

On the separability of subgroups of nilpotent groups by root classes of groups

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Abstract. Suppose that \mathcal{C} is a class of groups consisting only of periodic groups and $\mathfrak{P}(\mathcal{C})'$ is the set of prime numbers that do not divide the order of any element of a \mathcal{C} -group. It is easy to see that if a subgroup Y of a group X is \mathcal{C} -separable in this group, then it is $\mathfrak{P}(\mathcal{C})'$ -isolated in X . Let us say that X has the property \mathcal{C} -Sep if all of its $\mathfrak{P}(\mathcal{C})'$ -isolated subgroups are \mathcal{C} -separable. We find a condition that is sufficient for a nilpotent group N to have the property \mathcal{C} -Sep provided \mathcal{C} is a root class. We also prove that if N is torsion-free, then the indicated condition is necessary for this group to have \mathcal{C} -Sep.

1 Introduction

Let us begin with the definitions of the concepts appearing in the title of the article. A subgroup Y of a group X is said to be *separable in X by a class of groups \mathcal{C}* (\mathcal{C} -separable for brevity) if, for any element $x \in X \setminus Y$, there exists a homomorphism σ of X onto a group from \mathcal{C} such that $x\sigma \notin Y\sigma$ (see [17]). If the trivial subgroup of X is \mathcal{C} -separable, then X is called *residually a \mathcal{C} -group*.

The concept of a *root class of groups* has several equivalent definitions [22]. According to one of them, a class of groups \mathcal{C} is called a root class if it contains at least one non-trivial group and is closed under taking subgroups, extensions, and Cartesian products of the form $\prod_{y \in Y} X_y$, where $X, Y \in \mathcal{C}$ and X_y is an isomorphic copy of X for each $y \in Y$. Examples of root classes are the classes of all finite groups, finite p -groups (where p is a prime number), periodic \mathfrak{P} -groups of finite exponent (where \mathfrak{P} is a non-empty set of primes), solvable groups, and torsion-free groups. It is also easy to show that if the intersection of some number of root classes contains a non-trivial group, then it is again a root class.

The notion of a root class was introduced in [10] and allows one to prove many statements at once using the same reasoning. It turns out to be especially useful in studying the residual properties of free constructions of groups (see, e.g., [22–31]).

If X is such a construction and \mathcal{C} is a root class of groups, then the \mathcal{C} -separability of some subgroups of X is quite often one of the necessary and/or sufficient conditions for X to be residually a \mathcal{C} -group. Examples of assertions of this type can be found, in particular, in [1–3, 6, 13, 14, 24–26, 29]. These results become more constructive if the description of \mathcal{C} -separable subgroups is given for the groups from which the construction is composed.

Most of the known facts on the separability of subgroups concern the property of finite separability, i.e., separability by the class of all finite groups. There is also a series of assertions on the separability by the class of finite \mathfrak{P} -groups, where \mathfrak{P} is some (most often one-element) set of primes (see [4, 5, 7, 8, 14, 18, 21, 23]). The separability of subgroups by other classes of groups is not actually studied.

In this article, we consider the separability of subgroups of nilpotent groups by a root class of groups \mathcal{C} and give examples of using the obtained results in studying the property of being residually a \mathcal{C} -group (in the case of free constructions of groups). It follows from Proposition 4.3 below that if \mathcal{C} contains at least one non-periodic group and is closed under taking quotient groups, then it also contains all nilpotent groups of cardinality less than \aleph_∞ . It is clear that all subgroups of such groups are automatically \mathcal{C} -separable. So, in what follows, we consider only the case when the class \mathcal{C} consists of periodic groups.

Everywhere below, if \mathcal{C} is an arbitrary (not necessarily root) class of periodic groups, then we denote by $\mathfrak{P}(\mathcal{C})$ the set of all prime numbers that divide the order of an element of some \mathcal{C} -group. It turns out that $\mathfrak{P}(\mathcal{C})$ plays an important role in the study of \mathcal{C} -separability. To clarify this relationship, we need some notation and definitions.

Throughout the paper, if \mathfrak{P} is a set of primes, then \mathfrak{P}' denotes the set of all the prime numbers that do not belong to \mathfrak{P} . A subgroup Y of a group X is said to be \mathfrak{P}' -isolated in this group if, for any $x \in X$, $q \in \mathfrak{P}'$, it follows from the inclusion $x^q \in Y$ that $x \in Y$. If the trivial subgroup of X is \mathfrak{P}' -isolated, then X is said to have no \mathfrak{P}' -torsion. We note that if \mathfrak{P} contains all primes, then every subgroup is \mathfrak{P}' -isolated.

It is known (see Proposition 4.5 below) that if \mathcal{C} is a class of groups consisting only of periodic groups, then any \mathcal{C} -separable subgroup is $\mathfrak{P}(\mathcal{C})'$ -isolated. We say that a group X has the property \mathcal{C} -Sep if the inverse statement also holds, i.e., all the $\mathfrak{P}(\mathcal{C})'$ -isolated subgroups of X are \mathcal{C} -separable. In this article, we study the following question: what conditions are sufficient for a nilpotent group to have the property \mathcal{C} -Sep if \mathcal{C} is a root class of groups consisting only of periodic groups? The result of this research is the notion of a \mathcal{C} -bounded nilpotent group; its definition is given in the next section.

2 Main results

Suppose that \mathcal{C} is a class of groups consisting only of periodic groups and A is an abelian group. If $p \in \mathfrak{P}(\mathcal{C})$, then we refer to the p -power torsion subgroup of A as the *primary $\mathfrak{P}(\mathcal{C})$ -component* of this group. Let us say that A

- is *weakly \mathcal{C} -bounded* if, for any quotient group B of A , all the primary $\mathfrak{P}(\mathcal{C})$ -components of B are of finite exponent;
- is *\mathcal{C} -bounded* if, for any quotient group B of A , each primary $\mathfrak{P}(\mathcal{C})$ -component of B has a finite exponent and a cardinality not exceeding the cardinality of some \mathcal{C} -group.

We call a nilpotent group (weakly) *\mathcal{C} -bounded* if it has at least one finite central series with (weakly) \mathcal{C} -bounded abelian factors. The classes of \mathcal{C} -bounded abelian, \mathcal{C} -bounded nilpotent, weakly \mathcal{C} -bounded abelian, and weakly \mathcal{C} -bounded nilpotent groups are denoted below by $\mathcal{C}\text{-}\mathcal{BA}$, $\mathcal{C}\text{-}\mathcal{BN}$, $\mathcal{C}\text{-}w\mathcal{BA}$, and $\mathcal{C}\text{-}w\mathcal{BN}$ respectively.

Let us note that if the class \mathcal{C} is a root class, then a finite \mathcal{C} -group can have an arbitrarily large order (see Proposition 4.1 below), and therefore, any finitely generated nilpotent group is \mathcal{C} -bounded. It also follows from Proposition 4.3 that a weakly \mathcal{C} -bounded nilpotent group X is \mathcal{C} -bounded if \mathcal{C} contains infinite groups and the cardinality of X is less than \aleph_∞ .

The first result of this article is Theorem 2.1 that describes the abelian groups having the property $\mathcal{C}\text{-}\mathcal{Sep}$.

Theorem 2.1. *If \mathcal{C} is a root class of groups consisting only of periodic groups and X is an abelian group, then the following statements hold.*

1. *If X has the property $\mathcal{C}\text{-}\mathcal{Sep}$, then it is weakly \mathcal{C} -bounded.*
2. *If X is weakly \mathcal{C} -bounded, then it has the properties $\mathcal{C}\text{-}\mathcal{Sep}$ and $\mathcal{P}\text{-}\mathcal{Sep}$, where*

$$\mathcal{P} = \bigcup_{p \in \mathfrak{P}(\mathcal{C})} \mathcal{F}_p$$

and \mathcal{F}_p is the class of finite p -groups.

Thus, an abelian group has the property $\mathcal{C}\text{-}\mathcal{Sep}$ if and only if it is weakly \mathcal{C} -bounded. However, this property may not hold for weakly \mathcal{C} -bounded nilpotent groups: the corresponding example is constructed in Section 9. The main result of this article is Theorem 2.2 given below, which says that any \mathcal{C} -bounded nilpotent group enjoys $\mathcal{C}\text{-}\mathcal{Sep}$. In fact, this theorem states somewhat more, so its formulation must be preceded by several definitions.

Suppose again that \mathfrak{P} is an arbitrary set of primes, X is a group, and Y is a subgroup of X . It is easy to see that the intersection of any number of \mathfrak{P}' -isolated sub-

groups of X is in turn a \mathfrak{P}' -isolated subgroup, and therefore, there exists a smallest \mathfrak{P}' -isolated subgroup containing Y . It is called the \mathfrak{P}' -isolator of Y in X , and we denote it by $\mathfrak{P}'\text{-}\mathfrak{I}\mathfrak{s}(X, Y)$. We also denote by $\mathfrak{P}'\text{-}\mathfrak{R}\mathfrak{t}(X, Y)$ the set of \mathfrak{P}' -roots extracted in X from the elements of Y . More accurately, an element $x \in X$ belongs to $\mathfrak{P}'\text{-}\mathfrak{R}\mathfrak{t}(X, Y)$ if there exists a \mathfrak{P}' -number q such that $x^q \in Y$. Obviously, the set $\mathfrak{P}'\text{-}\mathfrak{R}\mathfrak{t}(X, Y)$ is contained in the subgroup $\mathfrak{P}'\text{-}\mathfrak{I}\mathfrak{s}(X, Y)$ and coincides with the latter if and only if it is itself a subgroup.

Theorem 2.2. *Suppose that \mathcal{C} is a root class of groups consisting only of periodic groups, X is a \mathcal{C} - \mathcal{BN} -group, and Y is a subgroup of X . Then the $\mathfrak{P}(\mathcal{C})'$ -isolator $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{I}\mathfrak{s}(X, Y)$ is \mathcal{C} -separable in X and coincides with the set $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{R}\mathfrak{t}(X, Y)$. In particular, X has the property \mathcal{C} - $\mathcal{C}\mathfrak{e}\mathfrak{p}$. If X is $\mathfrak{P}(\mathcal{C})'$ -torsion-free, then Y and $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{I}\mathfrak{s}(X, Y)$ have the same nilpotency classes.*

We note that the requirement for the group X from Theorem 2.2 to be $\mathfrak{P}(\mathcal{C})'$ -torsion-free is essential for the equality of the nilpotency classes of the groups Y and $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{I}\mathfrak{s}(X, Y)$. For example, if Y is an infinite cyclic group, Z is a finite nilpotent $\mathfrak{P}(\mathcal{C})'$ -group of class c , and $X = Y \times Z$, then $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{I}\mathfrak{s}(X, Y) = X$, and therefore, the nilpotency class of $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{I}\mathfrak{s}(X, Y)$ is equal to c , while Y is a group of class 1.

Theorem 2.1 shows that an abelian group with the property \mathcal{C} - $\mathcal{C}\mathfrak{e}\mathfrak{p}$ need not be \mathcal{C} -bounded. At the same time, the following assertion holds.

Corollary 2.3. *Let \mathcal{C} be a root class of groups consisting only of periodic groups. A torsion-free nilpotent group has the property \mathcal{C} - $\mathcal{C}\mathfrak{e}\mathfrak{p}$ if and only if it is \mathcal{C} -bounded.*

It turns out that the analogs of Theorem 2.2 and Corollary 2.3 do not hold even for supersolvable groups. For example, if \mathcal{C} is a root class of groups consisting only of periodic groups, then, by the main theorem from [30] and Proposition 8.7 given below, the supersolvable Baumslag–Solitar group

$$BS(1, -1) = \langle a, b; a^{-1}ba = b^{-1} \rangle$$

is residually a \mathcal{C} -group if and only if $2 \in \mathfrak{P}(\mathcal{C})$. At the same time, $BS(1, -1)$ is torsion-free, and therefore, its trivial subgroup is always $\mathfrak{P}(\mathcal{C})'$ -isolated.

Throughout the paper, if \mathcal{C} is a root class of groups consisting only of periodic groups, then we denote by $\mathcal{C}\text{-}\mathcal{BN}_{\mathfrak{P}(\mathcal{C})'}$ the class of all $\mathfrak{P}(\mathcal{C})'$ -torsion-free \mathcal{C} - \mathcal{BN} -groups. This class arises quite naturally because, by Theorem 2.2, only such a \mathcal{C} - \mathcal{BN} -group is residually a \mathcal{C} -group. The theorem given below serves as a partial generalization of Theorem 2.2.

Theorem 2.4. *Suppose that \mathcal{C} is a root class of groups consisting only of periodic groups and $\mathcal{N}\mathcal{C}$ is a class of nilpotent \mathcal{C} -groups. Suppose also that X is residually a $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{P}(\mathcal{C})}$ -group, Y is a subgroup of X , and there exists a homomorphism σ of X onto a $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{P}(\mathcal{C})}$ -group that acts injectively on Y . Then the following statements hold.*

1. *The $\mathfrak{P}(\mathcal{C})'$ -isolator $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{I}\mathfrak{S}(X, Y)$ is $\mathcal{N}\mathcal{C}$ -separable in X , coincides with the set $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{Rt}(X, Y)$, and has the same nilpotency class as Y . The group X is residually an $\mathcal{N}\mathcal{C}$ -group and therefore has no $\mathfrak{P}(\mathcal{C})'$ -torsion.*
2. *The homomorphism σ acts injectively on the subgroup $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{I}\mathfrak{S}(X, Y)$; as a result, the latter belongs to $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{P}(\mathcal{C})}$.*

We note that if $X \in \mathcal{C}\text{-}\mathcal{B}\mathcal{N}$, then any homomorphic image of X is nilpotent. Therefore, Theorem 2.2 actually states that the subgroup $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{I}\mathfrak{S}(X, Y)$ is separable in X by the class $\mathcal{N}\mathcal{C}$ of all nilpotent \mathcal{C} -groups. A similar argument does not work for residually $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{P}(\mathcal{C})}$ -groups, and as a result, the class $\mathcal{N}\mathcal{C}$ appears in Theorem 2.4 explicitly.

We also note that it is essential for the \mathcal{C} -separability of $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{I}\mathfrak{S}(X, Y)$ that there exists a homomorphism of X onto a $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{P}(\mathcal{C})}$ -group acting injectively on Y . For example, any non-abelian free group is residually a finitely generated torsion-free nilpotent group [11, 16], but it also contains a finitely generated isolated subgroup that is not separable by the class \mathcal{F}_p of all finite p -groups for any prime number p (see [4]).

3 Some applications

In this section, we formulate a number of assertions which concern the residual properties of free constructions of groups and can be deduced from known results using Theorem 2.4 and some other properties of $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}$ -groups. The free construction under consideration is always composed of residually \mathcal{C} -groups, and Theorem 2.4 completely generalizes Theorem 2.2 if they are applied to such groups. Therefore, it makes no sense to formulate separate corollaries from Theorem 2.2.

Throughout this section, we assume that \mathcal{C} is a root class of groups consisting only of periodic groups,

$$P = \langle A * B; U \rangle$$

is the free product of groups A and B with an amalgamated subgroup U , and

$$G^* = \langle G, t; t^{-1}Ht = K, \varphi \rangle$$

is the HNN-extension of a group G with subgroups H and K associated by an isomorphism $\varphi: H \rightarrow K$ (the definitions of these constructions can be found, e.g., in [15]). It is also assumed that $A \neq U \neq B$ and $H \neq G \neq K$.

Theorem 3.1. *Suppose that U is a retract of B , A is residually a \mathcal{C} - $\mathcal{BN}_{\mathfrak{P}(\mathcal{C})}$ -group, and there exists a homomorphism of A onto a \mathcal{C} - $\mathcal{BN}_{\mathfrak{P}(\mathcal{C})}$ -group that acts injectively on U . Then P is residually a \mathcal{C} -group if and only if B has the same property.*

If X is a group and Y is a normal subgroup of X , then the restrictions on Y of all inner automorphisms of X compose a subgroup of the automorphism group $\text{Aut } Y$, which we denote by $\text{Aut}_X(Y)$. Obviously, if U is normal in A and B , then it is normal in P and the group $\text{Aut}_P(U)$ is generated by its subgroups $\text{Aut}_A(U)$ and $\text{Aut}_B(U)$.

Theorem 3.2. *Suppose that U is normal in A and B , $\text{Aut}_P(U)$ is abelian or coincides with $\text{Aut}_A(U)$ or $\text{Aut}_B(U)$. Suppose also that A and B are residually \mathcal{C} - $\mathcal{BN}_{\mathfrak{P}(\mathcal{C})}$ -groups and have homomorphisms onto \mathcal{C} - $\mathcal{BN}_{\mathfrak{P}(\mathcal{C})}$ -groups that act injectively on U . Then P is residually a \mathcal{C} -group if and only if U is $\mathfrak{P}(\mathcal{C})'$ -isolated in A and B .*

Theorem 3.3. *Suppose that $K_0 = G$, $H_1 = H$, $K_1 = K$, and $H_{i+1} = H_i \cap K_i$, $K_{i+1} = H_{i+1}\varphi$ for all $i \geq 1$. Suppose also that H and K lie in the center of G and there exists $m \geq 1$ such that H_m and K_m are finitely generated. If G is residually a \mathcal{C} - $\mathcal{BN}_{\mathfrak{P}(\mathcal{C})}$ -group and has a homomorphism σ onto a \mathcal{C} - $\mathcal{BN}_{\mathfrak{P}(\mathcal{C})}$ -group that acts injectively on HK , then G^* is residually a \mathcal{C} -group if and only if*

- 1) $H_n = K_n$ for some $n > m$;
- 2) H and K are $\mathfrak{P}(\mathcal{C})'$ -isolated in G ;
- 3) $\bigcap_{N \in \Omega} N = 1$, where Ω is the family of subgroups of H_n defined as follows: $N \in \Omega$ if and only if H_n/N is a finite $\mathfrak{P}(\mathcal{C})$ -group, $N\varphi = N$, and the automorphism of H_n/N induced by φ has order a $\mathfrak{P}(\mathcal{C})$ -number.

If H and K are infinite cyclic subgroups, Theorem 3.3 acquires the following, simpler formulation.

Theorem 3.4. *Suppose that H and K are infinite cyclic subgroups lying in the center of G . Suppose also that G is residually a \mathcal{C} - $\mathcal{BN}_{\mathfrak{P}(\mathcal{C})}$ -group and has a homomorphism onto a \mathcal{C} - $\mathcal{BN}_{\mathfrak{P}(\mathcal{C})}$ -group that acts injectively on HK . Then G^* is residually a \mathcal{C} -group if and only if*

- 1) $H/H \cap K$ and $K/H \cap K$ are of the same order;

- 2) H and K are $\mathfrak{B}(\mathcal{C})'$ -isolated in G ;
- 3) $2 \in \mathfrak{B}(\mathcal{C})$, unless $H \cap K$ lies in the center of G^* .

In what follows, it is assumed that Γ is a non-empty undirected connected graph with a vertex set \mathcal{V} and an edge set \mathcal{E} (loops and multiple edges are allowed). Let us construct an oriented graph of groups $\mathcal{G}(\Gamma)$ over Γ . To do this, we first choose arbitrarily the directions for all edges of Γ and, for every edge $e \in \mathcal{E}$, denote by $e(1)$ and $e(-1)$ the vertices that are the ends of e . Then we assign to each vertex $v \in \mathcal{V}$ some group G_v and to each edge $e \in \mathcal{E}$ a group H_e and injective homomorphisms $\varphi_{+e}: H_e \rightarrow G_{e(1)}$, $\varphi_{-e}: H_e \rightarrow G_{e(-1)}$. The result is the graph of groups

$$\mathcal{G}(\Gamma) = (\Gamma, G_v (v \in \mathcal{V}), H_e (e \in \mathcal{E}), \varphi_{\varepsilon e} (e \in \mathcal{E}, \varepsilon = \pm 1)).$$

The following two theorems deal with the fundamental group $\pi_1(\mathcal{G}(\Gamma))$ of $\mathcal{G}(\Gamma)$, the definition of which can be found in [20]. Recall that if the graph Γ is a tree, then $\pi_1(\mathcal{G}(\Gamma))$ is said to be the *tree product* of the groups G_v ($v \in \mathcal{V}$) [12].

Theorem 3.5. *Suppose that Γ is a finite tree and $H_e\varphi_{\varepsilon e}$ is a proper central subgroup of $G_{e(\varepsilon)}$ for all $e \in \mathcal{E}$, $\varepsilon = \pm 1$. Suppose also that, for each $v \in \mathcal{V}$, G_v is residually a $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{B}(\mathcal{C})}$ -group and has a homomorphism onto a $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{B}(\mathcal{C})}$ -group that acts injectively on all the subgroups $H_e\varphi_{\varepsilon e}$ ($e \in \mathcal{E}$, $\varepsilon = \pm 1$, $v = e(\varepsilon)$). Then $\pi_1(\mathcal{G}(\Gamma))$ is residually a \mathcal{C} -group if and only if, for any $e \in \mathcal{E}$, $\varepsilon = \pm 1$, $H_e\varphi_{\varepsilon e}$ is $\mathfrak{B}(\mathcal{C})'$ -isolated in $G_{e(\varepsilon)}$.*

We say that $\mathcal{G}(\Gamma)$ is a *graph of groups with central trivially intersecting edge subgroups* if, for each $v \in \mathcal{V}$, the subgroup

$$H_v = \text{sgp}\{H_e\varphi_{\varepsilon e} \mid e \in \mathcal{E}, \varepsilon = \pm 1, v = e(\varepsilon)\}$$

lies in the center of G_v and is the direct product of the subgroups generating it.

Theorem 3.6. *Suppose that $\mathcal{G}(\Gamma)$ is a graph of groups with central trivially intersecting edge subgroups. Suppose also that, for each $v \in \mathcal{V}$, G_v is residually a $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{B}(\mathcal{C})}$ -group and has a homomorphism onto a $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{B}(\mathcal{C})}$ -group that acts injectively on H_v . Then the following statements hold.*

1. *If, for each $v \in \mathcal{V}$, H_v is $\mathfrak{B}(\mathcal{C})'$ -isolated in G_v , then $\pi_1(\mathcal{G}(\Gamma))$ is residually a \mathcal{C} -group.*
2. *Suppose that $\mathcal{G}(\Gamma)$ is finite and $H_e\varphi_{\varepsilon e} \neq G_{e(\varepsilon)}$ for all $e \in \mathcal{E}$, $\varepsilon = \pm 1$. Then $\pi_1(\mathcal{G}(\Gamma))$ is residually a \mathcal{C} -group if and only if, for any $e \in \mathcal{E}$, $\varepsilon = \pm 1$, $H_e\varphi_{\varepsilon e}$ is $\mathfrak{B}(\mathcal{C})'$ -isolated in $G_{e(\varepsilon)}$.*

The remaining part of the article is organized as follows. In Sections 5 and 6, we give equivalent definitions of a (weakly) \mathcal{C} -bounded abelian group, introduce the notion of a (weakly) \mathcal{C} -bounded solvable group, and establish a number of properties of \mathcal{C} -bounded groups. In Sections 7 and 8, all the theorems and corollaries formulated above are proved. Section 4 contains some auxiliary assertions used in the proofs. In Section 9, we construct the above-mentioned example of a weakly \mathcal{C} -bounded nilpotent group that does not have the property $\mathcal{C}\text{-}\mathfrak{Sep}$.

4 Some auxiliary statements

The next three assertions describe some families of groups that necessarily belong to a given root class.

Proposition 4.1 ([31, Proposition 8]). *Let \mathcal{C} be a root class of groups consisting only of periodic groups. A finite solvable group belongs to \mathcal{C} if and only if its order is a $\mathfrak{B}(\mathcal{C})$ -number. In particular, \mathcal{C} contains all finite p -groups for some prime number p .*

Proposition 4.2 ([29, Proposition 17]). *If \mathcal{C} is a root class of groups consisting only of periodic groups, then an arbitrary \mathcal{C} -group is of finite exponent.*

Proposition 4.3. *If \mathcal{C} is a root class of groups, X is a non-trivial \mathcal{C} -group, and c is the cardinality of X , then the following statements hold.*

1. *If \mathcal{C} contains at least one non-periodic group, then it includes all free solvable groups whose cardinalities do not exceed 2^c . In particular, \mathcal{C} contains any free solvable group of cardinality less than \aleph_∞ .*
2. *If \mathcal{C} consists only of periodic groups, then it includes all periodic solvable $\mathfrak{B}(\mathcal{C})$ -groups of finite exponent whose cardinalities do not exceed 2^c . In particular, together with some infinite group, \mathcal{C} contains any periodic solvable $\mathfrak{B}(\mathcal{C})$ -group that has a finite exponent and a cardinality less than \aleph_∞ .*

Proof. Let us prove statements 1. and 2. simultaneously.

Suppose that $Y^{(\alpha)}$ and $Y^{(\beta)}$ are a free solvable group and a periodic solvable $\mathfrak{B}(\mathcal{C})$ -group of finite exponent respectively, whose cardinalities do not exceed 2^c (hereinafter, groups with index β are considered only if \mathcal{C} consists of periodic groups and therefore the set $\mathfrak{B}(\mathcal{C})$ is defined). Suppose also that $Z^{(\alpha)}$ is an infinite cyclic group, q is the exponent of $Y^{(\beta)}$ (which is a $\mathfrak{B}(\mathcal{C})$ -number), and $Z^{(\beta)}$ is a cyclic group of order q . We note that, if \mathcal{C} includes at least one non-periodic group, then $Z^{(\alpha)} \in \mathcal{C}$ because \mathcal{C} is closed under taking subgroups; if \mathcal{C} consists only of periodic groups, then $Z^{(\beta)} \in \mathcal{C}$ by Proposition 4.1.

Let us put

$$\mathcal{J} = \prod_{x \in X} X_x, \quad D^{(k)} = \prod_{i \in \mathcal{J}} Z_i^{(k)},$$

where $k \in \{\alpha, \beta\}$, \prod stands for Cartesian product, X_x and $Z_i^{(k)}$ are isomorphic copies of X and $Z^{(k)}$ respectively for all $x \in X, i \in \mathcal{J}$. Then $\mathcal{J}, D^{(\alpha)}, D^{(\beta)} \in \mathcal{C}$ by the definition of a root class, and $2^c \leq \text{card } \mathcal{J} \leq 2^{c \cdot c}$.

Let $F^{(k)}$ be an arbitrary factor of the derived series of $Y^{(k)}$, $k \in \{\alpha, \beta\}$. Then $F^{(\alpha)}$ is a free abelian group and $F^{(\beta)}$ is a periodic abelian group whose exponent is finite and divides q . The latter means that $F^{(\beta)}$ has a finite number of p -power torsion subgroups. By the first Prüfer theorem, each of these subgroups can be decomposed into a direct product of cyclic subgroups whose orders also divide q . Therefore, it follows from the relations

$$\text{card } F^{(k)} \leq \text{card } Y^{(k)} \leq 2^c, \quad k \in \{\alpha, \beta\},$$

that $F^{(k)} \leq D^{(k)}$. Thus, $F^{(k)} \in \mathcal{C}$ and $Y^{(k)} \in \mathcal{C}$ because \mathcal{C} is closed under taking subgroups and extensions. □

Proposition 4.4 ([29, Proposition 3]). *If \mathcal{C} is an arbitrary class of groups, X is a group, and Y is a normal subgroup of X , then the following statements hold.*

1. *If X/Y is residually a \mathcal{C} -group, then Y is \mathcal{C} -separable in X .*
2. *If \mathcal{C} is closed under taking quotient groups and Y is \mathcal{C} -separable in X , then X/Y is residually a \mathcal{C} -group.*

Proposition 4.5 ([26, Proposition 5]). *If \mathcal{C} is a class of groups consisting only of periodic groups, X is a group, and Y is a \mathcal{C} -separable subgroup of X , then Y is $\mathfrak{B}(\mathcal{C})'$ -isolated in X .*

In what follows, if \mathcal{C} is a class of groups and X is a group, then $\mathcal{C}^*(X)$ denotes the family of normal subgroups of X such that $Y \in \mathcal{C}^*(X)$ if and only if $X/Y \in \mathcal{C}$.

Proposition 4.6 ([26, Proposition 1]). *If \mathcal{C} is a class of groups closed under taking subgroups and finite direct products, then, for any group X , the intersection of a finite number of subgroups from $\mathcal{C}^*(X)$ is again a subgroup from this family.*

Suppose that \mathcal{C} is a class of groups, X is a group, and Y is a subgroup of X . It is easy to see that, in X , the intersection of any number of \mathcal{C} -separable subgroups is again a \mathcal{C} -separable subgroup. Therefore, there exists the smallest \mathcal{C} -separable subgroup of X containing Y . We refer to this subgroup as the \mathcal{C} -closure of Y in X

and denote it by $\mathcal{C}\text{-}\mathfrak{CI}(X, Y)$. It follows from Proposition 4.5 that if \mathcal{C} consists only of periodic groups, then, for any X and Y , the following relation holds:

$$\mathfrak{B}(\mathcal{C})'\text{-}\mathfrak{S}\cong(X, Y) \leq \mathcal{C}\text{-}\mathfrak{CI}(X, Y).$$

Proposition 4.7 ([23, Proposition 3]). *Given an arbitrary class of groups \mathcal{C} , the equality*

$$\mathcal{C}\text{-}\mathfrak{CI}(X, Y) = \bigcap_{N \in \mathcal{C}^*(X)} YN$$

holds for any group X and its subgroup Y .

Proposition 4.8 ([23, Proposition 4]). *Suppose that \mathcal{C} is an arbitrary class of groups, X is residually a \mathcal{C} -group, and Y is a subgroup of X . If Y is a nilpotent group of class c , then $\mathcal{C}\text{-}\mathfrak{CI}(X, Y)$ is also a nilpotent group of class c .*

Proposition 4.9 ([9, Theorems 5.3, 5.6–5.8]). *If \mathfrak{P} is a set of primes, X is a locally nilpotent group, and Y is a subgroup of X , then the following statements hold.*

1. $\mathfrak{P}'\text{-}\mathfrak{Rt}(X, Y) = \mathfrak{P}'\text{-}\mathfrak{S}\cong(X, Y)$.
2. *If Y is \mathfrak{P}' -isolated in X , then the normalizer of Y in X is also \mathfrak{P}' -isolated in X .*
3. *If X is \mathfrak{P}' -torsion-free, then all the members of its upper central series are \mathfrak{P}' -isolated in X . The extraction of \mathfrak{P}' -roots is unique in such a group.*

Proposition 4.10 ([17, Lemma 2]). *If X is a nilpotent group of class c , then, for each $y \in X^{n^c}$, the equation $x^n = y$ is solvable in X .*

5 The classes of \mathcal{C} -bounded abelian, nilpotent, and solvable groups

Let \mathcal{C} be a class of groups consisting only of periodic groups, and let A be an abelian group. Consider the following set of conditions:

- (1) A is of finite rank;
- (2) \mathcal{C} an arbitrary quotient group of A does not contain p -quasicyclic subgroups for any $p \in \mathfrak{B}(\mathcal{C})$;
- (3) \mathcal{C} $A \in \mathcal{C}\text{-}w\mathcal{BA}$;
- (4) \mathcal{C} each primary $\mathfrak{B}(\mathcal{C})$ -component of A has a finite exponent and a cardinality not exceeding the cardinality of some \mathcal{C} -group;
- (5) \mathcal{C} $A \in \mathcal{C}\text{-}\mathcal{BA}$.

Proposition 5.1. *If \mathcal{C} is a root class of groups consisting only of periodic groups and A is an abelian group, then the following statements hold.*

1. $(2)_e \Rightarrow (1)$.
2. $(2)_e \Leftrightarrow (3)_e$.
3. $(3)_e \wedge (4)_e \Leftrightarrow (5)_e$.

Proof. 1. Assume that the rank of A is infinite. Then A contains a free abelian subgroup B of infinite rank, and this subgroup can be mapped homomorphically onto a p -quasicyclic group for any prime number p . Because A is abelian, any homomorphism of B can be extended to a homomorphism of A . Since \mathcal{C} contains non-trivial groups, $\mathfrak{B}(\mathcal{C}) \neq \emptyset$. Therefore, A does not satisfy $(2)_e$.

2. The sufficiency of the statement is obvious; let us verify the necessity. Assume that a quotient group B of A has elements b_1, b_2, b_3, \dots whose orders equal p, p^2, p^3, \dots respectively for some $p \in \mathfrak{B}(\mathcal{C})$. The group $N = \text{sgp}\{b_1, b_2, b_3, \dots\}$ is countable and, by the second Prüfer theorem, either decomposes into a direct product of cyclic subgroups, the orders of which can be arbitrarily large, or contains a non-trivial element b of infinite height. In the first case, N can be mapped homomorphically onto a p -quasicyclic group, and hence there exists a quotient group of A containing such a subgroup. Let us show that, in the second case, there is also a homomorphism of A onto a p -quasicyclic group and thus A does not satisfy $(2)_e$.

It is well known (see, e.g., [19, Section 4.1]) that B can be embedded into a divisible abelian group D and that the latter can be decomposed into a direct product of quasicyclic groups and groups isomorphic to the additive group of rational numbers. Since N is a periodic p -group, this decomposition has p -quasicyclic factors and there exists a homomorphism σ of D onto one of these factors mapping b to a non-trivial element. Since the height of b is infinite, the restriction of σ to B cannot have a finite image. Therefore, the composition of the natural homomorphism $A \rightarrow B$ and the specified restriction is the desired mapping.

3. As above, only the necessity has to be proved. Let M be a subgroup of A . It follows from statements 1. and 2. that M is of finite rank. Denote by N the subgroup generated by some maximal linearly independent subset of M . Since all the elements of M/N have finite order, the torsion subgroup $\tau(A/M)$ of A/M is a homomorphic image of the torsion subgroup $\tau(A/N)$ of A/N . Because $\tau(A/M)$ decomposes into the direct product of its p -power torsion subgroups, each of these subgroups is also a homomorphic image of $\tau(A/N)$ and hence is a quotient group of the corresponding p -power torsion subgroup of A/N . Therefore, it suffices to show that any primary $\mathfrak{B}(\mathcal{C})$ -component of A/N has cardinality not exceeding the cardinality of some \mathcal{C} -group.

So let $T/N = \tau_p(A/N)$ be a primary $\mathfrak{B}(\mathcal{C})$ -component of A/N . Denote by q the exponent of this group, which is finite by condition (3) $_{\mathcal{C}}$. Let σ be the endomorphism of T that raises each of its elements to the power q . Then $T\sigma \leq N$ and the kernel S of σ is contained in the corresponding primary $\mathfrak{B}(\mathcal{C})$ -component $\tau_p(A)$ of A . By condition (4) $_{\mathcal{C}}$, the cardinality of $\tau_p(A)$ does not exceed the cardinality c of some \mathcal{C} -group. As for N , it is finitely generated and therefore at most countable. Hence, if S is infinite, then the cardinalities of T and T/N do not exceed c . Otherwise, T is an extension of a finite group by a finitely generated one. Therefore, it is finitely generated, the quotient group T/N is finite and, by Proposition 4.1, has an order that does not exceed the order of some \mathcal{C} -group. \square

It turns out that, in some propositions given below and concerning \mathcal{C} -bounded groups, nilpotency is too strong a restriction and can be replaced by solvability. Therefore, we supplement the list of introduced concepts and call a solvable group (weakly) \mathcal{C} -bounded if it has at least one finite subnormal series with (weakly) \mathcal{C} -bounded abelian factors. Denote the classes of \mathcal{C} -bounded and weakly \mathcal{C} -bounded solvable groups by $\mathcal{C}\text{-}\mathcal{B}\mathcal{S}$ and $\mathcal{C}\text{-}w\mathcal{B}\mathcal{S}$ respectively.

Proposition 5.2. *If \mathcal{C} is a root class of groups consisting only of periodic groups, then the following statements hold.*

1. *The classes $\mathcal{C}\text{-}\mathcal{B}\mathcal{A}$, $\mathcal{C}\text{-}w\mathcal{B}\mathcal{A}$, $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}$, $\mathcal{C}\text{-}w\mathcal{B}\mathcal{N}$, $\mathcal{C}\text{-}\mathcal{B}\mathcal{S}$, and $\mathcal{C}\text{-}w\mathcal{B}\mathcal{S}$ are closed under taking subgroups, quotient groups, and finite direct products.*
2. *Let X be an abelian group. If $X \in \mathcal{C}\text{-}\mathcal{B}\mathcal{S}$ ($X \in \mathcal{C}\text{-}w\mathcal{B}\mathcal{S}$), then $X \in \mathcal{C}\text{-}\mathcal{B}\mathcal{A}$ (respectively $X \in \mathcal{C}\text{-}w\mathcal{B}\mathcal{A}$).*

Proof. 1. If A is an abelian group and B is a subgroup of A , then every homomorphic image of the quotient group A/B is simultaneously a homomorphic image of A , and every homomorphic image of B embeds into some homomorphic image of A . Hence, if A belongs to the class $\mathcal{C}\text{-}\mathcal{B}\mathcal{A}$ or $\mathcal{C}\text{-}w\mathcal{B}\mathcal{A}$, then the same class contains B and A/B .

Suppose that U and V are $\mathcal{C}\text{-}w\mathcal{B}\mathcal{A}$ -groups and P is their direct product. If Q is a subgroup of P , then P/Q is an extension of $UQ/Q \cong U/U \cap Q$ by

$$(P/Q)/(UQ/Q) \cong P/UQ = UV/UW \cong V/W(V \cap U) = V/W,$$

where W is the image of Q under the canonical projection of P onto V . It follows from the inclusions $U, V \in \mathcal{C}\text{-}w\mathcal{B}\mathcal{A}$ that all the primary $\mathfrak{B}(\mathcal{C})$ -components of $U/U \cap Q$ and V/W are of finite exponent. Hence, the primary $\mathfrak{B}(\mathcal{C})$ -components

of P/Q have the same property and therefore $P \in \mathcal{C}\text{-}w\mathcal{BA}$. Each primary $\mathfrak{P}(\mathcal{C})$ -component of P is the direct product of the corresponding primary $\mathfrak{P}(\mathcal{C})$ -components of U and V . Therefore, if $U, V \in \mathcal{C}\text{-}\mathcal{BA}$, then P satisfies (4) $_{\mathcal{C}}$ because \mathcal{C} is closed under taking finite direct products, and $P \in \mathcal{C}\text{-}\mathcal{BA}$ by Proposition 5.1.

Thus, statement 1. holds for the classes $\mathcal{C}\text{-}w\mathcal{BA}$ and $\mathcal{C}\text{-}\mathcal{BA}$. Suppose now that X and Y are nilpotent (solvable) groups,

$$1 = X_0 \leq X_1 \leq \dots \leq X_m = X \quad \text{and} \quad 1 = Y_0 \leq Y_1 \leq \dots \leq Y_n = Y$$

are their central (subnormal) series, and Z is a subgroup of X . Without loss of generality, we can assume that $m = n$. Then the subgroups $X_i \times Y_i$, $X_i \cap Z$, and (if Z is normal in X) $X_i Z/Z$ ($0 \leq i \leq n$) form central (subnormal) series of $X \times Y$, Z , and X/Z . The factors of these series are isomorphic respectively to the direct products, subgroups, and homomorphic images of X_{i+1}/X_i and Y_{i+1}/Y_i ($0 \leq i \leq n-1$). Therefore, it follows from the properties of the classes $\mathcal{C}\text{-}\mathcal{BA}$ and $\mathcal{C}\text{-}w\mathcal{BA}$ proved above that statement 1. also holds for the classes $\mathcal{C}\text{-}\mathcal{BN}$, $\mathcal{C}\text{-}\mathcal{BS}$, $\mathcal{C}\text{-}w\mathcal{BN}$, and $\mathcal{C}\text{-}w\mathcal{BS}$.

2. Let Y be a quotient group of X . By statement 1., each primary $\mathfrak{P}(\mathcal{C})$ -component T of Y is a $\mathcal{C}\text{-}w\mathcal{BS}$ -group, i.e., has a subnormal series with $\mathcal{C}\text{-}w\mathcal{BA}$ -factors. Any factor F of this series is a p -group for some $p \in \mathfrak{P}(\mathcal{C})$ and hence is of finite exponent. Therefore, the exponent of T is also finite and $X \in \mathcal{C}\text{-}w\mathcal{BA}$. If $X \in \mathcal{C}\text{-}\mathcal{BS}$, then $T \in \mathcal{C}\text{-}\mathcal{BS}$, $F \in \mathcal{C}\text{-}\mathcal{BA}$, and the cardinality of F does not exceed the cardinality of some \mathcal{C} -group. The group T also enjoys the last property because \mathcal{C} is closed under taking extensions. Therefore, $X \in \mathcal{C}\text{-}\mathcal{BA}$. \square

It follows from Proposition 5.2, in particular, that a nilpotent (solvable) group is (weakly) \mathcal{C} -bounded if and only if the factors of *any* of its central (solvable) series are (weakly) \mathcal{C} -bounded abelian groups.

6 \mathcal{C} -regularity and \mathcal{C} -quasiregularity

Suppose that \mathcal{C} is a class of groups and X is a group. Let us say that X is

- \mathcal{C} -quasiregular with respect to its subgroup Y if, for any subgroup $M \in \mathcal{C}^*(Y)$, there exists a subgroup $N \in \mathcal{C}^*(X)$ such that $N \cap Y \leq M$;
- \mathcal{C} -regular with respect to its normal subgroup Y if, for any subgroup $M \in \mathcal{C}^*(Y)$ normal in X , there exists a subgroup $N \in \mathcal{C}^*(X)$ such that $N \cap Y = M$.

These concepts often serve as parts of conditions sufficient for a free construction of group to be residually a \mathcal{C} -group. The main goal of this section is to prove that a $\mathcal{C}\text{-}\mathcal{BN}$ -group has the properties of \mathcal{C} -regularity and \mathcal{C} -quasiregularity for any root class \mathcal{C} consisting only of periodic groups.

Proposition 6.1. *Let \mathcal{C} be a root class of groups consisting only of periodic groups. If a periodic \mathcal{C} - $\mathcal{B}\mathcal{S}$ -group X has a finite exponent, which is a $\mathfrak{P}(\mathcal{C})$ -number, then $X \in \mathcal{C}$.*

Proof. By Proposition 5.2, an arbitrary factor F of some solvable series of X belongs to the class \mathcal{C} - $\mathcal{B}\mathcal{A}$. Since F has a finite exponent, which is a $\mathfrak{P}(\mathcal{C})$ -number, the number of its p -power torsion subgroups is finite, they all correspond to numbers from $\mathfrak{P}(\mathcal{C})$ and therefore have cardinalities not exceeding the cardinality of some \mathcal{C} -group. Hence, the cardinality of X also does not exceed the cardinality of some \mathcal{C} -group because \mathcal{C} is closed under taking extensions, and $X \in \mathcal{C}$ by Proposition 4.3. \square

Proposition 6.2. *Suppose that \mathcal{C} is a root class of groups consisting only of periodic groups, X is a \mathcal{C} - $w\mathcal{B}\mathcal{N}$ -group, and Y is a periodic subgroup of X . If the exponent m of Y is finite and is a $\mathfrak{P}(\mathcal{C})$ -number, then $Y \cap X^{m^n} = 1$ for some $n \geq 1$.*

Proof. Suppose that \mathfrak{S} is the set of all the prime divisors of m and

$$1 = X_0 \leq X_1 \leq \dots \leq X_n = X$$

is some central series of X . Since $X \in \mathcal{C}$ - $w\mathcal{B}\mathcal{N}$ and $\mathfrak{S} \subseteq \mathfrak{P}(\mathcal{C})$, it follows that, for any $i \in \{0, \dots, n-1\}$, $p \in \mathfrak{S}$, the relation $X_{i+1}/X_i \in \mathcal{C}$ - $w\mathcal{B}\mathcal{A}$ holds and the exponent $e_{i,p}$ of the p -power torsion subgroup of X_{i+1}/X_i is finite. Because \mathfrak{S} is also finite, the product

$$q = \prod_{\substack{0 \leq i \leq n-1, \\ p \in \mathfrak{S}}} e_{i,p}$$

is defined.

It is easy to see that, if $x \in X$ and the order of x is an \mathfrak{S} -number, then $x^q = 1$. Therefore, the equation $x^q = y$ is not solvable in X for any $y \in Y \setminus \{1\}$. But Proposition 4.10 says that it is solvable in X for every $y \in X^{q^c}$, where c is the nilpotency class of X . Hence, $Y \cap X^{q^c} = 1$. Since any prime divisor of q also divides m , we have $q \mid m^d$ for some $d \geq 1$, and the number $n = cd$ is the desired one. \square

Proposition 6.3. *Suppose that \mathcal{C} is a root class of groups consisting only of periodic groups and $\mathcal{N}\mathcal{C}$ is the class of all nilpotent \mathcal{C} -groups. Suppose also that X is a group, Y is a subgroup of X , and there exists a homomorphism σ of X onto a \mathcal{C} - $\mathcal{B}\mathcal{N}$ -group that acts injectively on Y . Then the following statements hold.*

1. For any subgroup $M \in \mathcal{C}^*(Y)$, there exists a subgroup $N \in \mathcal{N}\mathcal{C}^*(X)$ such that $N \cap Y \leq M$. In particular, X is \mathcal{C} -quasiregular with respect to Y .
2. Suppose that Y is normal in X . Then, for any subgroup $M \in \mathcal{C}^*(Y)$ normal in X , there exists a subgroup $N \in \mathcal{N}\mathcal{C}^*(X)$ such that $N \cap Y = M$. In particular, X is \mathcal{C} -regular with respect to Y .

Proof. 1. We fix some subgroup $M \in \mathcal{C}^*(Y)$ and assume first that $X \in \mathcal{C}\text{-}\mathcal{B}\mathcal{N}$. Denote the normalizer $N_X(Y)$ of Y in X by Y_1 and inductively the subgroup $N_X(Y_i)$ by Y_{i+1} . It is well known (see, e.g., [9, Theorem 2.9]) that $Y_n = X$ for some n . Let us argue by induction on n .

Suppose that $n = 1$, i.e., Y is normal in X , and denote by q the exponent of the \mathcal{C} -group Y/M , which is finite by Proposition 4.2. Then the subgroup Y^q is also normal in X and $Y^q \leq M$. The quotient group $\overline{X} = X/Y^q$ belongs to the class $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}$ by Proposition 5.2, and the exponent of the subgroup $\overline{Y} = Y/Y^q$ is a $\mathfrak{P}(\mathcal{C})$ -number. Therefore, it follows from Proposition 6.1 and Proposition 6.2 that $\overline{Y} \cap \overline{X}^r = 1$ for some $\mathfrak{P}(\mathcal{C})$ -number r and $\overline{X}/\overline{X}^r \in \mathcal{C}$. Let N be the preimage of \overline{X}^r under the natural homomorphism $X \rightarrow \overline{X}$. Then we have $N \in \mathcal{C}^*(X)$ and $N \cap Y \leq Y^q \leq M$. Thus, N is the required subgroup.

Suppose now that $n \geq 2$. Since Y is normal in Y_1 , then, as proved above, there exists a subgroup $N_1 \in \mathcal{C}^*(Y_1)$ satisfying the condition $N_1 \cap Y \leq M$. We apply the inductive hypothesis to the subgroups Y_1, N_1 and find a subgroup $N \in \mathcal{C}^*(X)$ such that $N \cap Y_1 \leq N_1$. Then

$$N \cap Y = N \cap Y_1 \cap Y \leq N_1 \cap Y \leq M,$$

and therefore, N is the desired subgroup.

In the general case (i.e., if X does not necessarily belong to $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}$), it follows from the above and the relations

$$X\sigma \in \mathcal{C}\text{-}\mathcal{B}\mathcal{N}, \quad Y\sigma/M\sigma \cong Y/M(Y \cap \ker \sigma) = Y/M \in \mathcal{C}$$

that there exists a subgroup $\overline{N} \in \mathcal{C}^*(X\sigma)$ satisfying the condition $\overline{N} \cap Y\sigma \leq M\sigma$. Since $X\sigma$ is nilpotent, $\overline{N} \in \mathcal{N}\mathcal{C}^*(X\sigma)$. Therefore, the preimage N of \overline{N} under σ belongs to $\mathcal{N}\mathcal{C}^*(X)$, and $N \cap Y \leq M$ because $Y \cap \ker \sigma = 1$. Hence, N is the required subgroup.

2. If a subgroup $M \in \mathcal{C}^*(Y)$ is normal in X , then $M\sigma$ is normal in $X\sigma$ and $X\sigma/M\sigma \in \mathcal{C}\text{-}\mathcal{B}\mathcal{N}$ by Proposition 5.2. As proved above, $Y\sigma/M\sigma \in \mathcal{C}$. Therefore, we can apply statement 1. to the group $\overline{X} = X\sigma/M\sigma$ and the subgroups $\overline{Y} = Y\sigma/M\sigma$ and $\{1\}$. It follows that there exists a subgroup $\overline{N} \in \mathcal{N}\mathcal{C}^*(\overline{X})$ such that $\overline{N} \cap \overline{Y} = 1$. Denote by N the preimage of \overline{N} under the composition of σ and the natural homomorphism $X\sigma \rightarrow \overline{X}$. It is easy to see that $N \in \mathcal{N}\mathcal{C}^*(X)$ and $N \cap Y = M$. Thus, N is the desired subgroup. \square

7 Proofs of Theorem 2.1–2.4 and Corollary 2.3

Proof of Theorem 2.1. 1. Assume that $X \notin \mathcal{C}\text{-}w\mathcal{BA}$. Then, by Proposition 5.1, X does not satisfy (2) $_{\mathcal{C}}$, and hence, there exists a subgroup Y of X such that the quotient group X/Y contains a p -quasicyclic subgroup for some $p \in \mathfrak{P}(\mathcal{C})$.

Let $T/Y = \{p\}'\text{-}\mathfrak{Zs}(X/Y, 1)$. Since $\{p\}'\text{-}\mathfrak{Zs}(X/Y, 1) = \{p\}'\text{-}\mathfrak{Rt}(X/Y, 1)$ by Proposition 4.9, the quotient group $X/T \cong (X/Y)/(T/Y)$ also contains a p -quasicyclic subgroup and therefore is not residually a \mathcal{D} -group, where \mathcal{D} is the class of all periodic $\mathfrak{P}(\mathcal{C})$ -groups of finite exponent. Because this class is closed under taking quotient groups, it follows from Proposition 4.4 that T is not \mathcal{D} -separable in X . Since $\mathcal{C} \subseteq \mathcal{D}$ by Proposition 4.2, T cannot be \mathcal{C} -separable in X . At the same time, the torsion subgroup of X/T is a p -group, so T is $\mathfrak{P}(\mathcal{C})'$ -isolated in X .

2. Let us choose a $\mathfrak{P}(\mathcal{C})'$ -isolated subgroup Y of X and show that the quotient group $A = X/Y$ is residually a \mathcal{P} -group. Then it will follow from Proposition 4.4 that Y is \mathcal{P} -separable in X . To do this, we fix a non-trivial element a of A and assume first that it belongs to a p -power torsion subgroup T of A for some prime number p .

As Y is $\mathfrak{P}(\mathcal{C})'$ -isolated in X , it follows that $p \in \mathfrak{P}(\mathcal{C})$ and, by condition (3) $_{\mathcal{C}}$, the exponent q of T is finite. It is easy to see that $A^q \cap T = 1$ and hence $aA^q \neq 1$. It remains to note that, by the first Prüfer theorem, the periodic p -group A/A^q decomposes into a direct product of cyclic subgroups and therefore is residually p -finite.

If a has a finite order r that is not a power of a prime number, the proof reduces to the case considered above. It is only necessary to choose some prime divisor p of r and pass to the quotient group $A/\{p\}'\text{-}\mathfrak{Zs}(A, 1)$.

Suppose now that the order of a is infinite. By Proposition 5.2, the quotient group $A/\langle a \rangle$ also satisfies (3) $_{\mathcal{C}}$, and therefore each of its primary $\mathfrak{P}(\mathcal{C})$ -components has a finite exponent. It follows that, for any $p \in \mathfrak{P}(\mathcal{C})$, there exists a number s such that $a \notin A^{p^s}$. Hence, we can use the same reasoning as in the first case, it only remains to note that $\mathfrak{P}(\mathcal{C}) \neq \emptyset$ (because \mathcal{C} is a root class), and choose an arbitrary number $p \in \mathfrak{P}(\mathcal{C})$.

Thus, any $\mathfrak{P}(\mathcal{C})'$ -isolated subgroup of X is \mathcal{P} -separable. Since $\mathfrak{P}(\mathcal{P}) = \mathfrak{P}(\mathcal{C})$ and $\mathcal{P} \subseteq \mathcal{C}$ by Proposition 4.1, it follows that X has the properties $\mathcal{P}\text{-}\mathfrak{Sep}$ and $\mathcal{C}\text{-}\mathfrak{Sep}$. \square

The next proposition generalizes [10, Theorem 2.1].

Proposition 7.1. *Let \mathcal{C} be a root class of groups consisting only of periodic groups. A $\mathcal{C}\text{-}\mathcal{BN}$ -group X is residually a \mathcal{C} -group if and only if it has no $\mathfrak{P}(\mathcal{C})'$ -torsion.*

Proof. The necessity follows from Proposition 4.5. To prove the sufficiency, we use induction on the nilpotency class of X and assume that the base of the induction is the case when $X = 1$. Let us fix an element $x \in X \setminus \{1\}$ and indicate a subgroup $N \in \mathcal{C}^*(X)$ such that $x \notin N$.

If $\mathcal{Z}(X)$ denotes the center of X , then, by the induction hypothesis and Proposition 4.9, we get that $X/\mathcal{Z}(X)$ is residually a \mathcal{C} -group. Therefore, we can assume that $x \in \mathcal{Z}(X)$. Since $\mathcal{Z}(X) \in \mathcal{C}\text{-}\mathcal{BA}$ by Proposition 5.2 and the trivial subgroup of $\mathcal{Z}(X)$ is $\mathfrak{P}(\mathcal{C})'$ -isolated, it follows from Theorem 2.1 that there exists a subgroup $Y \in \mathcal{C}^*(\mathcal{Z}(X))$ not containing x . Therefore, the element $\bar{x} = xY$ of the group $\bar{X} = X/Y$ is non-trivial and has a finite order, which is a $\mathfrak{P}(\mathcal{C})$ -number. By Propositions 5.2 and 4.1, \bar{X} is again a $\mathcal{C}\text{-}\mathcal{BN}$ -group and the cyclic subgroup U generated by \bar{x} belongs to \mathcal{C} . Hence, we can apply Proposition 6.3 to the group \bar{X} and the subgroups $U, \{1\}$, and find a subgroup $\bar{N} \in \mathcal{C}^*(\bar{X})$ such that $\bar{N} \cap U = 1$. Then $\bar{x} \notin \bar{N}$, and therefore, the preimage N of \bar{N} under the natural homomorphism $X \rightarrow \bar{X}$ is the required subgroup. \square

Proof of Theorem 2.2. Let us denote the subgroup $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{S}\mathfrak{s}(X, Y)$ by \mathfrak{S} for convenience. The equality $\mathfrak{S} = \mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{H}\mathfrak{t}(X, Y)$ follows from Proposition 4.9. If X has no $\mathfrak{P}(\mathcal{C})'$ -torsion, then it is residually a \mathcal{C} -group by Proposition 7.1 and the nilpotency classes of $\mathcal{C}\text{-}\mathcal{CI}(X, Y)$ and Y coincide by Proposition 4.8. It follows from Proposition 4.5 that $Y \leq \mathfrak{S} \leq \mathcal{C}\text{-}\mathcal{CI}(X, Y)$. Therefore, \mathfrak{S} has the same nilpotency class as Y . Thus, it remains to verify the \mathcal{C} -separability of \mathfrak{S} . As in the proof of Proposition 6.3, we use induction on the length n of the sequence of subgroups

$$\mathfrak{S} = \mathfrak{S}_0 \leq \mathfrak{S}_1 \leq \dots \leq \mathfrak{S}_n = X,$$

where \mathfrak{S}_{i+1} denotes the normalizer of \mathfrak{S}_i in X .

If $n = 1$, then \mathfrak{S} is normal in X , X/\mathfrak{S} is residually a \mathcal{C} -group by Proposition 7.1, and therefore, \mathfrak{S} is \mathcal{C} -separable in X by Proposition 4.4. Thus, we assume that $n \geq 2$, fix an arbitrary element $x \in X \setminus \mathfrak{S}$, and show that there exists a subgroup $N \in \mathcal{C}^*(X)$ satisfying the condition $x \notin \mathfrak{S}N$.

Since the subgroup $\mathfrak{S} = \mathfrak{S}_0$ is $\mathfrak{P}(\mathcal{C})'$ -isolated in X , \mathfrak{S}_1 has the same property by Proposition 4.9. Hence, if $x \notin \mathfrak{S}_1$, then, by the induction hypothesis, there exists a subgroup $N \in \mathcal{C}^*(X)$ such that $x \notin \mathfrak{S}_1N$. It follows that $x \notin \mathfrak{S}N$, and therefore, N is the required subgroup.

Let $x \in \mathfrak{S}_1$. Since \mathfrak{S} is $\mathfrak{P}(\mathcal{C})'$ -isolated in \mathfrak{S}_1 , it follows from the above that there exists a subgroup $M \in \mathcal{C}^*(\mathfrak{S}_1)$ satisfying the condition $x \notin \mathfrak{S}M$. By applying Proposition 6.3 to the group X and the subgroups \mathfrak{S}_1, M , we find a subgroup $N \in \mathcal{C}^*(X)$ such that $N \cap \mathfrak{S}_1 \leq M$. It is easy to see that $\mathfrak{S}N \cap \mathfrak{S}_1 \leq \mathfrak{S}M$ and hence $x \notin \mathfrak{S}N$. Thus, N is the desired subgroup. \square

Proof of Corollary 2.3. It follows from Theorem 2.2 that we only need to prove the necessity. Suppose that X is a torsion-free nilpotent group with the property $\mathcal{C}\text{-}\mathfrak{Sep}$,

$$1 = X_0 \leq X_1 \leq \dots \leq X_c = X \quad (*)$$

is its upper central series, and \mathcal{D} is the class of all periodic $\mathfrak{B}(\mathcal{C})$ -groups of finite exponent. By Proposition 4.2, $\mathcal{C} \subseteq \mathcal{D}$, and since $\mathfrak{B}(\mathcal{C}) = \mathfrak{B}(\mathcal{D})$, then X has the property $\mathcal{D}\text{-}\mathfrak{Sep}$. Let us show that all the factors of the series (*) also have this property.

Indeed, suppose that $i \in \{0, \dots, c-1\}$, Y/X_i is a $\mathfrak{B}(\mathcal{D})'$ -isolated subgroup of X_{i+1}/X_i , and $x \in X_{i+1} \setminus Y$. Since X is torsion-free, it follows from Proposition 4.9 that all the factors of the series (*) are isolated in X . Therefore, Y is $\mathfrak{B}(\mathcal{D})'$ -isolated in X and, by the property $\mathcal{D}\text{-}\mathfrak{Sep}$, is \mathcal{D} -separable in this group. Hence, there exists a subgroup $N \in \mathcal{D}^*(X)$ satisfying the condition $x \notin YN$. Then $xX_i \notin Y/X_i \cdot (NX_i \cap X_{i+1})/X_i$ and

$$\begin{aligned} NX_i &\in \mathcal{D}^*(X), \quad NX_i \cap X_{i+1} \in \mathcal{D}^*(X_{i+1}), \\ (NX_i \cap X_{i+1})/X_i &\in \mathcal{D}^*(X_{i+1}/X_i) \end{aligned}$$

because \mathcal{D} is closed under taking subgroups and quotient groups. Since x is chosen arbitrarily, it follows that Y/X_i is \mathcal{D} -separable in X_{i+1}/X_i .

Thus, for each $i \in \{0, \dots, c-1\}$, the quotient group X_{i+1}/X_i has the property $\mathcal{D}\text{-}\mathfrak{Sep}$ and, by Theorem 2.1, satisfies (3) $_{\mathcal{D}}$. Since $\mathfrak{B}(\mathcal{C}) = \mathfrak{B}(\mathcal{D})$, conditions (3) $_{\mathcal{C}}$ and (3) $_{\mathcal{D}}$ are equivalent. As noted above, X_i is isolated in X . Therefore, X_{i+1}/X_i is torsion-free and, by Proposition 5.1, satisfies (5) $_{\mathcal{C}}$. Therefore, $X \in \mathcal{C}\text{-}\mathcal{BN}$. \square

Proof of Theorem 2.4. 1. Suppose that

$$x \in \mathcal{N}\mathcal{C}\text{-}\mathfrak{CI}(X, Y), \quad V = \ker \sigma, \quad U \in \mathcal{C}\text{-}\mathcal{BN}_{\mathfrak{B}(\mathcal{C})}^*(X), \quad \text{and} \quad W = U \cap V.$$

It follows from Proposition 5.2 that the class $\mathcal{C}\text{-}\mathcal{BN}_{\mathfrak{B}(\mathcal{C})}$ is closed under taking subgroups and finite direct products. So $W \in \mathcal{C}\text{-}\mathcal{BN}_{\mathfrak{B}(\mathcal{C})}^*(X)$ by Proposition 4.6.

It is not difficult to show, by using Proposition 4.7, that

$$xV \in \mathcal{N}\mathcal{C}\text{-}\mathfrak{CI}(X/V, YV/V) \quad \text{and} \quad xW \in \mathcal{N}\mathcal{C}\text{-}\mathfrak{CI}(X/W, YW/W).$$

It follows from the nilpotency of the groups X/V , X/W and Theorem 2.2 that the equalities

$$\begin{aligned} \mathcal{N}\mathcal{C}\text{-}\mathfrak{CI}(X/V, YV/V) &= \mathcal{C}\text{-}\mathfrak{CI}(X/V, YV/V) \\ &= \mathfrak{B}(\mathcal{C})'\text{-}\mathfrak{It}(X/V, YV/V), \end{aligned}$$

$$\begin{aligned} \mathcal{N}\mathcal{C}\text{-}\mathcal{C}\mathcal{I}(X/W, YW/W) &= \mathcal{C}\text{-}\mathcal{C}\mathcal{I}(X/W, YW/W) \\ &= \mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{Rt}(X/W, YW/W) \end{aligned}$$

hold. Hence, there exist the least $\mathfrak{P}(\mathcal{C})'$ -numbers q and r satisfying the conditions $(xV)^q \in YV/V$ and $(xW)^r \in YW/W$. Let $y, z \in Y$ be elements such that $yV = x^qV$ and $zW = x^rW$. It follows from the last equality and the inclusion $W \leq V$ that $(xV)^r = zV \in YV/V$. By the choice of q and r , the equality $r = qs$ holds for some $\mathfrak{P}(\mathcal{C})'$ -number s . Hence, $y^sV = x^{qs}V = x^rV = zV$, and therefore, $y^{-s}z \in Y \cap V$. But $Y \cap V = 1$, so

$$z = y^s \quad \text{and} \quad (xW)^{qs} = (xW)^r = zW = (yW)^s.$$

Since the $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{P}(\mathcal{C})}$ -group X/W is nilpotent and has no $\mathfrak{P}(\mathcal{C})'$ -torsion, then, by Proposition 4.9, the extraction of $\mathfrak{P}(\mathcal{C})'$ -roots is unique in this group. Therefore, $(xW)^q = yW$ and $y^{-1}x^q \in W \leq U$. Since U is chosen arbitrarily and X is residually a $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{P}(\mathcal{C})}$ -group, $x^q = y \in Y$. Thus, the subgroup $\mathcal{N}\mathcal{C}\text{-}\mathcal{C}\mathcal{I}(X, Y)$ coincides with the set $\mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{Rt}(X, Y)$ and the subgroup $\mathfrak{S} = \mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{S}\mathfrak{s}(X, Y)$. This means, in particular, that \mathfrak{S} is $\mathcal{N}\mathcal{C}$ -separable in X .

The group X is residually a $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{P}(\mathcal{C})}$ -group, while a $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{P}(\mathcal{C})}$ -group is nilpotent and, by Proposition 7.1, is residually an $\mathcal{N}\mathcal{C}$ -group. Therefore, X is residually an $\mathcal{N}\mathcal{C}$ -group and so has no $\mathfrak{P}(\mathcal{C})'$ -torsion in accordance with Proposition 4.5. The nilpotency classes of $\mathcal{N}\mathcal{C}\text{-}\mathcal{C}\mathcal{I}(X, Y) = \mathfrak{S}$ and Y coincide by Proposition 4.8.

2. Suppose again that $V = \ker \sigma$, $\mathfrak{S} = \mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{S}\mathfrak{s}(X, Y)$, and $x \in \mathfrak{S} \cap V$. Since $\mathfrak{S} = \mathfrak{P}(\mathcal{C})'\text{-}\mathfrak{Rt}(X, Y)$, it follows that $x^q \in Y$ for some $\mathfrak{P}(\mathcal{C})'$ -number q . Then $x^q \in Y \cap V = 1$ and $x = 1$ because X has no $\mathfrak{P}(\mathcal{C})'$ -torsion. Hence, $\mathfrak{S} \cap V = 1$, the subgroup \mathfrak{S} is embedded into the $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{P}(\mathcal{C})}$ -group X/V and therefore itself belongs to the class $\mathcal{C}\text{-}\mathcal{B}\mathcal{N}_{\mathfrak{P}(\mathcal{C})}$. \square

8 Proof of Theorems 3.1–3.6

Let us use the notation introduced in Section 3.

Proposition 8.1 ([26, Theorem 2]). *Suppose that \mathcal{C} is an arbitrary root class of groups, $P = \langle A * B; U \rangle$, and U is a retract of B . If A and B are residually \mathcal{C} -groups, U is \mathcal{C} -separable in A , and A is \mathcal{C} -quasiregular with respect to U , then P is residually a \mathcal{C} -group.*

Proposition 8.2 ([29, Theorem 1]). *Suppose that \mathcal{C} is a root class of groups closed under taking quotient groups, $P = \langle A * B; U \rangle$, $A \neq U \neq B$, and U is normal*

in A and B . Suppose also that

$$\Omega = \{(R, S) \mid R \in \mathcal{C}^*(A), S \in \mathcal{C}^*(B), R \cap U = S \cap U\}.$$

If $\text{Aut}_P(U)$ is abelian or coincides with $\text{Aut}_A(U)$ or $\text{Aut}_B(U)$, then P is residually a \mathcal{C} -group if and only if

- 1) $\bigcap_{(R,S) \in \Omega} R = 1 = \bigcap_{(R,S) \in \Omega} S$;
- 2) U is \mathcal{C} -separable in A and B .

Proposition 8.3 ([25, Theorem 3]). *Suppose that \mathcal{C} is a root class of groups consisting only of periodic groups and closed under taking quotient groups,*

$$G^* = \langle G, t; t^{-1}Ht = K, \varphi \rangle, \quad K_0 = G, \quad H_1 = H, \quad K_1 = K, \\ \text{and } H_{i+1} = H_i \cap K_i, \quad K_{i+1} = H_{i+1}\varphi \quad \text{for all } i \geq 1.$$

Suppose also that G is residually a \mathcal{C} -group, H and K are proper central subgroups of G , and $H_n = H_{n+1}$ for some $n \geq 1$. If, for each $i \in \{0, 1, \dots, n-1\}$, the group K_i is \mathcal{C} -regular with respect to $H_{i+1}K_{i+1}$ and the subgroup

$$\mathfrak{B}(\mathcal{C})'\text{-}\mathfrak{I}\cong(K_i, H_{i+1}K_{i+1})$$

is \mathcal{C} -separable in K_i , then G^* is residually a \mathcal{C} -group if and only if

- 1) $H_n = K_n$;
- 2) the subgroup $E = \text{sgp}\{H_n, t\}$ is residually a \mathcal{C} -group;
- 3) H and K are $\mathfrak{B}(\mathcal{C})'$ -isolated in G .

Proposition 8.4 ([27, Theorem 3]). *Suppose that \mathcal{C} is a root class of groups consisting only of periodic groups and closed under taking quotient groups,*

$$G^* = \langle G, t; t^{-1}Ht = K, \varphi \rangle,$$

G is residually a \mathcal{C} -group, H and K are proper central infinite cyclic subgroups of G . If $H \cap K \neq 1$, then G^* is residually a \mathcal{C} -group if and only if

- 1) $H/H \cap K$ and $K/H \cap K$ have the same order;
- 2) H and K are \mathcal{C} -separable in G ;
- 3) \mathcal{C} contains a group of order 2, unless $H \cap K$ lies in the center of G^* .

Proposition 8.5 ([24, Theorem 3]). *Suppose that \mathcal{C} is a root class of groups closed under taking quotient groups and*

$$\mathcal{G}(\Gamma) = (\Gamma, G_v (v \in \mathcal{V}), H_e (e \in \mathcal{E}), \varphi_{\varepsilon e} (e \in \mathcal{E}, \varepsilon = \pm 1))$$

is a graph of groups with central trivially intersecting edge subgroups. If, for each $v \in \mathcal{V}$, G_v is residually a \mathcal{C} -group, H_v is \mathcal{C} -separable in G_v , and the latter is \mathcal{C} -regular with respect to H_v , then $\pi_1(\mathcal{G}(\Gamma))$ is residually a \mathcal{C} -group.

Proposition 8.6 ([24, Theorem 4]). *Suppose that \mathcal{C} is a root class of groups closed under taking quotient groups,*

$$\mathcal{G}(\Gamma) = (\Gamma, G_v (v \in \mathcal{V}), H_e (e \in \mathcal{E}), \varphi_{\varepsilon e} (e \in \mathcal{E}, \varepsilon = \pm 1)),$$

$H_e \varphi_{\varepsilon e}$ is a proper central subgroup of $G_{e(\varepsilon)}$ for all $e \in \mathcal{E}$, $\varepsilon = \pm 1$, and at least one of the following conditions hold:

- 1) *$\mathcal{G}(\Gamma)$ is a finite graph of groups with central trivially intersecting edge subgroups and, for each $v \in \mathcal{V}$, G_v is \mathcal{C} -regular with respect to H_v ;*
- 2) *Γ is a finite tree and, for any $e \in \mathcal{E}$, $\varepsilon = \pm 1$, $G_{e(\varepsilon)}$ is \mathcal{C} -regular with respect to $H_e \varphi_{\varepsilon e}$.*

Then $\pi_1(\mathcal{G}(\Gamma))$ is residually a \mathcal{C} -group if and only if all $G_v (v \in \mathcal{V})$ are residually \mathcal{C} -groups and, for any $e \in \mathcal{E}$, $\varepsilon = \pm 1$, $H_e \varphi_{\varepsilon e}$ is \mathcal{C} -separable in $G_{e(\varepsilon)}$.

The following proposition allows us to omit the requirement for the class of groups to be closed under taking quotient groups in the statements of certain assertions.

Proposition 8.7. *Suppose that we know a necessary or sufficient condition for a group X to be residually a \mathcal{C} -group, where \mathcal{C} is an arbitrary root class of groups consisting only of periodic groups and closed under taking quotient groups. If the parts of this condition related to \mathcal{C} depend only on the set $\mathfrak{B}(\mathcal{C})$ and condition (5) $_{\mathcal{C}}$, then the same condition holds for an arbitrary root class consisting of periodic groups.*

Proof. By the definition, a class of groups is root if and only if it is closed under the operations of taking a subgroup (\mathbb{S}), an extension (\mathbb{E}), and a Cartesian degree (\mathbb{D}). In addition to them, we also consider the operation of taking a quotient group (\mathbb{F}).

Suppose that \mathcal{D} is the class of all periodic $\mathfrak{B}(\mathcal{C})$ -groups of finite exponent, $\mathfrak{S}\mathcal{D}$ is the class of solvable \mathcal{D} -groups, and \mathcal{C}_1 is a subclass of $\mathfrak{S}\mathcal{D}$, in which the cardinality of each group does not exceed the cardinality of some \mathcal{C} -group (not necessarily the same for all groups from \mathcal{C}_1). Suppose also that \mathcal{C}_2 is the

class of groups obtained from \mathcal{C} -groups by a finite number of the operations \mathbb{S} , \mathbb{E} , \mathbb{D} , and \mathbb{F} . It is clear that \mathcal{C}_2 is closed under the listed operations. Since \mathcal{C} is closed under \mathbb{S} , \mathbb{E} , and \mathbb{D} , while \mathcal{D} and $\mathcal{S}\mathcal{D}$ are also closed under \mathbb{F} , which does not increase the cardinality of a group, it follows that \mathcal{C}_1 is closed under all four operations.

Obviously, $\mathcal{C} \subseteq \mathcal{C}_2$ and $\mathfrak{P}(\mathcal{C}_1) = \mathfrak{P}(\mathcal{C})$. Since \mathcal{D} is closed under \mathbb{F} and $\mathcal{C} \subseteq \mathcal{D}$ by Proposition 4.2, it follows that $\mathcal{C} \subseteq \mathcal{C}_2 \subseteq \mathcal{D}$, and therefore,

$$\mathfrak{P}(\mathcal{C}) \subseteq \mathfrak{P}(\mathcal{C}_2) \subseteq \mathfrak{P}(\mathcal{D}) = \mathfrak{P}(\mathcal{C}).$$

It follows from the inclusion $\mathcal{C}_1 \subseteq \mathcal{S}\mathcal{D} \cap \mathcal{C}\text{-}\mathcal{B}\mathcal{S}$ and Proposition 6.1 that $\mathcal{C}_1 \subseteq \mathcal{C}$. Thus,

$$\mathfrak{P}(\mathcal{C}_1) = \mathfrak{P}(\mathcal{C}) = \mathfrak{P}(\mathcal{C}_2) \quad \text{and} \quad \mathcal{C}_1 \subseteq \mathcal{C} \subseteq \mathcal{C}_2,$$

whence $(5)_{\mathcal{C}_1} \Rightarrow (5)_{\mathcal{C}} \Rightarrow (5)_{\mathcal{C}_2}$. Since the operation \mathbb{F} does not increase the cardinality, for each \mathcal{C}_2 -group X , there exists a \mathcal{C} -group Y whose cardinality is not less than the cardinality of X . Hence, $(5)_{\mathcal{C}} \Leftrightarrow (5)_{\mathcal{C}_2}$. Let us show that a similar statement holds for the classes \mathcal{C} and \mathcal{C}_1 .

Suppose that Y is a \mathcal{C} -group, $p \in \mathfrak{P}(\mathcal{C})$ (since \mathcal{C} is root, then $\mathfrak{P}(\mathcal{C}) \neq \emptyset$), Z_y ($y \in Y$) is a cyclic group of order p , and P is the direct product of the groups Z_y ($y \in Y$). Then $Z_y \in \mathcal{C}$ in accordance with Proposition 4.1 and $P \in \mathcal{C}$ by the definition of a root class. Hence, $P \in \mathcal{C}_1$ and, moreover, the cardinality of P is not less than the cardinality of Y . Therefore, $(5)_{\mathcal{C}_1} \Leftrightarrow (5)_{\mathcal{C}}$.

Thus, if X is residually a \mathcal{C} -group, then it is residually a \mathcal{C}_2 -group and satisfies the necessary condition (depending only on $\mathfrak{P}(\mathcal{C})$ and $(5)_{\mathcal{C}}$). If the sufficient condition holds (which also depends only on $\mathfrak{P}(\mathcal{C})$ and $(5)_{\mathcal{C}}$), then X is residually a \mathcal{C}_1 -group and therefore is residually a \mathcal{C} -group. \square

Proposition 8.7 allows us to assume that, in Theorems 3.2–3.6, the class \mathcal{C} is closed under taking quotient groups. Therefore, Theorems 3.5, 3.6, and also Theorem 3.1, follow from Theorem 2.4 and Propositions 6.3, 8.5, 8.6, and 8.1.

Proof of Theorem 3.2. By Theorem 2.4, A and B are residually \mathcal{C} -groups and U is \mathcal{C} -separable in these groups if and only if it is $\mathfrak{P}(\mathcal{C})'$ -isolated in them. Let us show that, for any subgroups $L \in \mathcal{C}^*(A)$ and $M \in \mathcal{C}^*(B)$, one can find subgroups $R \in \mathcal{C}^*(A)$ and $S \in \mathcal{C}^*(B)$ such that $R \leq L$, $S \leq M$, and $R \cap U = S \cap U$. Then the fact that A and B are residually \mathcal{C} -groups will mean that condition 1) of Proposition 8.2 is satisfied. In combination with Proposition 8.7, this proves the required statement.

So let $L \in \mathcal{C}^*(A)$, and let $M \in \mathcal{C}^*(B)$. Because \mathcal{C} is closed under taking subgroups, it follows from the relations

$$U/L \cap U \cong UL/L \leq A/L \in \mathcal{C}$$

that $L \cap U \in \mathcal{C}^*(U)$. Similarly, $M \cap U \in \mathcal{C}^*(U)$, and by Proposition 4.6, we have $L \cap M \cap U \in \mathcal{C}^*(U)$. Let q be the exponent of the group $U/(L \cap M \cap U)$, which is finite by Proposition 4.2, and let $N = U^q$. Then $U/N \in \mathcal{C}\text{-}\mathcal{B}\mathcal{N} \cap \mathcal{C}$ in accordance with Propositions 5.2 and 6.1. Since N is normal in A and B , and these groups are \mathcal{C} -regular with respect to U by Proposition 6.3, there exist subgroups $V \in \mathcal{C}^*(A)$ and $W \in \mathcal{C}^*(B)$ satisfying the equalities $V \cap U = N = W \cap U$. If $R = V \cap L$ and $S = W \cap M$, then $R \leq L$, $S \leq M$,

$$R \cap U = V \cap L \cap U = N \cap L = N = N \cap M = W \cap M \cap U = S \cap U,$$

and by Proposition 4.6, $R \in \mathcal{C}^*(A)$ and $S \in \mathcal{C}^*(B)$. Therefore, R and S are the required subgroups. \square

Proof of Theorem 3.4. If $H \cap K = 1$, then conditions 1) and 3) are satisfied automatically and the statement to be proved is a special case of Theorem 3.6. Let $H \cap K \neq 1$. By Theorem 2.4, G is residually a \mathcal{C} -group and each subgroup lying in HK is \mathcal{C} -separable in G if and only if it is $\mathfrak{B}(\mathcal{C})$ -isolated in this group. The inclusion $2 \in \mathfrak{B}(\mathcal{C})$ holds if and only if \mathcal{C} contains a group of order 2. As above, the class \mathcal{C} can be considered closed under taking quotient groups. Therefore, the required statement follows from Proposition 8.4. \square

To prove Theorem 3.3, we need the following two assertions in addition to Proposition 8.3.

Proposition 8.8 ([25, Proposition 5.7]). *Let \mathcal{C} be a root class of groups consisting only of periodic groups and closed under taking quotient groups. Suppose that Y is an abelian group, $\varphi \in \text{Aut } Y$, and X is the split extension of Y by an infinite cyclic group Z such that the conjugation by the generator of Z acts on Y as φ . Suppose also that Ω is a family of subgroups of Y defined as follows: $N \in \Omega$ if and only if $N \in \mathcal{C}^*(Y)$, $N\varphi = N$, and the order of the automorphism φ_N of Y/N induced by φ is finite and is a $\mathfrak{B}(\mathcal{C})$ -number. Then X is residually a \mathcal{C} -group if and only if $\bigcap_{N \in \Omega} N = 1$.*

Proposition 8.9 ([25, Proposition 6.8]). *Suppose that \mathcal{C} is a root class of groups consisting only of periodic groups and closed under taking quotient groups,*

$$G^* = \langle G, t; t^{-1}Ht = K, \varphi \rangle, \quad K_0 = G, \quad H_1 = H, \quad K_1 = K,$$

and $H_{i+1} = H_i \cap K_i, \quad K_{i+1} = H_{i+1}\varphi$ for all $i \geq 1$.

Suppose also that H and K are proper central subgroups of G , and there exists $m \geq 1$ such that H_m and K_m are finitely generated. If G^ is residually a \mathcal{C} -group, then $H_n = H_{n+1}$ for some n .*

Proof of Theorem 3.3. It follows from Proposition 8.7 that \mathcal{C} can be considered closed under taking quotient groups. Let us show that the conditions of Proposition 8.3 hold.

By Proposition 5.2, $H_i, K_i \in \mathcal{C}\text{-}\mathcal{BA}$ for all $i \geq 1$. Besides, σ acts injectively on $H_1 K_1$. Therefore, it follows from Proposition 6.3 and Theorem 2.4 that, for any $i \geq 0$, the group K_i is \mathcal{C} -regular with respect to $H_{i+1} K_{i+1}$ and the $\mathfrak{B}(\mathcal{C})'$ -isolator $\mathfrak{B}(\mathcal{C})'\text{-}\mathfrak{S}\cong(K_i, H_{i+1} K_{i+1})$ is \mathcal{C} -separable in K_i . The fact that G is residually a \mathcal{C} -group follows from Theorem 2.4 if G is residually a $\mathcal{C}\text{-}\mathcal{BN}_{\mathfrak{B}(\mathcal{C})}$ -group, and from the assumption that \mathcal{C} is closed under taking subgroups if G^* is residually a \mathcal{C} -group.

If $H_n = K_n$ for some $n > m$, then the restriction of φ on H_n is an automorphism of the latter, and therefore, $E = \text{sgp}\{H_n, t\}$ is a split extension of H_n by the infinite cyclic subgroup $\langle t \rangle$. If $N \in \mathcal{C}^*(H_n)$, then the periodic \mathcal{C} -group H_n/N is a finite $\mathfrak{B}(\mathcal{C})$ -group because H_m and K_m are finitely generated. Conversely, if H_n/N is a finite $\mathfrak{B}(\mathcal{C})$ -group, then $H_n/N \in \mathcal{C}$ in accordance with Proposition 4.1. Therefore, by Proposition 8.8, condition 3) of Theorem 3.3 holds if and only if $E = \text{sgp}\{H_n, t\}$ is residually a \mathcal{C} -group.

Finally, we note that $H_n = H_{n+1}$ for some $n > m$. This is obvious if $H_n = K_n$, and follows from Proposition 8.9 if G^* is residually a \mathcal{C} -group (this proposition does not state that $n > m$, but it is clear that if $H_n = K_n$ for some $n \geq 1$, then $H_l = K_l$ for all $l \geq n$). Thus, Theorem 3.3 follows from Proposition 8.3. \square

9 The example

Suppose that \mathcal{C} is a root class of groups consisting only of periodic groups. Let us show that if the cardinalities of all \mathcal{C} -groups are less than the cardinality of some set \mathcal{J} (and hence there are weakly \mathcal{C} -bounded abelian groups that are not \mathcal{C} -bounded), then there exists a $\mathfrak{B}(\mathcal{C})'$ -torsion-free weakly \mathcal{C} -bounded nilpotent group, which is not residually a \mathcal{C} -group and hence does not have the property $\mathcal{C}\text{-}\mathfrak{Sep}$.

Since \mathcal{C} contains non-trivial groups, $\mathfrak{B}(\mathcal{C}) \neq \emptyset$. Suppose that $p \in \mathfrak{B}(\mathcal{C})$, Y is the direct product of two cyclic groups of order p with generators y, z , and χ is the automorphism of Y defined by the rule $y\chi = yz, z\chi = z$. It is easy to see that the order of χ is equal to p , and therefore, we can define the split extension X of Y by the cyclic group of order p with a generator x , in which the conjugation by x acts on Y as χ . Let, for each $i \in \mathcal{J}$,

$$X_i = \langle x_i, y_i, z_i; x_i^p = y_i^p = z_i^p = [x_i, z_i] = [y_i, z_i] = 1, [y_i, x_i] = z_i \rangle$$

be an isomorphic copy of X , and let Γ be the star graph with a central vertex w and the set \mathcal{J} as a set of leaves. We define a graph of groups $\mathcal{G}(\Gamma)$ by associating

the vertex w with the group $Z = \langle z; z^p = 1 \rangle$, the vertex $i \in \mathcal{I}$ with the group X_i , and the edge connecting w and i with the group Z and the homomorphisms acting according to the rules: $z \mapsto z, z \mapsto z_i$.

Let D be the generalized direct product associated with $\mathcal{G}(\Gamma)$, i.e., the quotient group of the direct product of the groups X_i ($i \in \mathcal{I}$) and Z by the normal closure of the set $\{z_i z^{-1} \mid i \in \mathcal{I}\}$. Since Γ is a tree, Z is embedded into D by the identity mapping of the generator [28, Theorem 1]. Therefore, the relation $z \neq 1$ holds in D .

It is easy to see that the presentation of D can be reduced to the form

$$\begin{aligned} \langle x_i, y_i, z \ (i \in \mathcal{I}); x_i^p = y_i^p = z^p = 1, \\ [x_i, z] = [y_i, z] = 1, [y_i, x_i] = z \ (i \in \mathcal{I}), \\ [x_k, x_l] = [y_k, y_l] = [x_k, y_l] = 1 \ (k, l \in \mathcal{I}, k \neq l) \rangle, \end{aligned}$$

which implies that the sequence $1 \leq Z \leq D$ is a central series of D with factors of exponent p (here Z is still the subgroup generated by z). It is clear that D has no $\mathfrak{B}(\mathcal{C})'$ -torsion and the factors of the specified series are weakly \mathcal{C} -bounded, i.e., $D \in \mathcal{C}\text{-}w\mathcal{B}\mathcal{N}$. Let us show, however, that D is not residually a \mathcal{C} -group and therefore it is the required group.

Indeed, let σ be a homomorphism of D onto some \mathcal{C} -group T . Then the cardinality of \mathcal{I} exceeds the cardinality of T , and hence, there exist $k, l \in \mathcal{I}$ such that $k \neq l$ and $x_k \sigma = x_l \sigma$. It follows that $z \sigma = [y_k \sigma, x_k \sigma] = [y_k \sigma, x_l \sigma] = 1$, and since σ is chosen arbitrarily, D is not residually a \mathcal{C} -group.

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