are $\pi(\mathcal{C})$-bounded, and since $\pi \subseteq \pi(\mathcal{C})$, all their primary components are finite. Thus, K_l/H_{l+1} and K_l/K_{l+1} are also finite.

By Proposition 6.1, the HNN-extensions

$$\mathfrak{G}_l = \operatorname{HNN}(G, H_{l+1}, K_{l+1}, \varphi), \quad \mathfrak{K}_l = \operatorname{HNN}(K_l, H_{l+1}, K_{l+1}, \varphi)$$

are residually $\mathcal{C}$-groups. If $H_{l+1} = K_l$ or $K_{l+1} = K_l$, then $H_{l+1} \geq K_{l+1}$ or $H_{l+1} \leq K_{l+1}$ and it follows from Proposition 5.3 applied to $\mathfrak{G}_l$ that $H_{l+1} = K_{l+1}$. Hence, we can put $n = l + 1$. Let $H_{l+1} \neq K_l \neq K_{l+1}$. Then, by Proposition 5.6 applied to $\mathfrak{K}_l$, there exists a subgroup $Q \in \mathcal{C}^*(K_l)$ such that $Q \leq H_{l+1} \cap K_{l+1}$ and $Q\varphi = Q$. The $\mathcal{C}$-group K_l/Q is finite: this is obvious if (α) holds, and follows from Propositions 6.6, 6.7 if (β) takes place. By Proposition 6.2, $Q \leq H_i$ for any $i \geq l + 1$. Thus, there exists $n \geq l + 1$ such that $H_n = H_{n+1}$. □

Proposition 6.9. [18, Proposition 10] *If $\mathcal{C}$ is a root class of groups consisting only of periodic groups, then any finite solvable $\pi(\mathcal{C})$-group belongs to $\mathcal{C}$.*

Proposition 6.10. *If $\mathcal{C}$ is a root class of groups consisting only of periodic groups and $\pi(\mathcal{C})$ contains all prime numbers, then all the subgroups of an arbitrary $\pi(\mathcal{C})$-bounded solvable group are $\mathcal{C}$-separable.*

Proof. Let X be a $\pi(\mathcal{C})$-bounded solvable group, and let Y be a subgroup of X. By Theorem 6 from [10], Y is $\mathcal{F}$-separable in X, where $\mathcal{F}$ is the class of all finite groups. Since any homomorphic image of X is a solvable group, then Y turns out to be $\mathcal{FS}$-separable in X, where $\mathcal{FS}$ is the class of all finite solvable groups. By Proposition 6.9, $\mathcal{FS} \subseteq \mathcal{C}$. Hence, Y is $\mathcal{C}$-separable. □

Proposition 6.11. [19, Proposition 8] *If $\mathcal{C}$ is a root class of groups consisting only of periodic groups and X is a $\pi(\mathcal{C})$-bounded nilpotent group, then every $\pi(\mathcal{C})'$-isolated subgroup of X is $\mathcal{C}$-separable in this group.*

Proof of Corollaries 3 and 4. Let us verify that the conditions of Theorem 2 hold.

By Proposition 6.7, H_i and K_i ($i \geq 1$) are $\pi(\mathcal{C})$-bounded abelian groups. Therefore, it follows from Propositions 6.10 and 6.11 that all the $\pi(\mathcal{C})'$-isolated subgroups of K_i ($i \geq 0$) are $\mathcal{C}$-separable in this group. In particular, if $N \in \mathcal{C}^*(P_{i+1})$, then the subgroup $\mathcal{I}_{\pi(\mathcal{C})'}(K_i, N)$ is $\mathcal{C}$-separable in K_i.

The fact that G is residually a $\mathcal{C}$-group follows from

- Proposition 6.10 if $\pi(\mathcal{C})$ contains all primes and G is a $\pi(\mathcal{C})$-bounded solvable group;
- Proposition 6.11 and the absence of $\pi(\mathcal{C})'$-torsion in G if the last group is $\pi(\mathcal{C})$-bounded nilpotent;
- the assumption that $\mathcal{C}$ is closed under taking subgroups if $\mathfrak{G}$ is residually a $\mathcal{C}$-group.

Finally, $H_n = H_{n+1}$ for some $n \geq 1$: it is obvious if $H_n = K_n$, and is guaranteed by Proposition 6.8 if $\mathfrak{G}$ is residually a $\mathcal{C}$-group. Thus, the necessity of the statements of both

corollaries follows from Theorem 2. We only need to note that if $\mathfrak{G}$ is residually a $\mathcal{C}$-group, then G is also residually a $\mathcal{C}$-group and, by Proposition 6.3, has no $\pi(\mathcal{C})'$-torsion.

The group E is $\pi(\mathcal{C})$-bounded solvable as an extension of the $\pi(\mathcal{C})$-bounded abelian group H_n by the infinite cyclic group $\langle t \rangle$, which is also $\pi(\mathcal{C})$-bounded abelian. If $\pi(\mathcal{C})$ contains all primes, then H and K are $\pi(\mathcal{C})'$-isolated in G and, by Proposition 6.10, E is residually a $\mathcal{C}$-group. Therefore, the sufficiency of the statements of the corollaries also follows from Theorem 2. $\quad\square$

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